# Lecture 6

# Truss optimization (contd.)

ME260 Indian Institute of Science

**Structural Optimization: Size, Shape, and Topology**

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# Outline of the lecture

Size optimization of multi-bar trusses

Adjoint method of sensitivity analysis

Optimality criterion

What we will learn:

Solution to the size optimization of trusses

Algorithm for the optimality criteria method

# Size optimization of a truss

$$\min_{\mathbf{A}} \; MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \qquad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \qquad \sum_{i=1}^{N} \rho A_i l_i - W^* \le 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \le 0$$

Data: $\rho, W^*, l_{i=1,2\ldots,N}, N, P_{i=1,2,\ldots,N}, E$

*MC* = mean compliance = work done by the external forces

*MC* is a measure of stiffness:
*the lower the value of MC, the stiffer the truss.*

**K** depends on areas of cross section of all truss elements and Young's modulus.

# Size optimization of a truss

$$\underset{\mathbf{A}}{\text{Min}} \; MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \quad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \quad \sum_{i=1}^{N} \rho A_i l_i - W^* \leq 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

Data: $\rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$

**Lagrangian**

$$L = \mathbf{p}^T \mathbf{u} + \boldsymbol{\lambda}^T \left( \mathbf{Ku} - \mathbf{p} \right) + \mu \left( \rho \mathbf{a}^T \mathbf{l} - W^* \right)$$

**Necessary conditions**

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left( \mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0$$

$$i = 1, 2, \cdots, N$$

$$\mathbf{Ku} = \mathbf{p}, \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

$$\mu \left( \rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$$

# Tally the number of unknowns and the number of equations

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left( \mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \quad i = 1, 2, \cdots, N$$

$N$ equations

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$

$n$ equations

$$\rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

$$\mu \left( \rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$$

1 equation

| | | |
|---|---|---|
| $\mathbf{A}$ | $N$ | unknowns |
| $\mathbf{u}$ | $n$ | unknowns |
| $\boldsymbol{\lambda}$ | $n$ | unknowns |
| $\mu$ | 1 | unknown |

$n$ = twice the number of vertices in the truss = total number of DoF

($N$+$n$+1) equations in ($N$+2$n$+1) unknowns.

Ouch… what do we do? Where are $n$ more equations?

# Finding $n$ more equations

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left( \mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \qquad i = 1, 2, \cdots, N \qquad N \text{ equations}$$

Let us see hidden $n$ more equations by recasting the above equation.

$$\left( \mathbf{p}^T + \boldsymbol{\lambda}^T \mathbf{K} \right) \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i = 0$$

Do you see $n$ more equations yet?

No? Let us see what we can compute and what we cannot if we know the areas of cross section.

# Here are *n* more equations

$$\left(\mathbf{p}^T + \boldsymbol{\lambda}^T \mathbf{K}\right)\frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u} + \mu\rho l_i = 0 \quad \Rightarrow \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u} + \mu\rho l_i = 0$$

$$\left(\mathbf{p}^T + \boldsymbol{\lambda}^T \mathbf{K}\right) = \mathbf{0}^T \quad \text{or} \quad \mathbf{K}\boldsymbol{\lambda} = -\mathbf{p}$$

This clever way of finding *n* more equations is called the **adjoint method of sensitivity analysis**.

$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{p}$ This is called the adjoint (equilibrium) equation.

$\boldsymbol{\lambda}$ is called the adjoint (state) variable.

# Now, we are set to solve the problem.

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left( \mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \quad i = 1, 2, \cdots, N$$

$N$ equations

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$

$n$ equations

$$\rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

$$\mu \left( \rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$$

1 equation

$$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{p}$$

$n$ equations

| | | |
|---|---|---|
| $\mathbf{A}$ | $N$ | unknowns |
| $\mathbf{u}$ | $n$ | unknowns |
| $\boldsymbol{\lambda}$ | $n$ | unknowns |
| $\mu$ | 1 | unknown |

($N$+2$n$+1) equations in ($N$+2$n$+1) unknowns.

