Lecture 2a

Finite-variable optimization vs. Calculus of variations

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Classification of optimization problems

Finite-variable optimization vs. calculus of variations

A sample problem that illustrates the difference between the above two and their relation two structural optimization

What we will learn:

What are the "tags" to describe an optimization problem?

What exactly is the difference between finite-variable optimization and calculus of variations?

How do we pose Fermat's least-time problem in two ways?

How do we pose structural mechanics and structural optimization problems?

Types of optimization problems

There are many, many types of optimization problems.

The types arise because of…

- How many objective functions you have.
- Types of objective function and constraints.
- Types of variables.
- Nature of optimization we want to do.
  - Global
  - Local

We will examine important ones, one at a time.

We do this at the outset just so we understand what calculus of variations is.

First note that… calculus of variations is also optimization.

In fact, the theory of calculus of variations got developed much before the "usual" optimization theory got developed.

Classification based on the objective function

One or more objective functions

- Working with a single objective is easy.
  - Even in life!
  - Most of the theory of optimization is focused on dealing with a single objective function.
- Multi-objective optimization is hard.
  - Weights can be given but then how do you give the weights when you do not know which objective is more important for you?
  - The best thing to do is to move the less important objective to a constraint.
- Pareto optimum
  - Pareto optimum concept is an important concept in multi-objective optimization.
  - Pareto optimum is one where you can improve on objective function without hurting another.
  - Often, Pareto optimum will be a set; that is there will be many Pareto optima.
    - Pareto optimum set can be continuous or discontinuous.
    - Generating the entire Pareto set is difficult in practice.

Global or local (more later)

- Local optimum is one in a small vicinity of the optimum point.
  - It is like you are the smartest in your class. You are a local maximum.
- Global optimum is one that considers the entire domain of the objective function.
  - If your school beauty is a local maximum, Miss Universe if the global optimum.

Classification based on constraints

Without or with constraints

- If there are no constraints, it becomes an **unconstrained optimization** problem.
- With constraints, it is **constrained optimization** problem.

Equality or inequality

- Constraints can be equalities.
    - Governing equations are usually equality constraints in structural optimization.
- They can be inequalities
    - Inequalities arise mostly due to resource and performance constraints in structural optimization.

Constraints reduce the permissible values of the optimization variables.

- Constraints constrain the space of optimization variables.

Feasible space

- The space of optimization variables where all constraints are satisfied is called the **feasible space**.
- In constrained optimization, we need to search for the optimum of the objective function only in the feasible space.
- Constructing feasible space is often impractical but we can certainly search within it.

Classification based on objective and constraints

Linear programing

- Both objective function and constraints are linear.

Quadratic programming

- Objective function is quadratic and constraints are linear.

Nonlinear programming

- Both objective and constraints are nonlinear.

Geometric programming

- Objective and constraints are posynomials.
- Posynomials are polynomials with positive coefficients.
  - Positive + Polynomial = posynomial! A portmanteaus word.
  - Exponents in posynomials can be real numbers, positive or negative, in the context of geometric programming.

Convex optimization

- The objective function is convex and so is the feasible sapce.

