Lecture 10a

Genesis of Calculus of Variations

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

We will discuss how the field of calculus of variations came into existence.

We will discuss what problems defined the field.

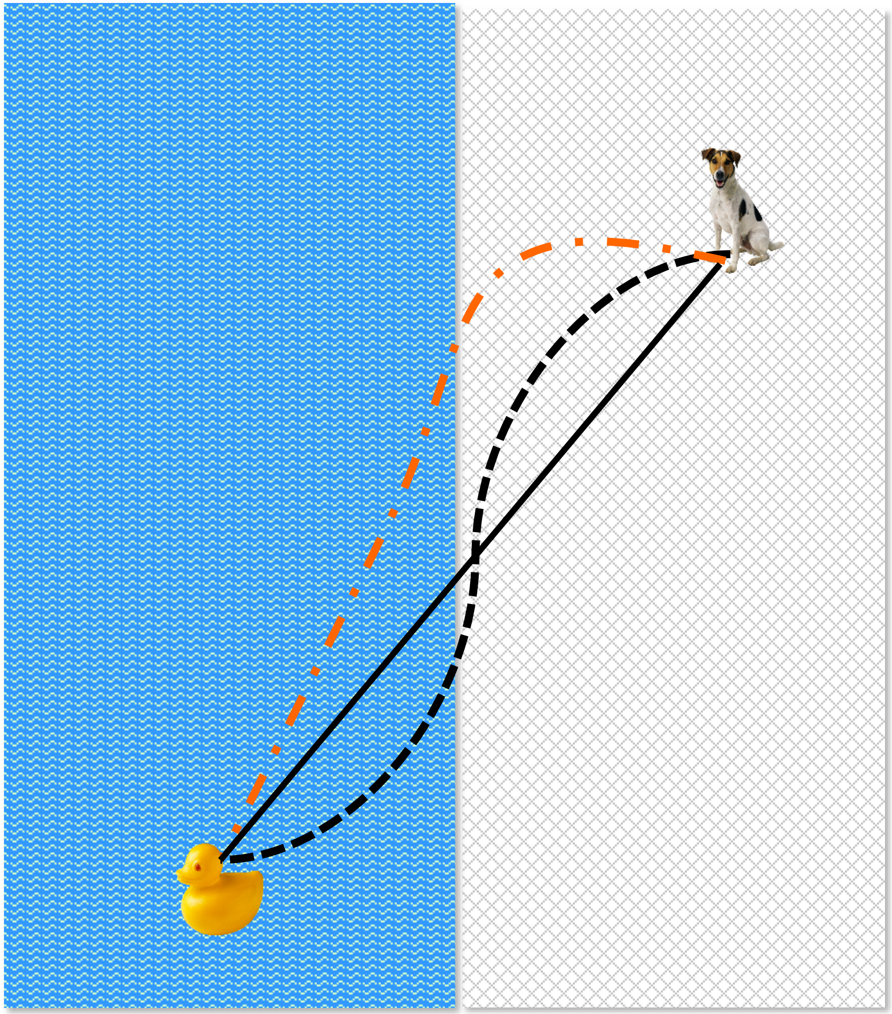
We will mention the names of individuals and their seminal works that created the field of calculus of variations.

What we will learn:

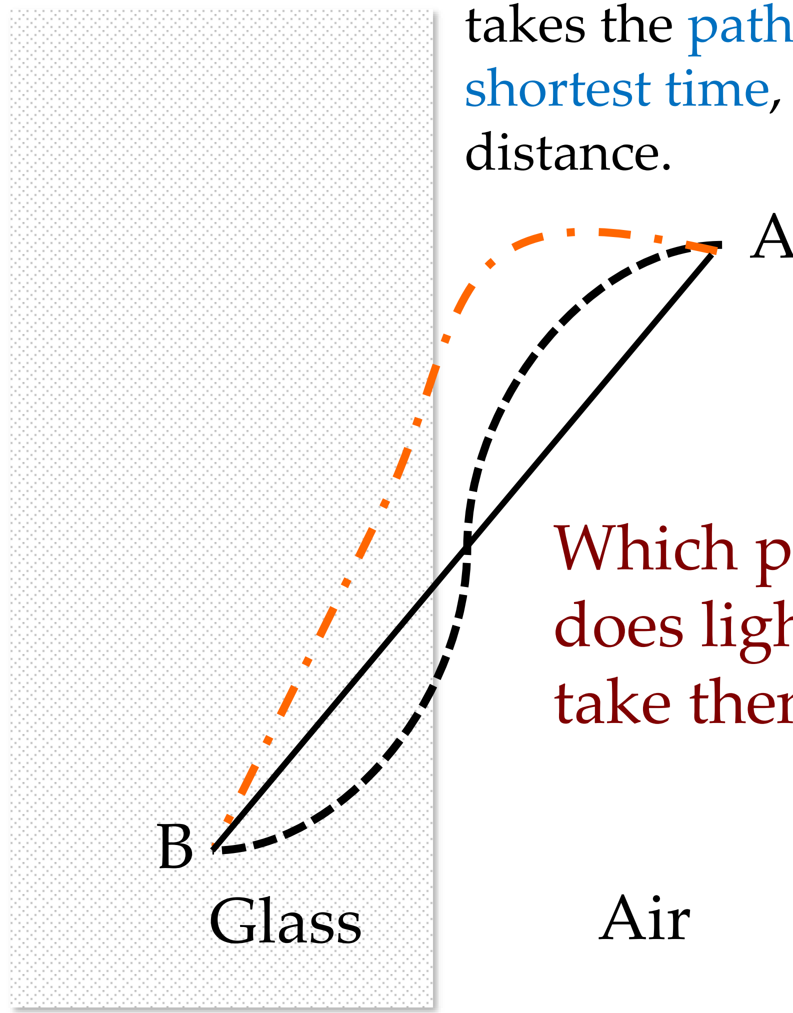
A brief history of calculus of variations from the ancient (BC) times to 20th century

The importance of calculus of variations in many fields and its fundamental nature

