Lecture 8

The dual problem of size optimization of statically determinate trusses

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

G. K. Ananthasuresh

Professor, Mechanical Engineering, Indian Institute of Science, Bengaluru

suresh@iisc.ac.in

Outline of the lecture

Clayperon's theorem

Maxwell's rule for statically determinate trusses

Dual problem for statically determinate trusses

What we will learn:

Maxwell's rule with Calladine's modification

Posing the dual problem of size optimization of statically determinate trusses

Unit virtual (dummy) load method

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$

Imagine virtual force $\delta\mathbf{p}^T = \{\overset{1}{0} \quad 0 \quad \overset{i}{1} \quad \cdots \quad \cdots \quad 0 \quad \overset{n}{0}\}$

$$EVW = \delta\mathbf{p}^T \ \mathbf{u}^*$$

$$IVW = \left(\mathbf{K}\delta\mathbf{u}\right)^T \mathbf{u}^* = \delta\mathbf{u}^T\mathbf{K}\mathbf{u}^*$$

$$EVW = IVW \Rightarrow \delta\mathbf{p}^T \ \mathbf{u}^* = \delta\mathbf{u}^T\mathbf{K}\mathbf{u}^*$$

$$\Rightarrow u_i = \left(\delta\mathbf{u}^T\mathbf{K}\right)\mathbf{u}^*$$

$\mathbf{u}^*$ = displacements due to applied real loads

$\delta\mathbf{u}$ = displacements due to the unit virtual load

Clayperon's theorem

At static equilibrium, the mean compliance is equal to twice the strain energy.

What? $MC = 2 * SE$

How? $EVW = \mathbf{p}^T \ \delta\mathbf{u}$ $\qquad IVW = \left(\mathbf{K}\mathbf{u}^*\right)^T \delta\mathbf{u} = \delta\mathbf{u}^T\mathbf{K}\mathbf{u}$

$$EVW = IVW$$

$$\Rightarrow \mathbf{p}^T \ \delta\mathbf{u} = \delta\mathbf{u}^T\mathbf{K}\mathbf{u}^*$$

Make virtual displacement equal to real equilibrium displacement.

$$\mathbf{p}^T \ \mathbf{u}^* = \mathbf{u}^{*T}\mathbf{K}\mathbf{u}^* \Rightarrow MC = 2 * SE$$

Does this make sense?

An implication from the Clayperon's theorem.

What?

$$\underset{\mathbf{u}}{\text{Min}}\ \underset{\mathbf{a}}{\text{Max}}\ PE = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{p}^T\mathbf{u}$$

Subject to

$$\mu: \qquad \rho\mathbf{a}^T\mathbf{l} - W^* \le 0$$

Data: $\rho, W^*, l_{i=1,2\ldots,N}, N, P_{i=1,2,\ldots,N}, E$

How?

$$PE = SE + WP = SE - MC$$

$$PE^* = SE^* - MC^* = -SE^*$$

Minimizing SE (or MC) is equivalent to maximizing PE w.r.t. to the design variable.

A small detour

ON MAXWELL’S RULE FOR TRUSSES

Statically determinate trusses

Statically determinate trusses are those in which internal forces can be computed from equations of statics without having to solve for dixplacements.

➔ *Internal forces do not depend on areas of cross section of the truss members.*

Statically determinate trusses satisfy the Maxwell's rule.

