Lecture 5

Truss optimization

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

G. K. Ananthasuresh

Professor, Mechanical Engineering, Indian Institute of Science, Bengaluru

suresh@iisc.ac.in

Outline of the lecture

Analysis of trusses

Two-bar truss for stiffness and strength

Optimization of multi-bar trusses

What we will learn:

A quick view of finite element analysis

Size optimization of trusses in the framework of finite-variable optimization

Structural analysis

Structural analysis is necessary but not sufficient for structural optimization.

Do you agree with this statement?

Corollaries

You cannot optimize a structure without analyzing it.

By only analyzing it, you cannot optimize a structure.

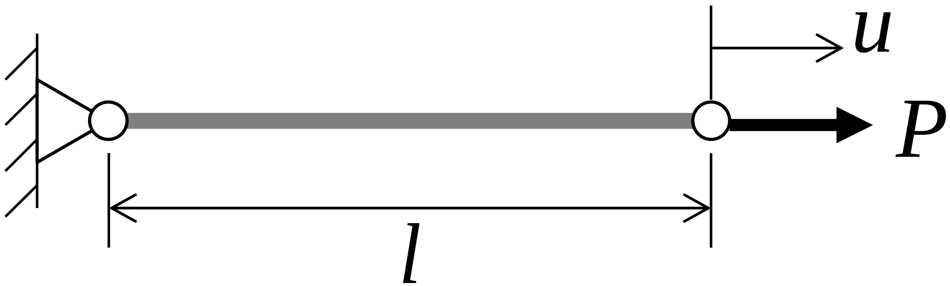
Three key steps in structural analysis.

Kinematics (geometry, displacement, velocity, acceleration, strain)

Constitutive relationships (material properties, stress-strain relationship)

Balance laws (physics)

Three steps in structural analysis



$$u = \frac{Pl}{AE}$$

Where does this come about?

Kinematics $\varepsilon = \dfrac{u}{l}$

Constitutive relationships $\sigma = E\varepsilon$

Balance laws $A\sigma = P$

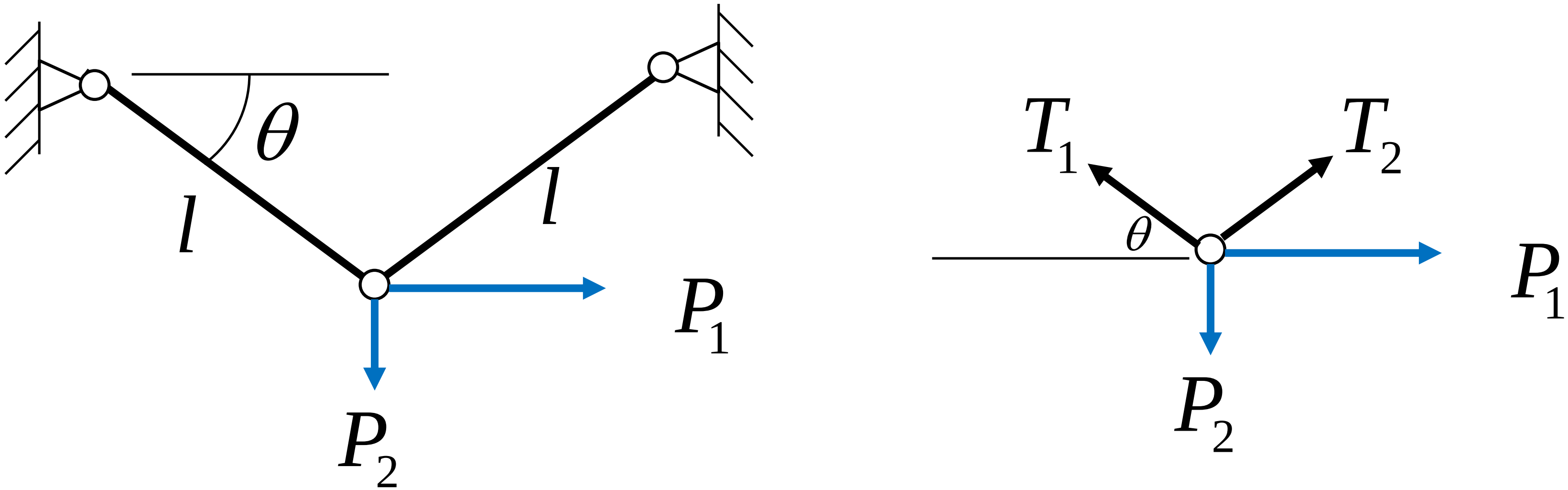
$$A\sigma = P$$

$$\Rightarrow AE\varepsilon = P$$

$$\Rightarrow AE\frac{u}{l} = P$$

$$\Rightarrow u = \frac{Pl}{AE}$$

Consider the two-bar truss.