Fermat's problem

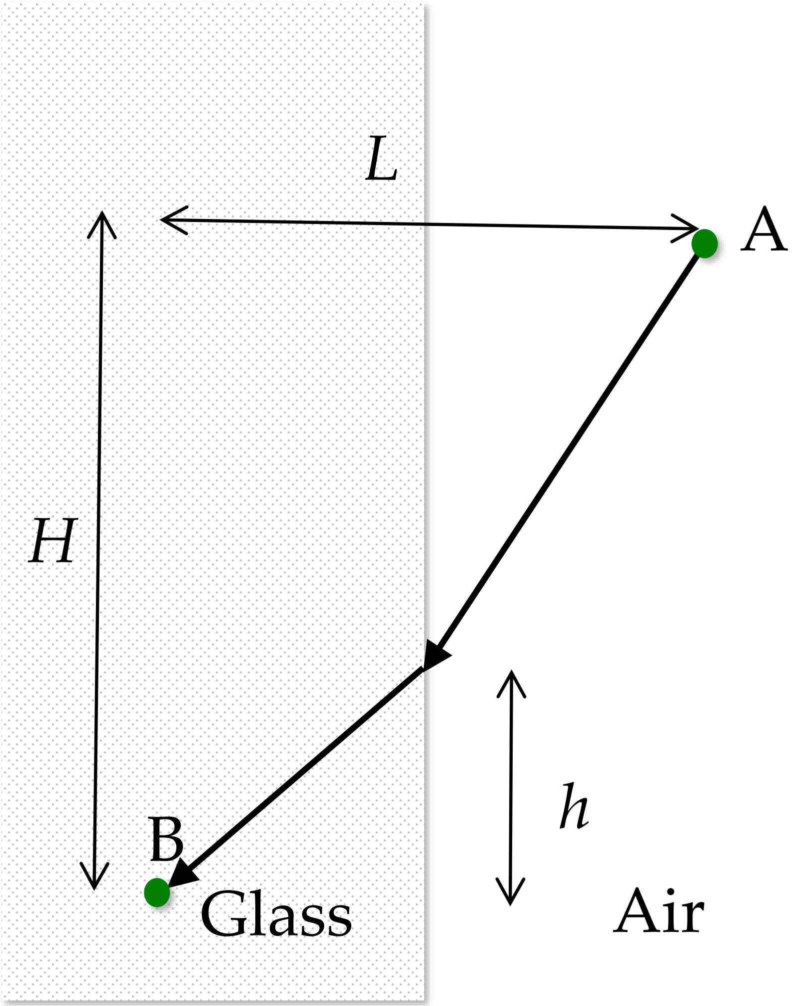


Light travels faster in air than in water. So, to go from A to B, light takes the path of shortest time, not distance.



Which path does light ray take then?

Fermat's conjecture



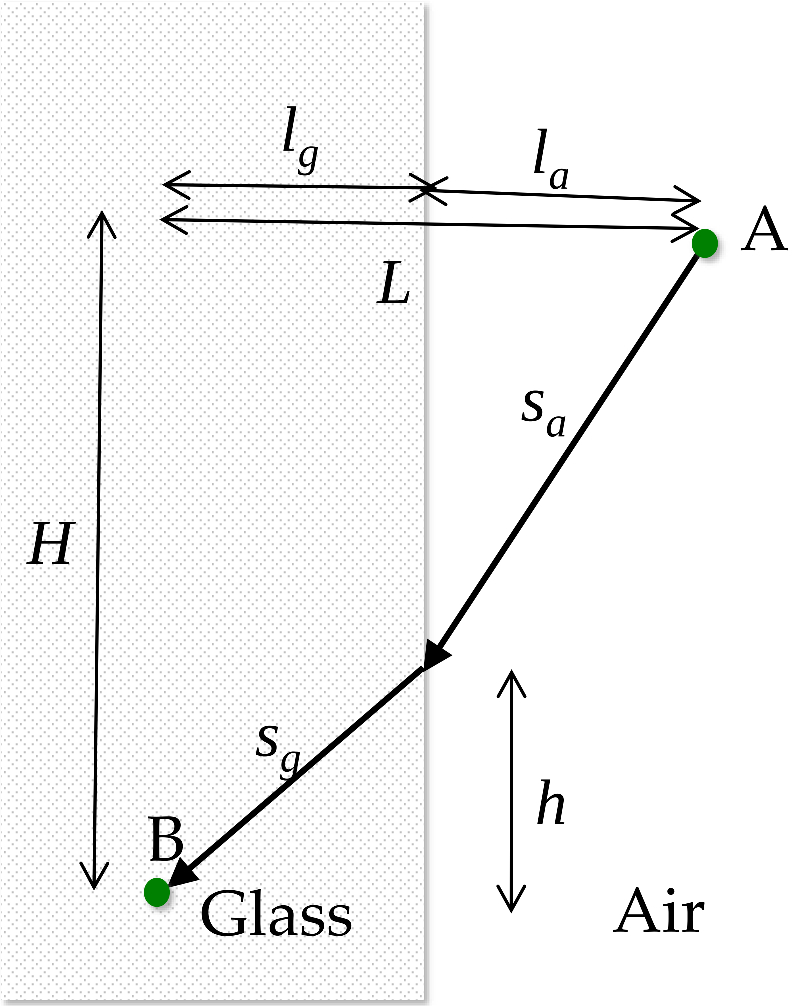
17th century amateur mathematician, Fermat (1601-1665), had conjectured that light rays take the shortest-time paths and **not** shortest-distance paths.

This is clear from the fact that light rays bend when they pass from one medium to another but will travel straight in one medium.

Here, light will take a slightly longer path in air than in water because it can travel faster in air.

How do we find h?

Bending of light ray (refraction)



Let v_a be the speed of light in air and v_g be that in glass.