Non-smooth optimization

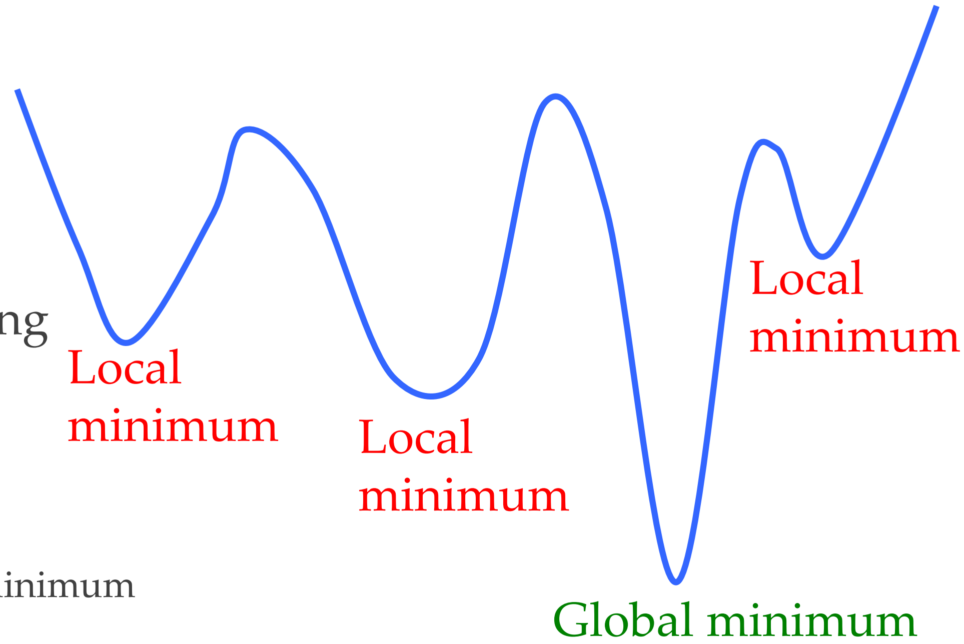
- Where either objective function or the constraints, or both, are not differentiable.

And many more types!

Classification based on the nature of optimization

Global or local

- Local when we are happy with a local minimum.
  - That is smallest in the vicinity of a point.
- Global when we want to find the smallest among all minima.
- There are no easy methods to find the global minimum.
  - Only special types of problems allow finding the global minimum



Deterministic or stochastic

- Deterministic means that you have the same thing any number of times you try it.
- Stochastic means that there is also an element of randomness in addition to deterministic nature.
- An optimization problem can be stochastic if the variables involved are stochastic.
- But, one can use non-deterministic methods to solve a deterministic problem.
  - Genetic algorithms, Simulated annealing methods, Monte Carlo search, etc.

Classification based on types of variables

Continuous or discrete

Discrete

- Binary
  - Only 0 or 1 are allowed.
- Integer
- Discrete sets
  - Bearing sizes, screw threads, etc.; you cannot have whatever you want. They will be certain pre-specified values.