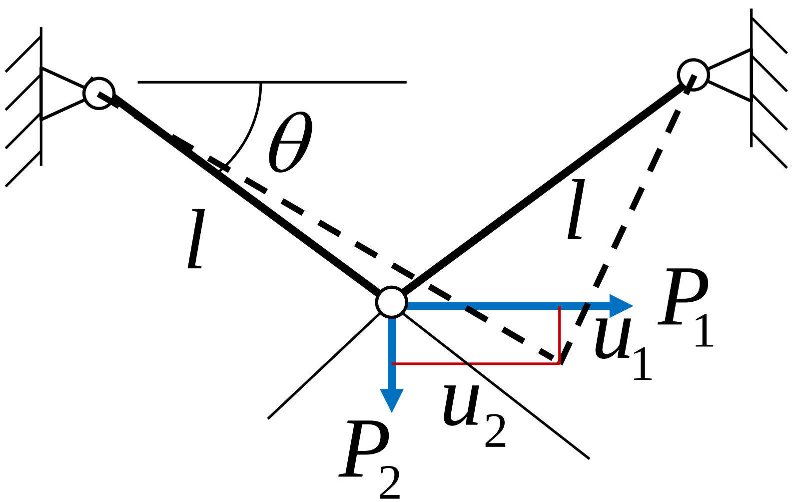


Balance laws

(force balance)

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\cos\theta \\ \sin\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

Consider the two-bar truss.



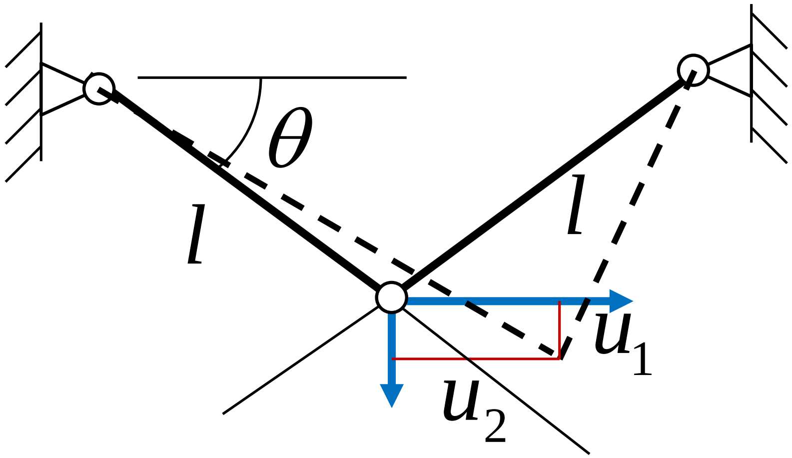
$$T_1 = A_1\sigma_1 = A_1 E\varepsilon_1 = A_1 E\frac{\delta_1}{l}$$

$$T_2 = A_1\sigma_2 = A_2 E\varepsilon_2 = A_2 E\frac{\delta_2}{l}$$

Constitutive relationships
(stress-strain relation)

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} \frac{A_1 E}{l} & 0 \\ 0 & \frac{A_2 E}{l} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

Consider the two-bar truss.



Project the displacement of the node onto the truss elements's undeformed direction.

$$\delta_1 = u_1 \cos\theta + u_2 \sin\theta$$

$$\delta_2 = -u_1 \cos\theta + u_2 \sin\theta$$

Kinematics

(geometry)

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Put the three together.

Kinematics (geometry)

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Balance laws (force balance)

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\cos\theta \\ \sin\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