Maxwell's rule

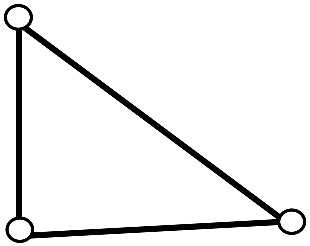
2D $2v - 3 - b = 0$

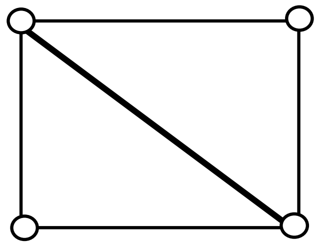
3D $3v - 6 - b = 0$

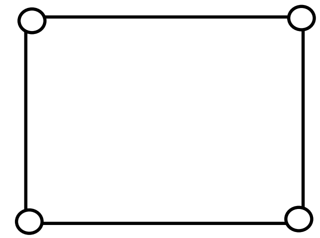
v = number of vertices

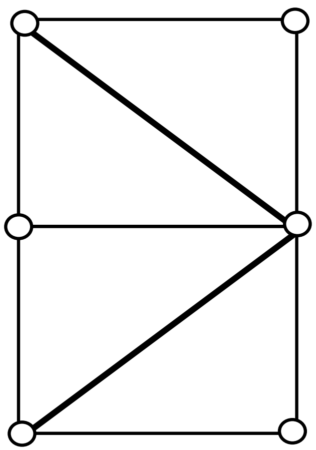
b = number of bars

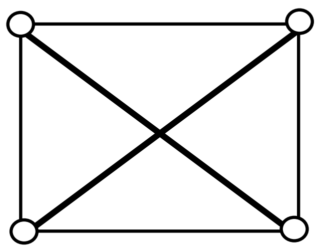
Try Maxwell's rule on these

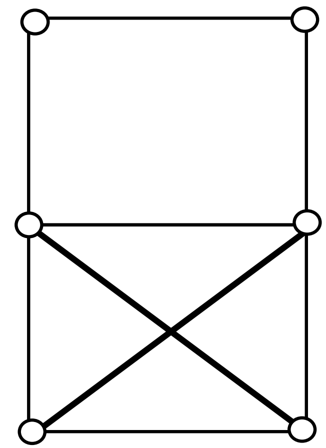












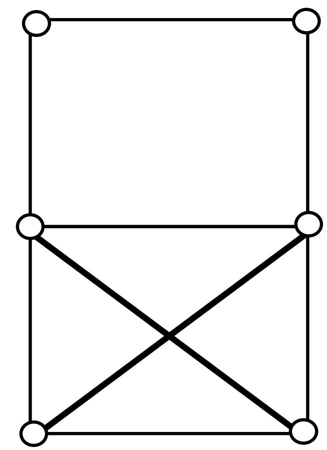
Maxwell's rule modified by Calladine

2D

$$2v - 3 - b = DoF - SoSS$$

3D

$$3v - 6 - b = DoF - SoSS$$



v = number of vertices

b = number of bars

DoF = number of degrees of freedom

$SoSS$ = number of states of self-stress

How do you interpret Maxwell's rule?

2D $2v - 3 - b = 0$

Equilibrium matrix, Bar forces, Forces at vertices

$$\mathbf{H}_{2v \times b}\mathbf{p}_{b \times 1} = \mathbf{f}_{2v \times 1}$$

Compatibility matrix, Disp. at vertices, Bar elongations

$$\mathbf{C}_{b \times 2v}\mathbf{u}_{2v \times 1} = \mathbf{e}_{b \times 1}$$

$$\mathbf{p}^T \delta\mathbf{e} = \mathbf{f}^T \delta\mathbf{u}$$

$$\Rightarrow \mathbf{p}^T \mathbf{C}\,\delta\mathbf{u} = \mathbf{p}^T \mathbf{H}^T \delta\mathbf{u}$$

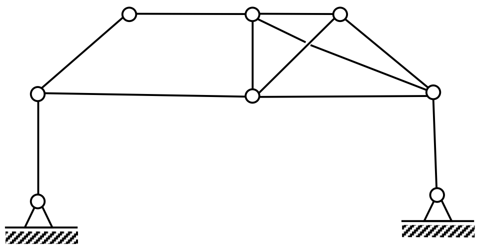
$$\Rightarrow \mathbf{C} = \mathbf{H}^T$$

Rank-deficiency of **C** indicates DoF
Rank-deficiency of **H** indicates SoSS.