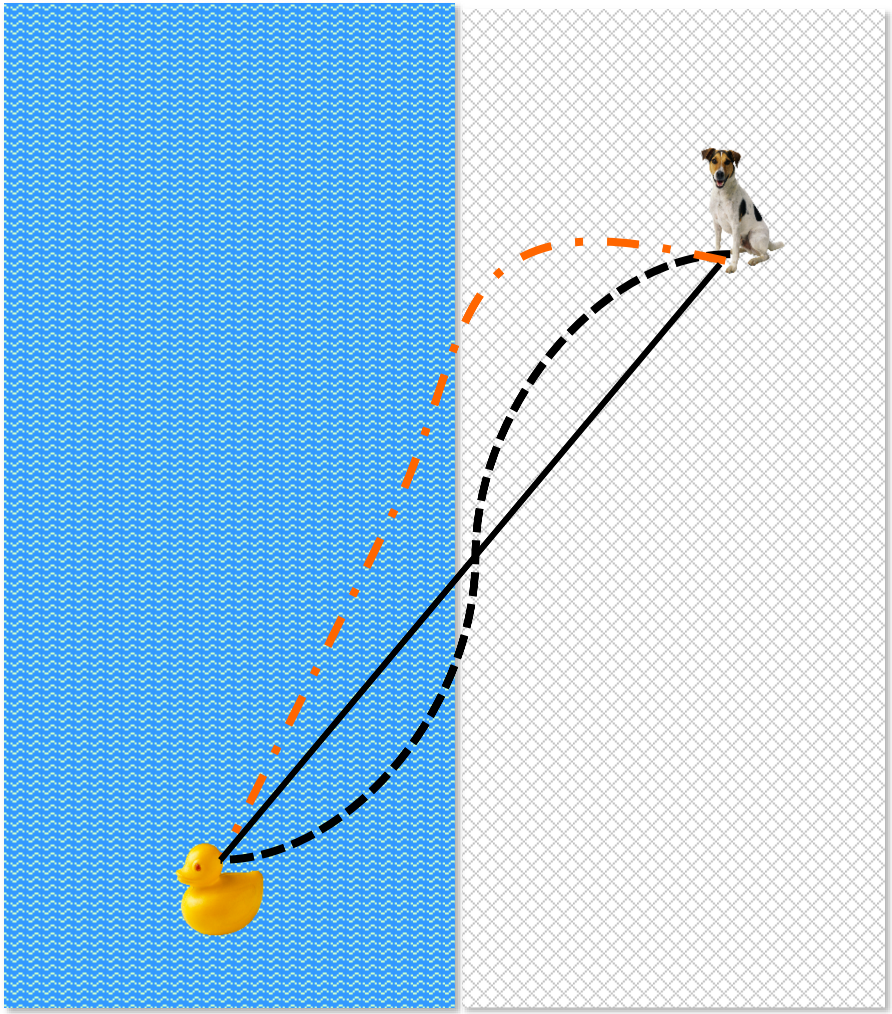
Deterministic or stochastic

- Discussed in the previous slide
- Uncertainties bring about stochastic nature in structural optimization.

Finite variables or functions themselves!

- Variables are finite if they are like $x_1, x_2, x_3, \ldots, x_n$.
- What if variables are not finite?
- What if variables are **functions themselves**?
- This is what brings us to calculus of variations

Consider this optimization problem:



A dog is sitting next to a swimming pool and his owner threw a rubber duck into the pool. The dog can **run on the pool-tiles twice as fast as it can swim** in water.

What path should the dog take to touch the duck in the **shortest time**?

It is clearly an optimization problem.

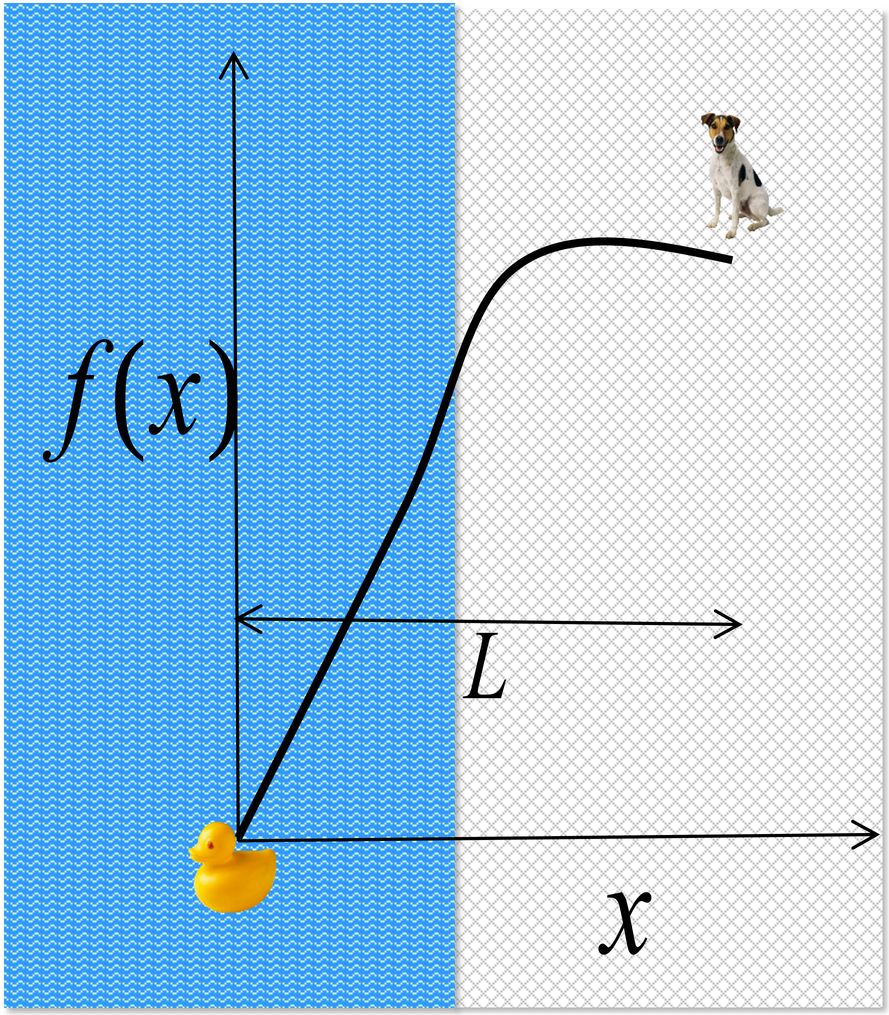
Straight line is not the solution! We need shortest-time path and not the shortest path.