Then, time of going from A to B =

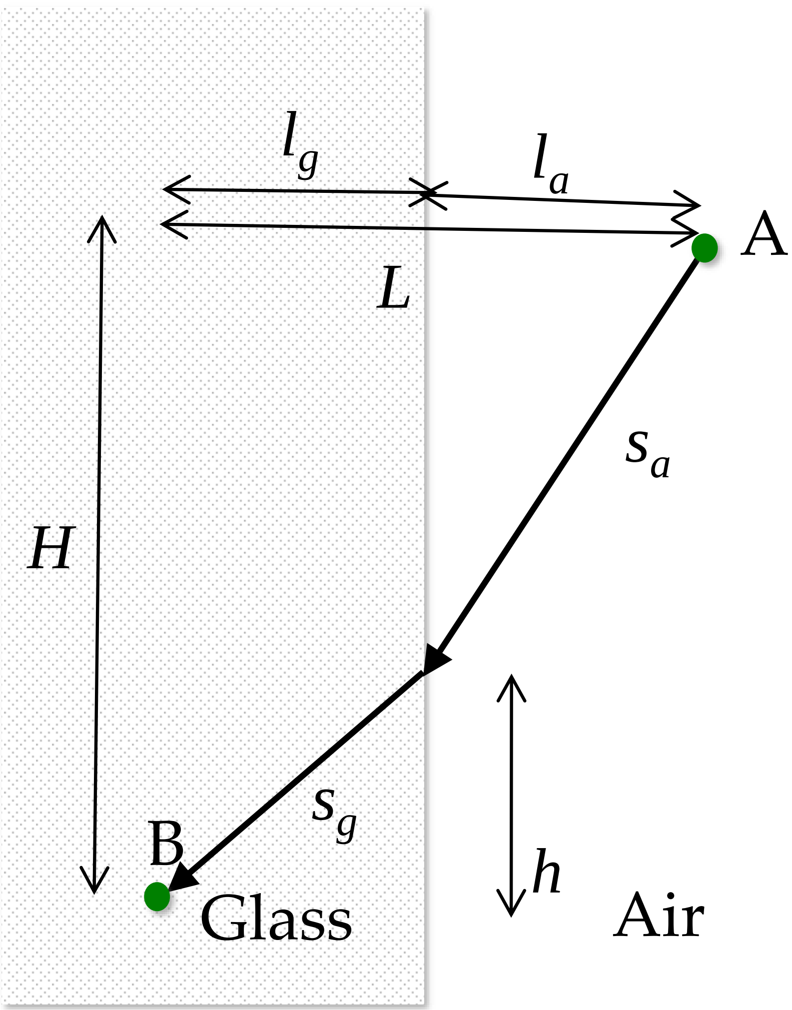
$$T = \frac{s_a}{v_a} + \frac{s_g}{v_g}$$

$$s_a = \sqrt{l_a^2 + (H - h)^2} \qquad s_g = \sqrt{l_g^2 + h^2}$$

$$\min_h \; T = \frac{\sqrt{l_a^2 + (H-h)^2}}{v_a} + \frac{\sqrt{l_g^2 + h^2}}{v_g}$$

$$Data: \; l_a, l_g, H, v_a, v_g$$

Finding *h*



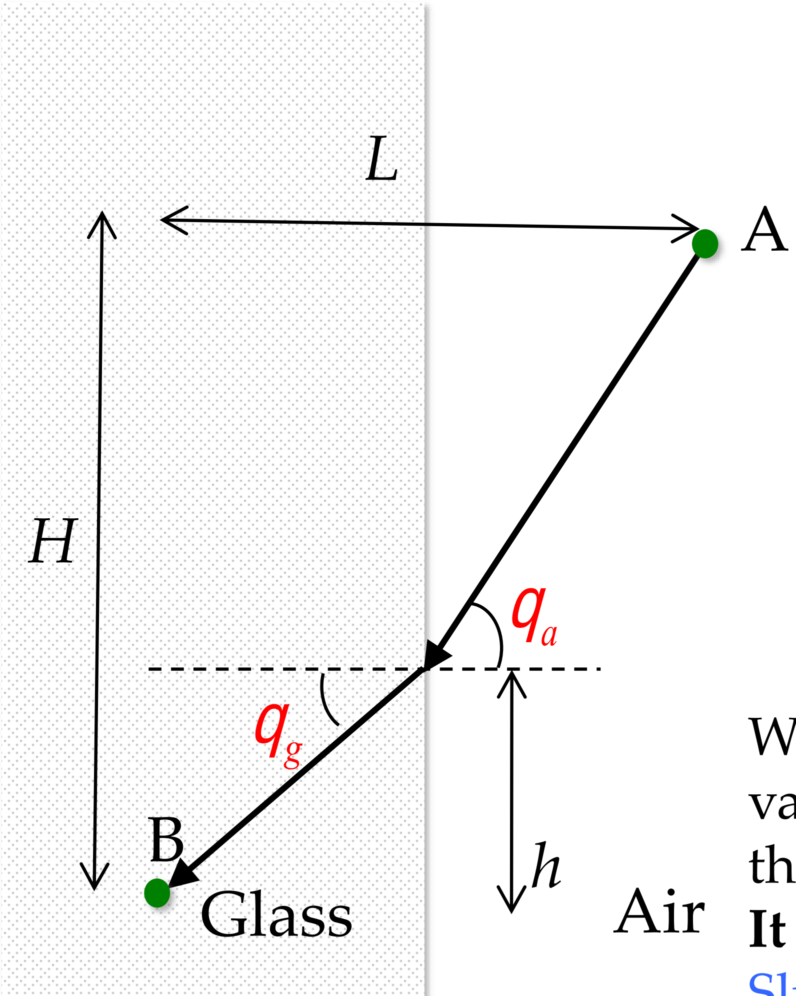
$$\operatorname*{Min}_{h}\ T = \frac{\sqrt{l_a^2 + (H-h)^2}}{v_a} + \frac{\sqrt{l_g^2 + h^2}}{v_g}$$

$$\frac{dT}{dh} = 0$$ (first derivative is zero for a minimum)

$$\Rightarrow \frac{-2(H-h)}{v_a\sqrt{l_a^2 + (H-h)^2}} + \frac{2h}{v_g\sqrt{l_g^2 + h^2}} = 0$$

$$\Rightarrow \frac{(H-h)}{v_a\sqrt{l_a^2 + (H-h)^2}} = \frac{h}{v_g\sqrt{l_g^2 + h^2}}$$

Light ray follow's Snell's rule; the rule follows from optimization.



