Lecture 7a

Truss optimization algorithm

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Size optimization (contd.)

Obeying the bounds on areas of cross section

What we will learn:

Modifying the algorithm to satisfy the bounds on the areas of cross section

Inner and outer loops in the algorithm

Size optimization of a truss

$$\underset{\mathbf{A}}{\text{Min}} \;\; MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \qquad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \qquad \sum_{i=1}^{N} \rho A_i l_i - W^* \leq 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

$$\text{Data:} \;\; \rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$$

MC = mean compliance = work done by the external forces

MC is a measure of stiffness:
the lower the value of MC, the stiffer the truss.

K depends on areas of cross section of all truss elements and Young's modulus.

Necessary conditions

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \Rightarrow \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i = 0$$

$i = 1, 2, \cdots, N$ $\quad N$ equations

$\mathbf{K}\mathbf{u} = \mathbf{p}$ $\quad n$ equations

$\rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$

$\mu \left(\rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$ $\quad$ 1 equation

$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{p}$ $\quad n$ equations

$\mathbf{A}$	N	unknowns
$\mathbf{u}$	n	unknowns
$\boldsymbol{\lambda}$	n	unknowns
μ	1	unknown

(N+2n+1) equations in (N+2n+1) unknowns.

We also have two inequalities too: $\rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$ and $\mu \geq 0$

Algorithm

Step 0 Assume initial guess for all areas of cross section so that the weight constraint is satisfied.

$$A_i^{(o)}, i = 1, 2, \cdots, N$$

Step 1 Solve for the displacements using the loads.

$$\mathbf{Ku} = \mathbf{p}$$

Step 2 Solve $\left(\rho \mathbf{a}^T \mathbf{l} - W^*\right) = 0$ to find μ

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Step 2 and farther…

$$\left(\rho\mathbf{a}^T\mathbf{l}-W^*\right)=0 \Rightarrow \sum_{i=1}^{N}\rho A_i l_i = W^*$$

$$\Rightarrow \sum_{i=1}^{N}\frac{1}{\mu\rho l_i}\left(\mathbf{u}^T\frac{\partial\mathbf{K}}{\partial A_i}\mathbf{u}\right)A_i^{(k)}\rho l_i = W^*$$

$$\Rightarrow \mu = \frac{\displaystyle\sum_{i=1}^{N}\left(\mathbf{u}^T\frac{\partial\mathbf{K}}{\partial A_i}\mathbf{u}\right)A_i^{(k)}}{W^*}$$

Step 3 Update areas of cross section.

$$A_i^{(k+1)}=\frac{1}{\mu\rho l_i}\left(\mathbf{u}^T\frac{\partial\mathbf{K}}{\partial A_i}\mathbf{u}\right)A_i^{(k)}$$

Go to Step 1 until convergence.

Do you see any other issues here?

What if any A_is are out of bounds?

$$A_{lb} \le A_i \le A_{ub}$$ Practical bounds

Can areas of cross-section they become negative?

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \underbrace{\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right)}_{\text{Could this be negative?}} A_i^{(k)} \quad \text{where} \quad \mu = \underbrace{\frac{\sum_{i=1}^{N} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{W^*}}_{\text{Could this be negative?}}$$

Staying within bounds

$$A_{lb} \le A_i \le A_{ub} \qquad A_i^{(k+1)} = \frac{1}{\mu \rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) A_i^{(k)}$$

If $\quad A_i^{(k+1)} > A_{ub}\,,\quad A_i^{(k+1)} = A_{ub}$

Else if $\quad A_i^{(k+1)} < A_{lb}\,,\quad A_i^{(k+1)} = A_{lb}$

Else $\quad$ leave $A_i^{(k+1)}$ as it is.

Three bins for areas of cross section

L Lower bound — I Intermediate — U Upper bound

$A_i = A_{ub}$ $\qquad$ $A_{lb} \le A_i \le A_{ub}$ $\qquad$ $A_i = A_{ub}$

$$\sum_{i=1}^{N} \rho A_i l_i = W^* \Rightarrow \sum_{i \in L} A_{lb} \rho l_i + \sum_{i \in I} \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)} \rho l_i + \sum_{i \in U} A_{ub} \rho l_i = W^*$$

$$\Rightarrow \mu = \frac{\displaystyle\sum_{i=1}^{N} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{\displaystyle W^* - \sum_{i \in L} A_{lb} \rho l_i - \sum_{i \in U} A_{ub} \rho l_i}$$

Will this binning be complete in one go?
No, it is also iterative.

Let us modify the algorithm.

Inner loop to satisfy the optimality condition for the entire problem.

Step 0 $A_i^{(o)}, i = 1, 2, \cdots, N \qquad k = 0$

Step 1 Solve $\mathbf{Ku} = \mathbf{p}$ using $A_{i=1,2,\cdots,N}^{(k)}$

$k = k + 1$

Step 2 Guess μ

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Step 2a Check the bounds on $A_i^{(k+1)}s$ and update $\mu^{(i+1)}$

$j = j + 1$

Step 2b

$$\mu^{(j+1)} = \frac{\sum_{i \in I}^{N} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{W^* - \sum_{i \in L} A_{lb} \rho l_i - \sum_{i \in U} A_{ub} \rho l_i}$$

Inner loop to satisfy the volume constraint and bounds on A_is.

Step 3 Re-compute $A_i^{(k+1)} = \frac{1}{\mu^{(j+1)} \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$

Step 4 Go back to Step 2a until binning is unchanged.

Step 5 Go back to Step 1 until convergence.

Re-statement of the problem

$$\underset{\mathbf{A}}{\text{Min}}\ MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \qquad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \qquad \sum_{i=1}^{N} \rho A_i l_i - W^* \le 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \le 0$$

$$\left.\begin{aligned} \gamma_{ub}: &\qquad A_i - A_{ub} \le 0 \\ \gamma_{lb}: &\qquad A_{lb} - A_i \le 0 \end{aligned}\right\} i = 1, 2, \cdots, N$$

Data: $\rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$

We have $2N$ more inequality constraints. If we wish we can compute the corresponding Lagrange multipliers after the numerical algorithm is implemented and solved. How? Write the Lagrangian and the optimality conditions.

Re-stated optimality criterion

$$\boxed{\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \gamma_{ub_i} - \gamma_{lb_i} + \mu \rho l_i = 0}$$

The end note

Truss optimization

Checking the bounds on areas of cross section and binning them into three groups.

Computing the value of Lagrange multiplier of the weight constraint in the iterative inner loop, to satisfy the constraint.

Maintain the iterative outer loop as before, to satisfy the optimality criterion.

Complete algorithm for size optimization of trusses

You should implement it to fully understand the algorithm.

Thanks