What are the optimization variables here?

The variable here is the continuous function that represents the path of the curve to be taken by the dog.

It is a calculus of variations problem!

Calculus of variations problem



Speed of the dog = $v(x)$

$$v(x) = \frac{ds}{dt} = \frac{\sqrt{dx^2 + df^2}}{dt}$$

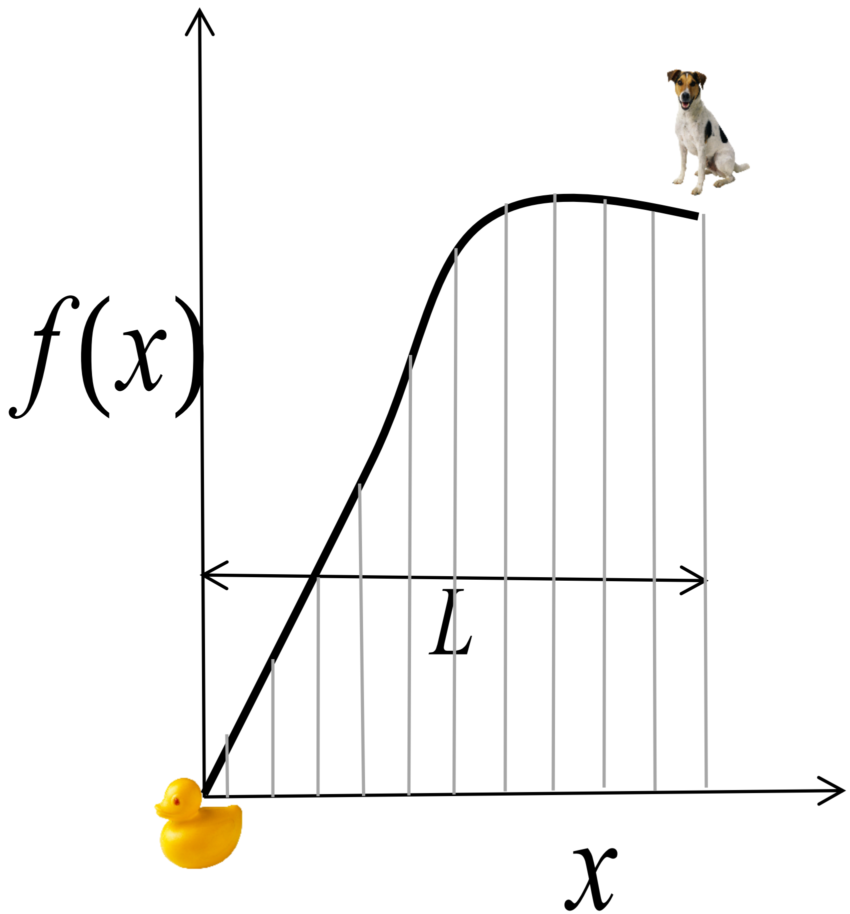
$$\Rightarrow \; dt = \frac{\sqrt{dx^2 + df^2}}{v(x)}$$

So, $T = \int_0^L \frac{\sqrt{1+\left(\frac{df}{dx}\right)^2}}{v(x)} dx$

$$\underset{f(x)}{\text{Minimize}} \; T = \int_0^L \frac{\sqrt{1+\left(\frac{df}{dx}\right)^2}}{v(x)} dx$$

The variable is the function, $f(x)$. The objective function depends on the derivative of this function.