$$\Rightarrow \frac{(H-h)}{v_a\sqrt{l_a^2+(H-h)^2}} = \frac{h}{v_g\sqrt{l_g^2+h^2}}$$

$$\Rightarrow \frac{\sin\theta_a}{v_a} = \frac{\sin\theta_g}{v_g}$$

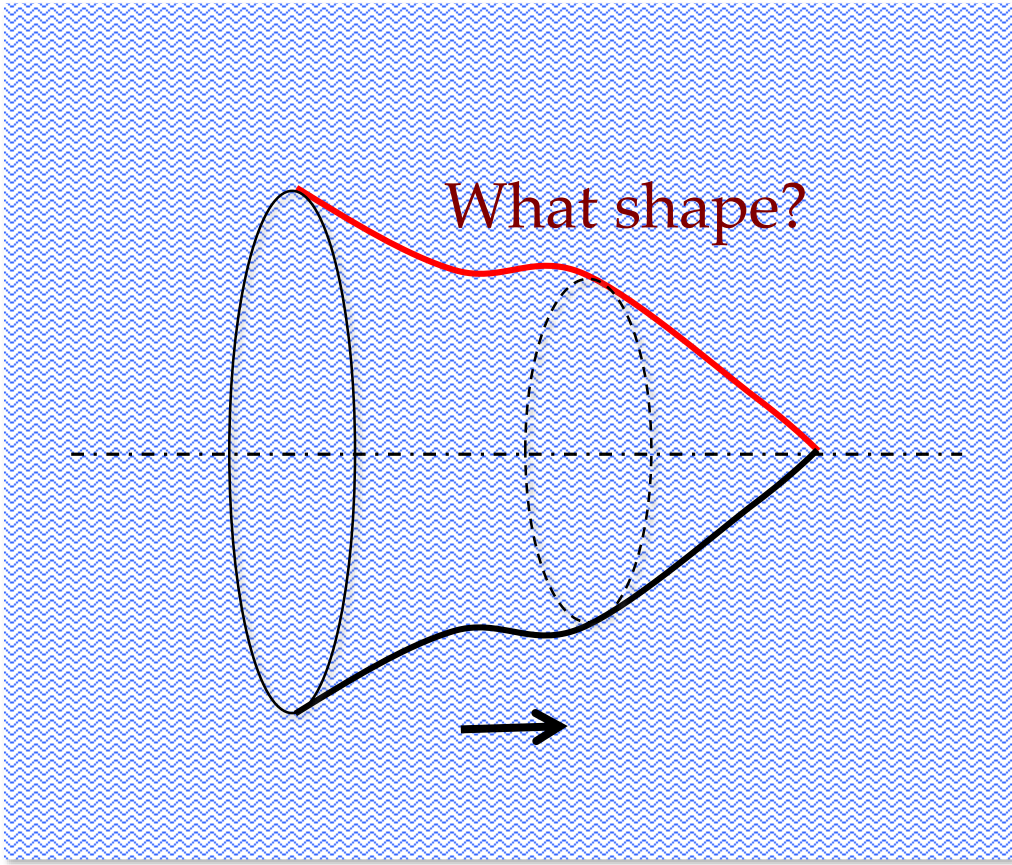
Snell's law of refraction.

So, light rays optimize the time taken for them to go from a point to another. Reflection too follows the same optimal path. **Try it.**

We solved a calculus of variations problem as a finite variable optimization problem because we assumed that light follows straight paths in air and glass. **It is a non-smooth path.** We will re-visit it later in Slide 21 in Lecture 15 from the viewpoint of calculus of variations.

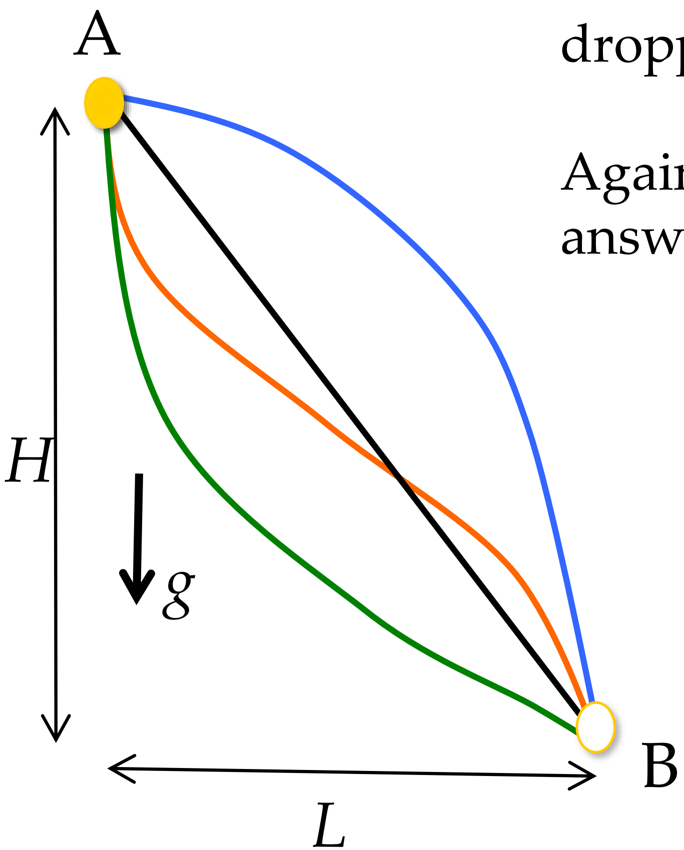
Newton's problem of least fluid resistance

Newton had proposed, around 1685, and solved the the problem of finding the surface of revolution for a body moving at constant velocity in a fluid medium to have the least resistance from the fluid.



This problem did not become popular because of the assumptions made by Newton about the fluid resistance.
However, it led to remarkable advances, much later in the 20th century, in optimal control and non-smooth solutions.

The next classical calculus of variations problem



Which curve between A and B will provide a path for a mass to fall in the least time under the effect of gravity if dropped at A from the rest position?

Again straight line (the least-distance path) is not the answer.