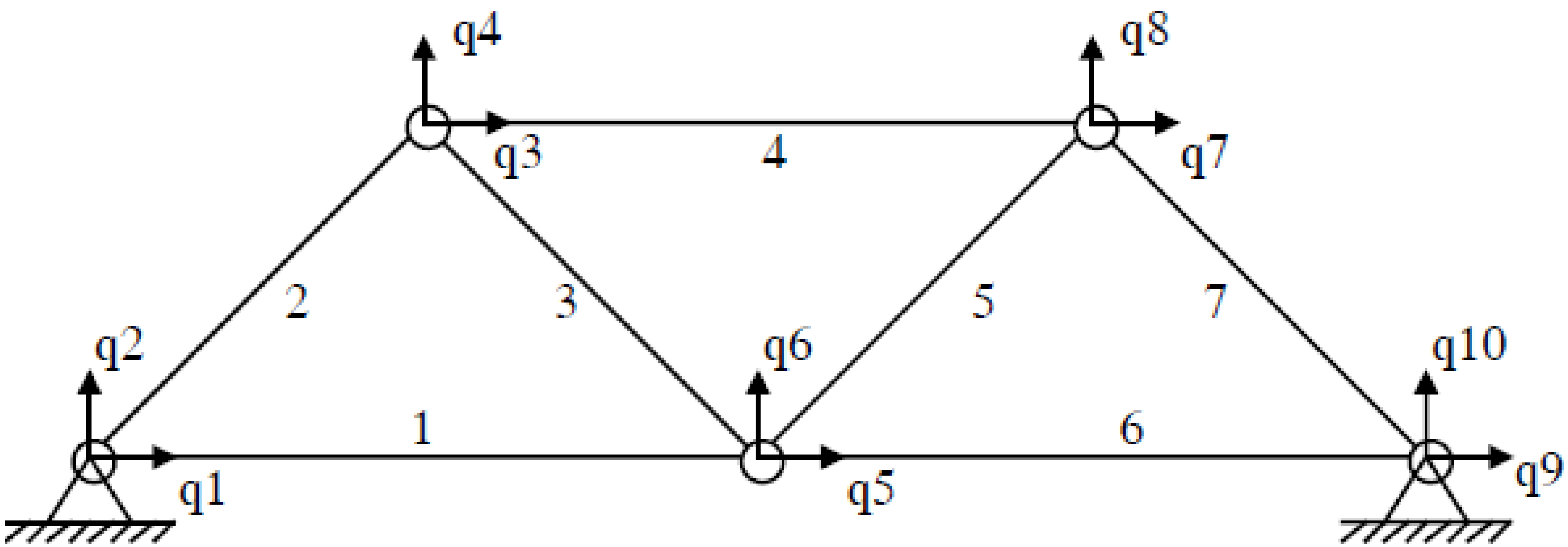
Constitutive relationships (stress-strain relation)

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} \frac{A_1 E}{l} & 0 \\ 0 & \frac{A_2 E}{l} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

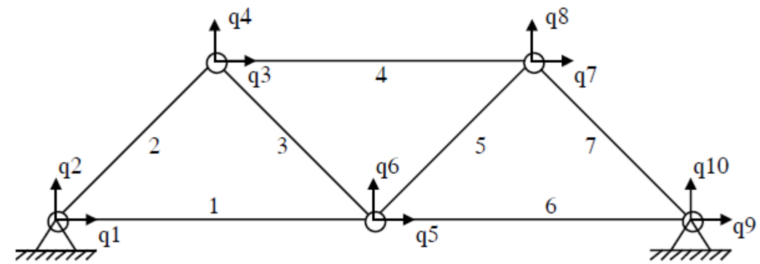
$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\cos\theta \\ \sin\theta & \sin\theta \end{bmatrix}}_{\mathbf{B}^T} \underbrace{\begin{bmatrix} \frac{A_1 E}{l} & 0 \\ 0 & \frac{A_2 E}{l} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\cos\theta & \sin\theta \end{bmatrix}}_{\mathbf{B}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \mathbf{P} = \left[\mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\right]\mathbf{u} = \mathbf{K}\mathbf{u} = \mathbf{K}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Consider a 2D truss.



Consider a 2D truss.