Discretization of the "function" variable



Imagine that the span of length L is discretized into small intervals. Then, $f(x)$ can also be imagined as different heights.

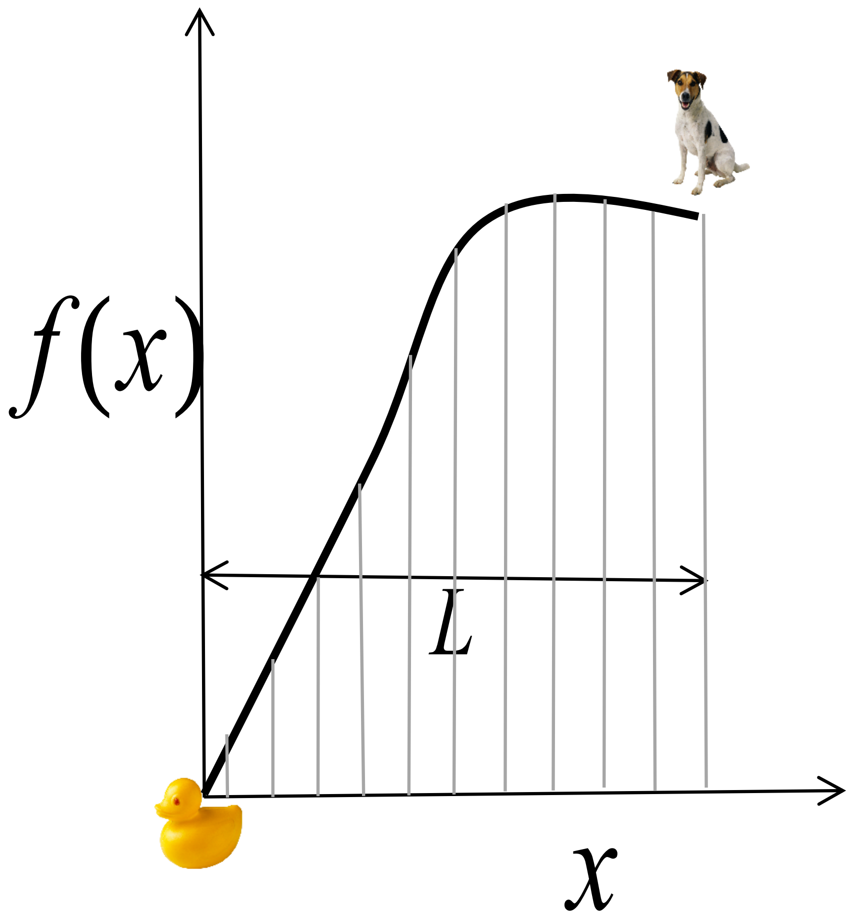
Let us denote the heights with

$$f_1, f_2, f_3, \ldots, f_n$$

It now, becomes a **finite-variable optimization problem**.

But then, we have to take a very, very fine intervals to get the smooth curve, $f(x)$.

"function" variable becoming "finite" variables…



Calculus of variations

$$\underset{f(x)}{\text{Minimize}}\; T = \int_0^L \frac{\sqrt{1+\left(\dfrac{df}{dx}\right)^2}}{v(x)}\,dx$$

Finite-variable optimization

$$\underset{f_1,\, f_2,\, f_3,\, \ldots,\, f_n}{\text{Minimize}}\; T(f_1, f_2, f_3, \ldots, f_n)$$

Finite-variable optimization vs. calculus of variations

Variables are finite in number. Each variable may be continuous (i.e., a real number) or discrete (as in binary, integer, etc.).