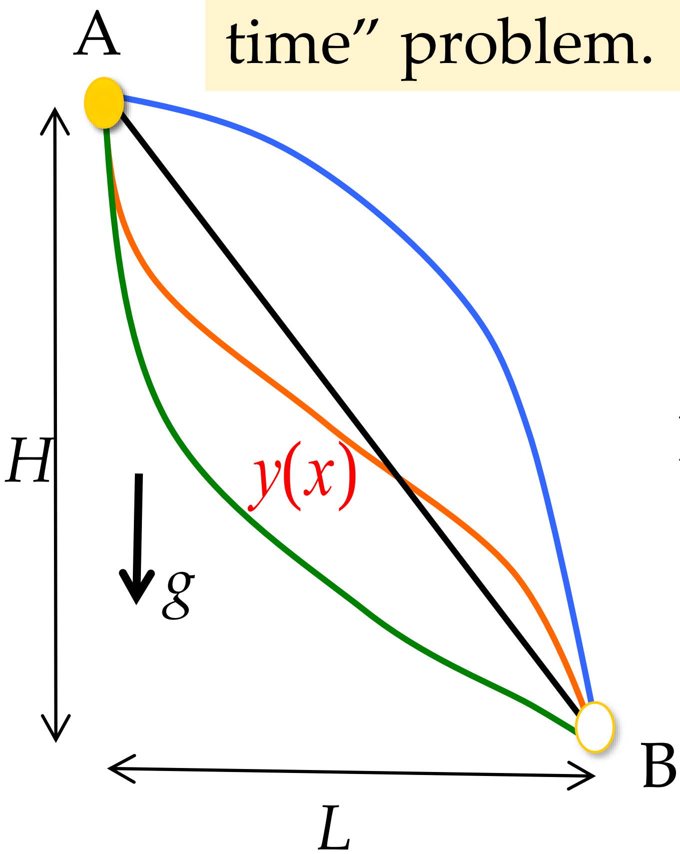
This problem was posed by Johann Bernoulli in 1696 as a challenge to mathematics of that time.

Apparently, five people including Bernoulli himself solved it. Others are Bernoulli's brother Jacob, Leibnitz, L'Hospital, and Newton.

This problem gave birth to calculus of variations.

bracistoVcronoV problem

It is called the **brachistochrone** problem or the "minimum time" problem.

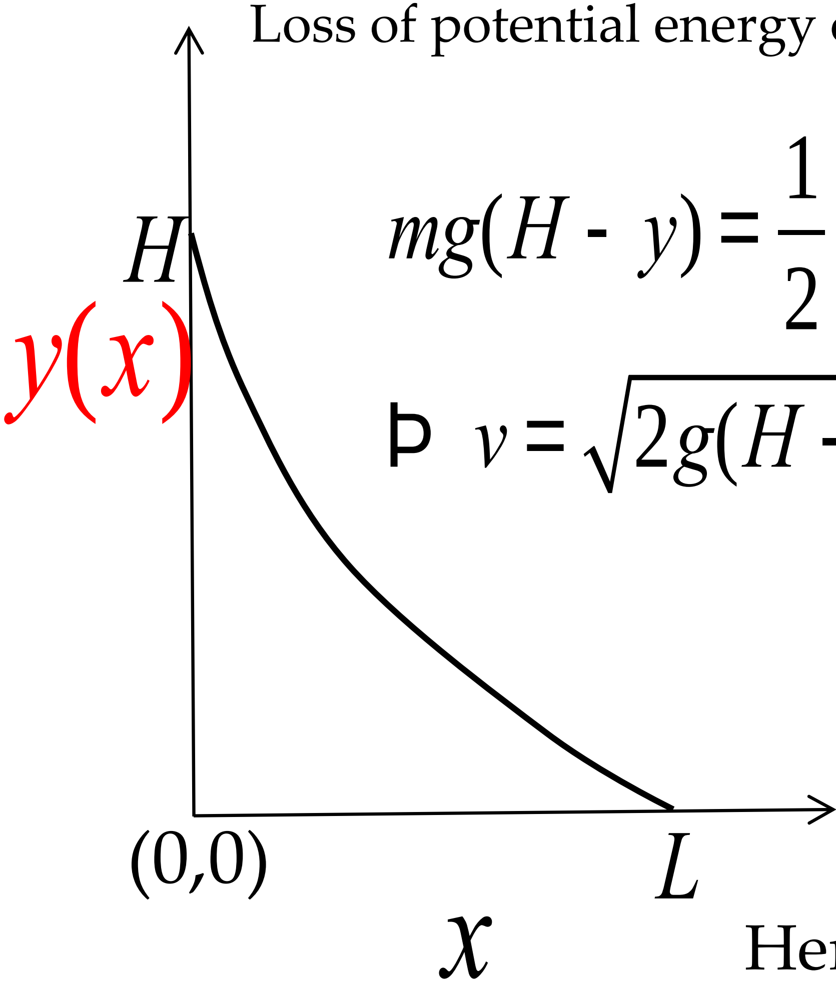


The problem is similar to the refracting light ray problem. Here is why:

The speed changes from A to B continuously here; whereas it has two speeds in different zones in the light ray problem.

$$\underset{y(x)}{\text{Minimize}} \quad T = \int_0^L \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{v(y)}\,dx$$