We also have two inequalities too: $\rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$ and $\mu \geq 0$

# Solving...

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left( \mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \quad i = 1, 2, \cdots, N$$

$N$ equations

$\mathbf{Ku} = \mathbf{p}$ $n$ equations

$\rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$

$\mu\left( \rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$ 1 equation

$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{p}$ $n$ equations

$$\boldsymbol{\lambda} = -\mathbf{u}$$

State variable and adjoint variable are equal.

# Solving (contd.)

$$-\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u} + \mu\rho l_i = 0 \Rightarrow \mu = \frac{1}{\rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) \qquad i = 1, 2, \cdots, N$$

$N$ equations

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$

$n$ equations

$$\rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

$$\mu\left(\rho \mathbf{a}^T \mathbf{l} - W^*\right) = 0$$

$$\mu \geq 0$$

1 equation and two inequalities

$$\frac{1}{\rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) = ?$$

What is this, physically?

# Interpretation: $\frac{1}{\rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) = se_i$

$$SE = \frac{1}{2}\left(\mathbf{u}^T \mathbf{K}\mathbf{u}\right) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} k_{jk} u_j u_k$$

$$SE_i = \frac{1}{2}\sum_{j \in DoF}\sum_{k \in DoF} k_{jk} u_j u_k$$

$$se_i = \frac{SE_i}{\rho l_i A_i} = \frac{1}{2\rho l_i A_i}\sum_{j \in DoF}\sum_{k \in DoF} k_{jk} u_j u_k$$

Strain energy density of the $i^{\text{th}}$ element

$$\frac{\partial SE}{\partial A_i} = \frac{1}{2}\sum_{j \in DoF}\sum_{k \in DoF} \frac{k_{jk}}{A_i} u_j u_k$$

This is only partial sensitivity; not the total sensitivity to which we will return to later.

Since the strain energy density of all elements cannot be zero, we say that $\mu$ cannot be zero. Hence, the weight constraint is active.

# Solving (contd.): Optimality criterion

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$

$$\left(\rho \mathbf{a}^T \mathbf{l} - W^*\right) = 0$$

$$\mu = \frac{1}{\rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right)$$

This implies that the strain energy density is constant for all elements.

How do we find $A_i$s?

Since $$1 = \frac{1}{\mu \rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right)$$

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) A_i^{(k)}$$

$k$ is the iteration number in optimization.

# Fixed point method of solution

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i}\left( \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Multiplicative version of the optimality criteria method.

Additive version

$$A_i^{(k+1)} = A_i^{(k)} + \left( -\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i \right)^{(k)}$$

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$$f(x) = 0$$

$$\mathrm{g}(x) = f(x) + x$$

$$\tan x - x^2 - 1 = 0$$

Example

$$x^{(k+1)} = g(x^{(k)}) = f(x^{(k)}) + x^{(k)}$$

# Algorithm

Step 0 Assume initial guess for all areas of cross section so that the weight constraint is satisfied.

$$A_i^{(o)}, i = 1, 2, \cdots, N$$

Step 1 Solve for the displacements using the loads.

$$\mathbf{Ku} = \mathbf{p}$$

Step 2 Solve $\left(\rho \mathbf{a}^T \mathbf{l} - W^*\right) = 0$ to find $\mu$

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left( \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

# Step 2 and farther…

$$\left(\rho \mathbf{a}^T \mathbf{l} - W^*\right) = 0 \Rightarrow \sum_{i=1}^{N} \rho A_i l_i = W^*$$

$$\Rightarrow \sum_{i=1}^{N} \frac{1}{\mu \rho l_i} \left( \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)} \rho l_i = W^*$$

$$\Rightarrow \mu = \frac{\sum_{i=1}^{N} \left( \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{W^*}$$

Step 3 Update areas of cross section.

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left( \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Do you see any other issues here?

Go to Step 1 until convergence.

# The end note

Truss optimization

- Size optimization: problem statement and necessary conditions
- Counting unknowns and equations
- Rational behind the adjoint method of sensitivity analysis
- **Algorithm for size optimization of trusses**
- We are we missing anything else in the algorithm? Think.

Thanks