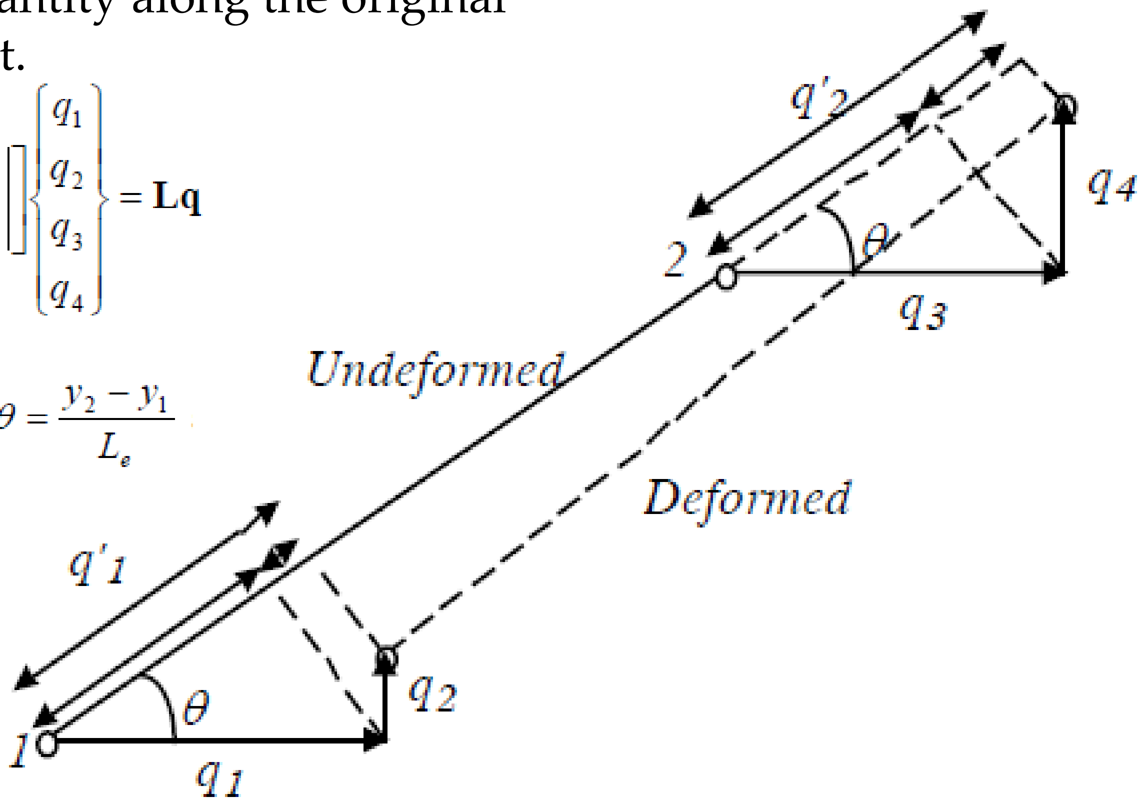
$$q_1' = q_1 \cos\theta + q_2 \sin\theta$$

$$q_2' = q_3 \cos\theta + q_4 \sin\theta$$

Prime (‘) represents quantity along the original direction of the element.

$$\mathbf{q}' = \begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \mathbf{Lq}$$

$$l = \cos\theta = \frac{x_2 - x_1}{L_e} \text{ and } m = \sin\theta = \frac{y_2 - y_1}{L_e}$$



Element stiffness matrix

$$\mathbf{K}'_e = \frac{A_e E_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$\mathbf{K}_e = \mathbf{L}^T \mathbf{K}'_e \mathbf{L}$$

$$SE_e = \frac{1}{2}\{q_1 \quad q_2 \quad q_3 \quad q_4\}[\mathbf{K}_e]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

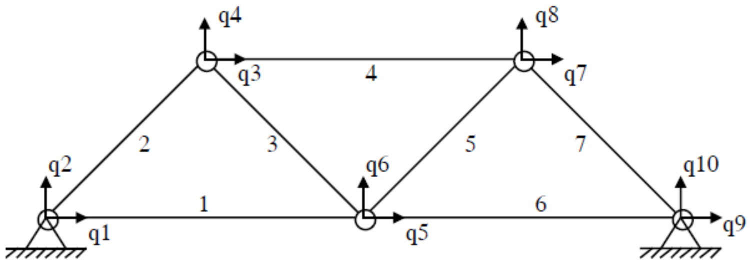
Strain energy = SE

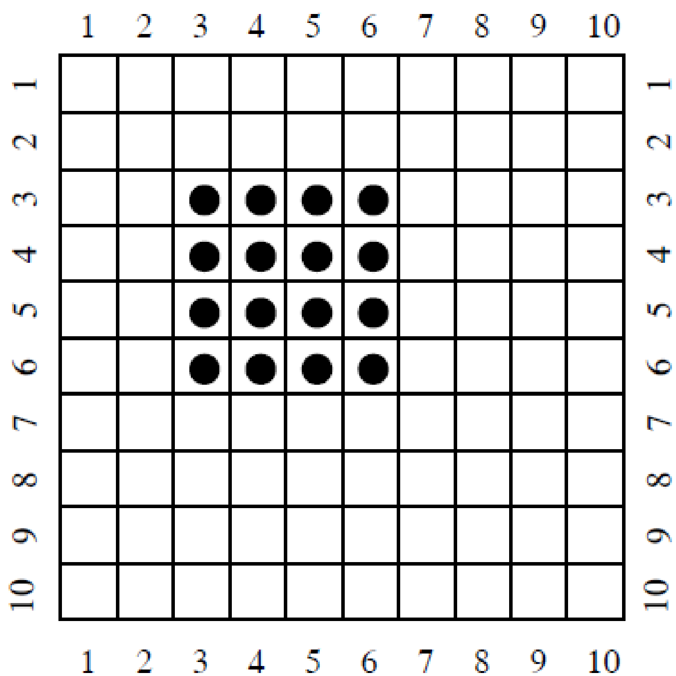
$$SE_e = \frac{1}{2}k\Delta^2 = \frac{1}{2}\frac{A_e E_e}{L_e}\Delta^2 = \frac{1}{2}\frac{A_e E_e}{L_e}\left(q'_2 - q'_1\right)^2$$