Objective function and constraints will be functions of the finite number of variables.

Variable is an unknown function.

There can be many such functions that are unknown. That is, finite number of functions can be variables.

Objective function and constraints will be **functions of functions.**

Such a thing is called a ***functional***. More later.

We need to know the nature of "function" variables and the "functionals".
We will review the basic notion of function spaces later.
They form the basis for calculus of variations.

Calculus of variations is analytical… not computational

At the outset, it is useful to note that calculus of variations is analytical in the sense that everything will be in symbols and not numbers. Hence, it is not computational.

Calculus of variations, i.e., optimization with functions as variables, gives us differential equations to solve for those unknown functions.

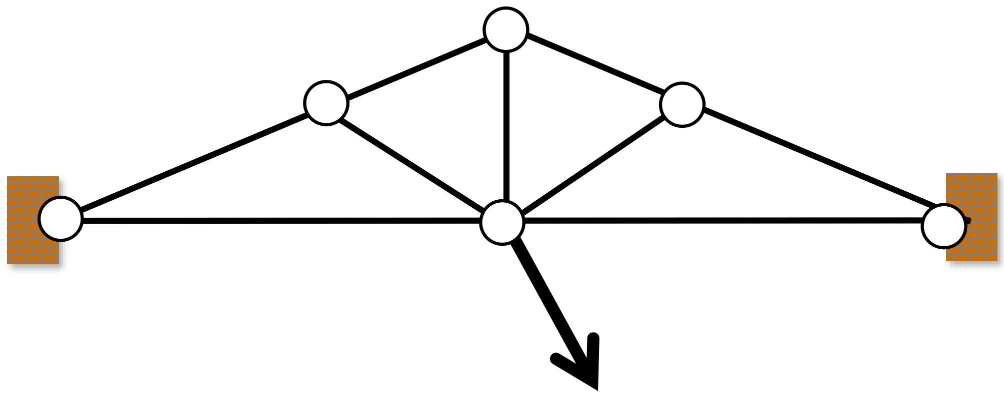
Calculus of variations also gives us boundary conditions along with the differential equations.

It does not tell us how to compute a solution. It just gives equations using which we can compute the unknown function using other methods.

Many problems in geometry, physics, chemistry, mathematics, engineering, economics, etc., can be posed as calculus of variations.

Calculus of variations is also crucial for structural optimization.

Role of finite-variable optimization in structural mechanics



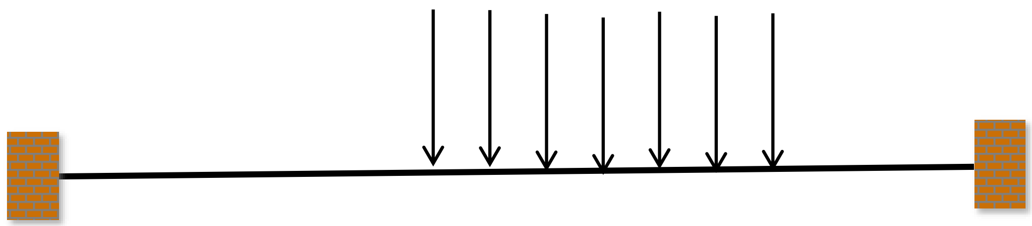
How do the vertices of the truss move under the applied load?

Since we have four movable vertices, we have eight unknown variables.

The structure deforms to minimize potential energy, which is a function of eight displacements.

So, it is a finite-variable optimization problem.