Finding the speed of a falling mass along *y(x)*



Loss of potential energy of the mass = kinetic energy gained by the mass

$$mg(H - y) = \frac{1}{2} m v^2$$

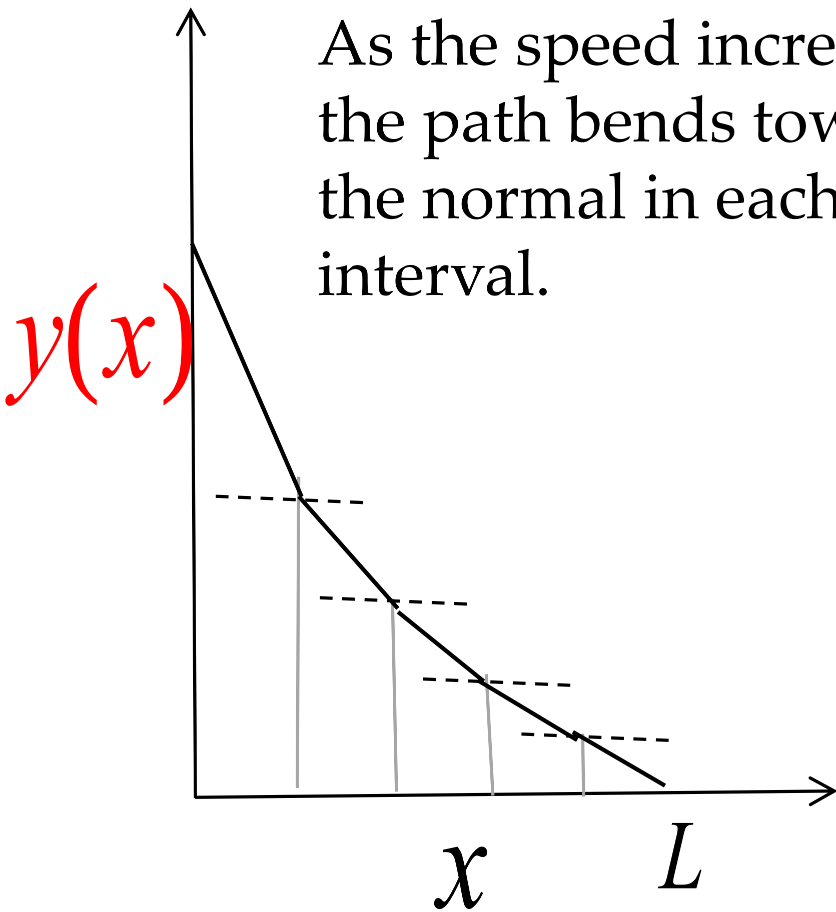
$$\Rightarrow \quad v = \sqrt{2g(H - y)}$$

Now, the problem becomes…

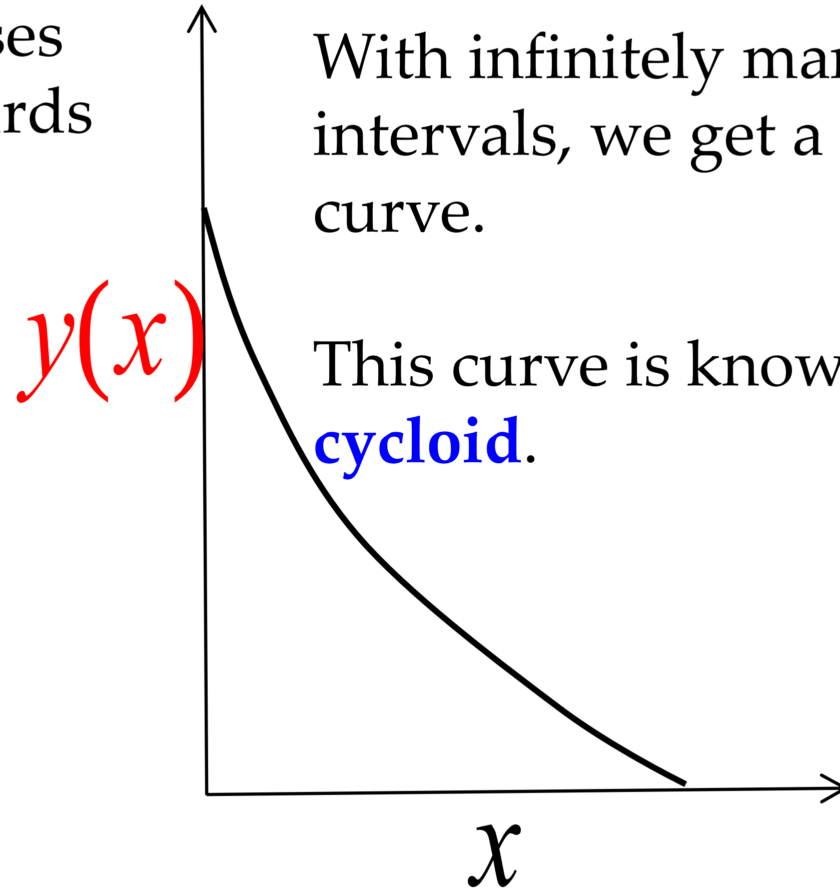
$$\underset{y(x)}{\text{Minimize}} \quad T = \int_0^L \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2g(H - y)}} dx$$

Here, the objective function depends on *y(x)* and its derivative, *(dy/dx)*.

Light-refraction and brachistochrone problem



As the speed increases the path bends towards the normal in each interval.



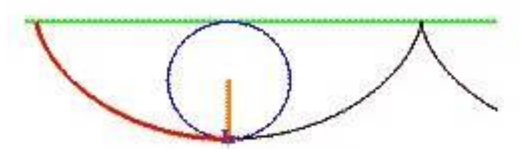
With infinitely many small intervals, we get a smooth curve.