$$= \frac{1}{2}\{q'_1 \quad q'_2\}\frac{A_e E_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} q'_1 \\ q'_2 \end{Bmatrix}$$

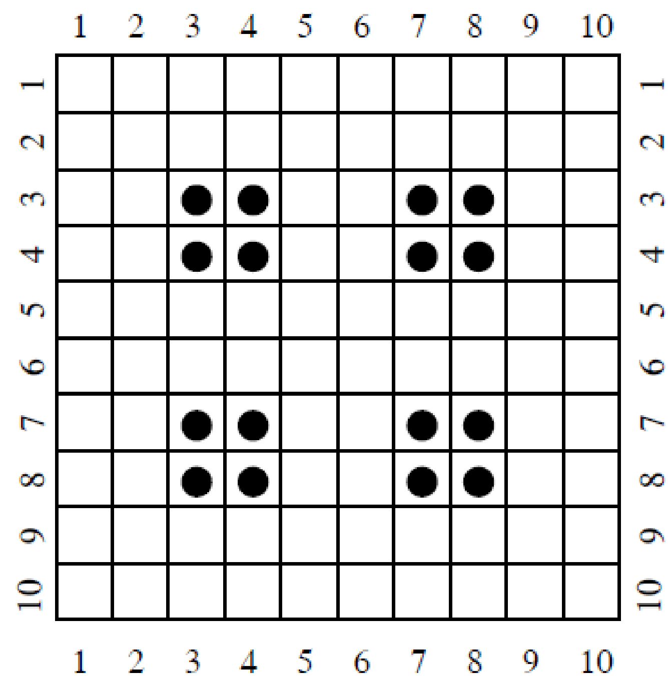
$$\mathbf{K}_e = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Assembly of the global stiffness matrix





Contribution of element 3 to the system stiffness matrix



Contribution of element 4 to the system stiffness matrix

Size optimization of a truss

$$\underset{\mathbf{A}}{\text{Min}} \; MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \qquad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \qquad \sum_{i=1}^{N} \rho A_i l_i - W^* \leq 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

Data: $\rho, W^*, l_{i=1,2\ldots,N}, N, P_{i=1,2,\ldots,N}, E$

MC = mean compliance = work done by the external forces

MC is a measure of stiffness: *the lower the value of MC, the stiffer the truss.*

K depends on areas of cross section of all truss elements and Young's modulus.

Size optimization of a truss

$$\underset{\mathbf{A}}{\text{Min}}\ MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \quad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \quad \sum_{i=1}^{N} \rho A_i l_i - W^* \le 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \le 0$$

Data: $\rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$

Lagrangian

$$L = \mathbf{p}^T \mathbf{u} + \boldsymbol{\lambda}^T \left(\mathbf{Ku} - \mathbf{p} \right) + \mu \left(\rho \mathbf{a}^T \mathbf{l} - W^* \right)$$

Necessary conditions

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l = 0$$

$$i = 1, 2, \cdots, N$$

$$\mathbf{Ku} = \mathbf{p}, \quad \rho \mathbf{a}^T \mathbf{l} - W^* \le 0$$

$$\mu \left(\rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$$

Tally the number of unknowns and the number of equations

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l = 0 \qquad i = 1, 2, \cdots, N$$

N equations

$\mathbf{Ku} = \mathbf{p}$ $\quad n$ equations

$\rho \mathbf{a}^T \mathbf{l} - W^* \le 0$

$\mu \left(\rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$ $\quad$ 1 equation

$\mathbf{A}$	N	unknowns
$\mathbf{u}$	n	unknowns
$\boldsymbol{\lambda}$	n	unknowns
μ	1	unknown

(N+n+1) equations in (N+2n+1) unknowns.

Ouch… what do we do? Where are n more equations?

The end note

Truss optimization

Structural analysis steps

- Kinematics
- Constitutive relationships
- Balance laws

Two-bar analysis

A general 2D truss; stiffness matrix

Posing the size optimization of a truss

We are short of equations…what do we do?

Thanks