Role of calculus of variations in structural mechanics

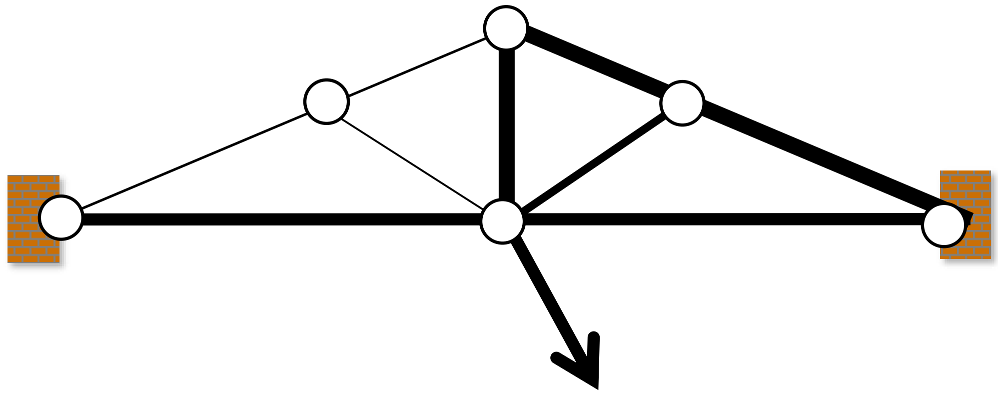


What shape does this beam take under the applied load?

The deformed profile of the beam is a function and that is the variable here.

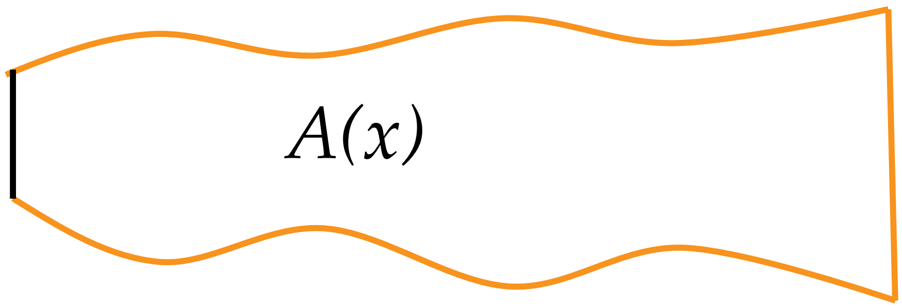
So, it is a calculus of variations problem.

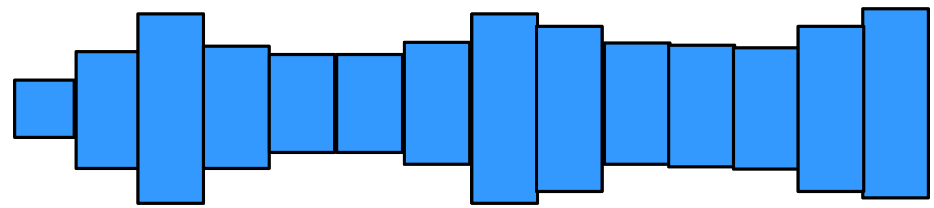
Role of finite-variable optimization in structural design



We have nine bars in this truss. By making their cross-section areas variables in optimization, we can maximize the stiffness of this truss for given volume of material under given loading. So, it is a finite-variable optimization problem.

Role of calculus of variations in structural design





Also posed as a finite-variable optimization

What should be the width profile of the beam for being stiffest under given load for a given volume of material?

The width profile along the length of the beam is a function and that is the variable here.

So, it is a calculus of variations problem.

The end note

Optimization problems

- Unconstrained
- Constrained
  - Equalities
  - Inequalities
  - Both
- Single-objective
- Multi-objective
  - Linear
    - Linear programming
  - Nonlinear
    - Quadratic programming
    - Nonlinear programming
    - Convex programming
    - Geometric programming
    - Semi-definite programming
    - More…
  - Smooth
  - Non-smooth
- Finite variables
  - Continuous
  - Discrete
    - Binary
    - Integer
    - Discrete real variable set

  Finite-variable optimization
- "function" variables
  - **Calculus of variations**
- Global
- Local
- Deterministic
- Stochastic

Thanks