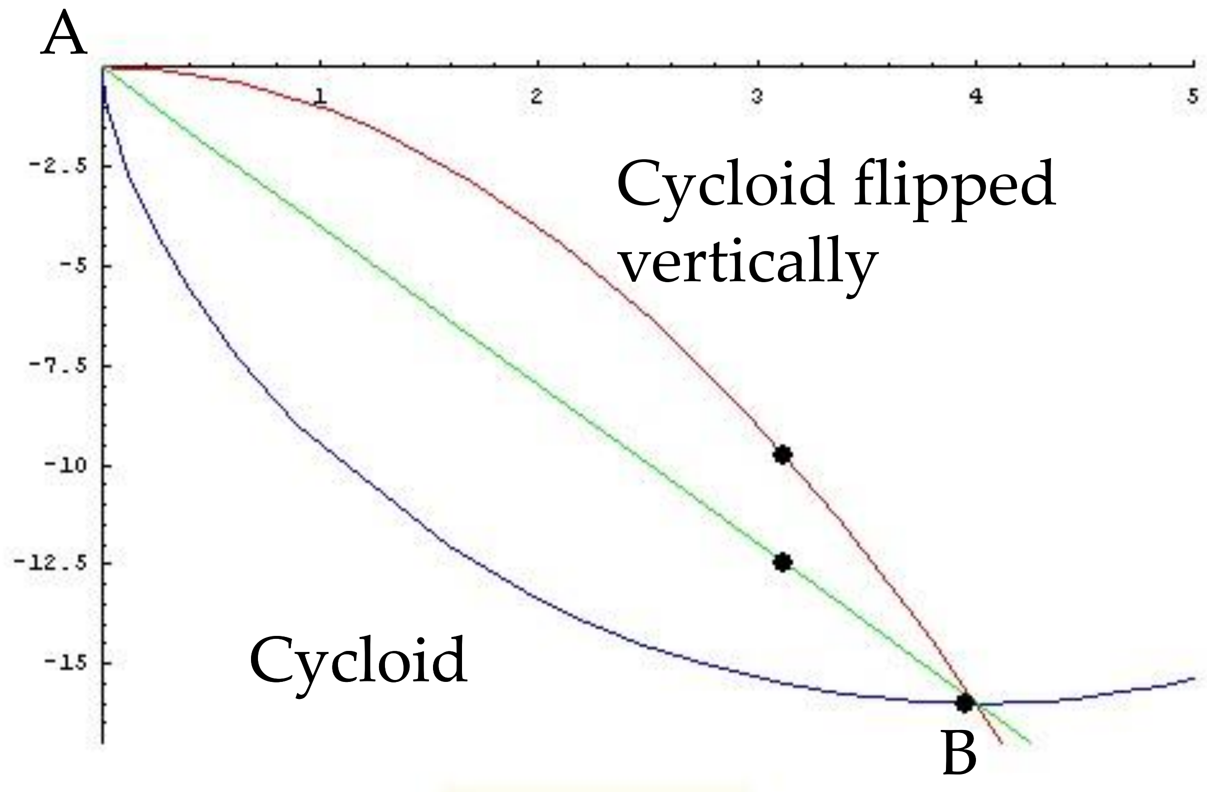
This curve is known as **cycloid**.

Cycloid: the brachistochrone curve

http://curvebank.calstatela.edu

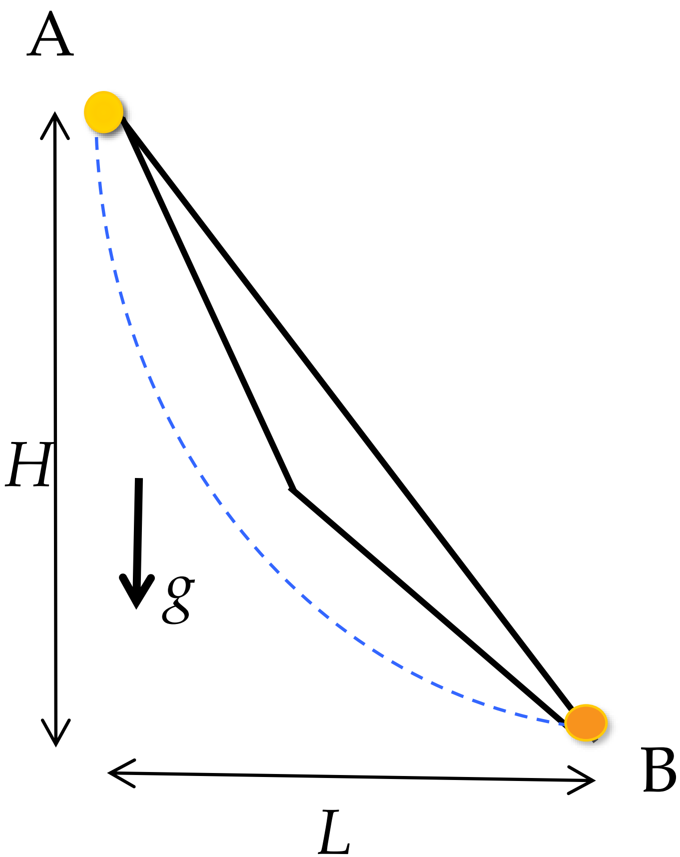
A cycloid is generated by a point on a circle that rolls on a straight line.





Three masses dropped from rest at A at the same time to slide along three curves. The one on the cycloid reaches B first.

A century before that… Galileo



Galileo (1564-1642) has contemplated the problem of least time of fall under gravity along paths of straight line and circular arc.

He knew that double-inclined plane takes less time than a single inclined plane between the same two points. And triple-inclined planes, and so on.

He then asked which circular arc, among infinite possible ones, between two points will have the least time.

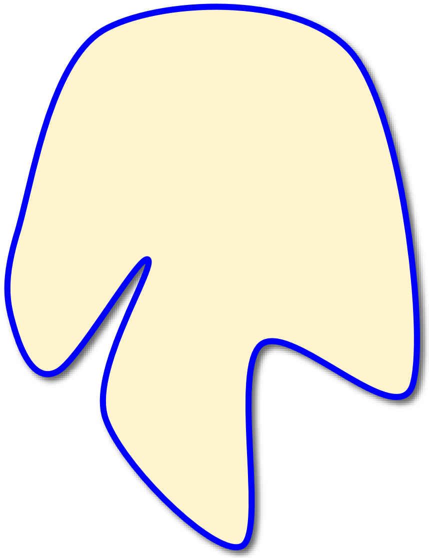
- Can you guess/find the solution for this?
- Hint: use finite-variable optimization.

Much earlier… in the ancient world

Hero of Alexandria (lived sometime during 150 BC and 300 AD) had thought of light rays taking the path of least time.

- Hero had pondered over the principle of reflection of light rays. But then it is also the path of least distance.

Pappus of Alexandria (290 AD to 350 AD) had thought of the so-called isoperimetric problems.



What shape of a closed rope of given length will enclose maximum enclosed area?

In other words…

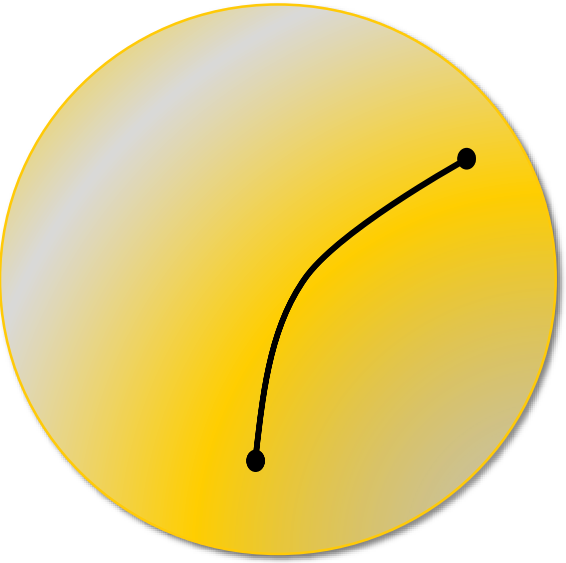
Of all closed curves of equal perimeter, which one has the largest area?

We will consider the mathematical statement of this problem in the framework of calculus of variations later on.

Entry of Euler (1707-1783)

Leonhard Euler got interested in geodesic curves and isoperimetic problems and started to develop the theory of minimal curves and surfaces)1744).

- We can say, he started to develop the general theory of calculus of variations.
- His methods were geometric in nature.
- He also used, we can say with hind sight, the technique of finite-variable optimization approach, to solve calculus of variations.



Geodesic: the curve of least length between two points on a given surface.

The curve of least length between two points in a flat plane is a straight line.
But on a sphere, we cannot draw straight lines. So, what curve gives the least length between two points on a sphere?
And on a cylndrical surface?
And cone? A hyperboloid?
There are many problems of geodesics.
Civil engineers need geodesics when they plan a road in rough terrain.

From curves and surfaces to mechanics

Middle of 18th century.

Connection between mechanics (that is motion of bodies) and calculus of variations was made by Euler, Maupertuis, Leibniz, and others.

- The so-called principle of least action says that Nature minimizes something called an "action".

By this time, Euler had developed a necessary condition for calculus of variations problem.

- This minimization condition is the differential equation that is used to solve for the unknown function of the curve or surface.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Euler had developed this equation to solve minimal curves and surfaces.

We will derive this fundamental equation later.

A minimal surface problem

Which surface of least area is bounded by a given closed curve in 3D?

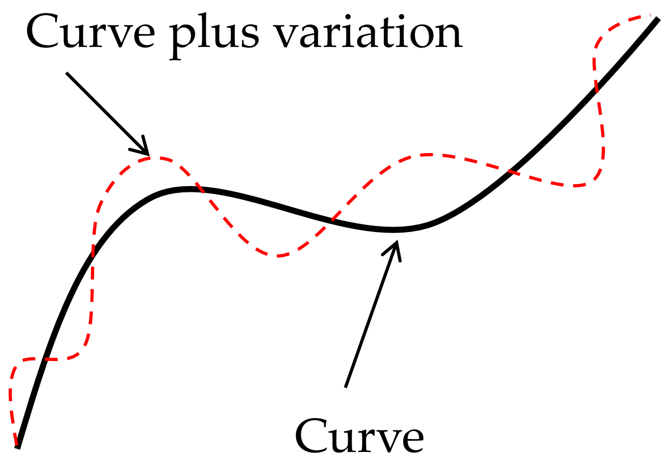
Soap films solve this problem instantly!

Enters Lagrange (1736-1813)

In 1755, Joseph-Louis Lagrange, then less than 20 years old, developed an elegant method to solve calculus of variations instead of the tedious geometrical methods of Euler.

So, its was Lagrange, who invented the concept of variation.

It was Euler, who with due credit to Lagrange, embraced the method and christened the field "Calculus of Variations".



Variation is simply a slight perturbation of a curve.
The *variated* is shown with dashed red line here.
By considering the perturbed curve, i.e., the variation of the curve, one can check how the objective function changes when the curve is perturbed.
When the objective function does not change, to first order, with variation, then we have an optimal curve.
Variations of non-optimal curves give finite changes in the objective function.

Legendre (1752-1833) and Jacobi (1804-1851)

Legendre, Jacobi, and others developed theory of sufficient conditions.

Euler-Lagrange developed the necessary conditions for calculus of variations.

Mikhail Ostrogradsky (1801-1862) too had contributed to the theory of calculus of variations.

Optimization problems are solved using necessary and sufficient conditions.
Necessary conditions are , well, necessary.
Sufficient conditions are, as their name implies, sufficient.
Ponder over this too:
Necessary conditions are not sufficient!
Sufficient conditions are not necessary!
There are some conditions that are necessary and sufficient.

Du Bois-Reymond's contribution

There is something called a fundamental lemma of calculus of variations.

Du Bois-Reymond (1831-1889) proved it.

The lemma is indispensable in calculus of variations.

The fundamental lemma of calculus of variations

If $\int_a^b f(x)\, h(x)\, dx = 0$

and $h(a) = h(b) = 0$ but

$h(x)$ is arbitrary, then

$f(x) = 0$ for all $x \in [a,b]$

It is not difficult, in fact it is rather easy, to understand this lemma; it almost looks obvious. Mathematicians of the past must have implicitly assumed the truth of the lemma.

Hamilton, Riemann, Dirichlet, Weierstrass, et al.

Many mathematicians and mechanicians developed calculus of variations rigorously.

Pontryagin (1908-1988) developed optimal control theory using calculus of variations.

Richard Feynman (1918-1988) used calculus of variations in quantum electrodynamics.

Economists, writers, and philosophers started to embrace the concept of minimality in everything.

By then there was a strong mathematical foundation for believing that notion.

Thus, calculus of variations got established as a powerful mathematical tool with applications in many, many fields of basic and social sciences, and engineering.

The end note

Genesis of calculus of variations

- Isoperimetric problems (Pappus of Alexandria and others) (290-350)
- Fermat on refraction of light; principle of minimum time (1601-1665)
- Newton's problem of shape for minimal fluid resistance (late 17th century)
- Brachistochrone problem (1696)
- Euler (1707-1783)
- Lagrange (1736-1813)
- Legendre (1752-1833) and Jacobi (1801-1851); sufficient conditions
- Du Bois Raymond (1801-1862)and the fundamental lemma
- 20th century
  Hamilton, Riemann, Dirichlet, Weierstrass, et al.
  Pontryagin developed optimal control theory.
  Feynman applied to quantum electro dynamics.

Around (1744)
Euler-Lagrange-Ostrogradski necessary conditions
(1801-1862)

Thanks