Null-space of **C** indicates instantaneous rigid-body modes.

Null-space of **H** indicates self-stress modes.

DoF and SoSS



$$2v - 3 - b = DoF - SoSS$$

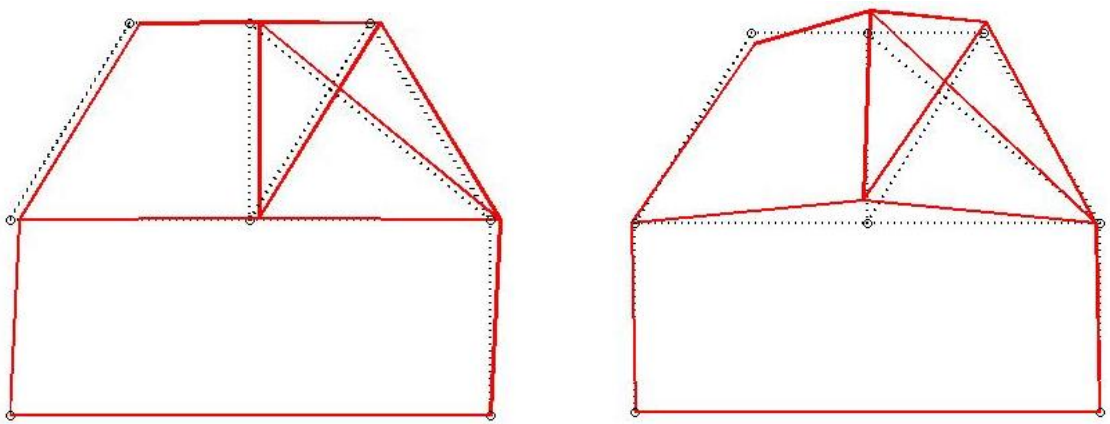
$$16 - 3 - 12 = 1 = 2 - 1$$

$$\mathbf{C}_{12\times16}\mathbf{u}_{16\times1} = \mathbf{e}_{12\times1}$$

Rank deficiency = 2 (not counting rigid-body modes)
→ 2 DoF

$$\mathbf{H}_{16\times12}\mathbf{p}_{12\times1} = \mathbf{f}_{16\times1}$$

Rank deficiency = 1
➔ 1 SoSS



Null-space "modes" of **C**.

We can also use the stiffness matrix (finite element framework)

Compatibility matrix | Disp. at vertices | Bar elongations

$$\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{e}_{b\times 1}$$

Equilibrium matrix | Bar forces | Forces at vertices

$$\mathbf{H}_{2v\times b}\mathbf{p}_{b\times 1} = \mathbf{f}_{2v\times 1}$$

$$\mathbf{p}_{b\times 1} = \mathbf{D}_{b\times b}\mathbf{e}_{b\times 1}$$

$$\mathbf{K}_{2v\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{f}_{2v\times 1}$$

$$\Rightarrow \mathbf{p}_{b\times 1} = \mathbf{D}_{b\times b}\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1}$$

$$\Rightarrow \mathbf{H}_{2v\times b}\mathbf{D}_{b\times b}\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{H}_{2v\times b}\mathbf{p}_{b\times 1}$$

$$\Rightarrow \mathbf{H}_{2v\times b}\mathbf{D}_{b\times b}\mathbf{C}_{b\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{f}_{2v\times 1}$$

Rank deficiency of the stiffness matrix

$$\mathbf{K}_{2v\times 2v}\mathbf{u}_{2v\times 1} = \mathbf{f}_{2v\times 1}$$

Summary

Compatibility and equilibrium matrices give correct but only instantaneous DoF and SoSS.

Stiffness matrix can also be used for finding instantaneous (infinitesimal) DoF.

Size optimization of statically determinate trusses

$$\underset{\mathbf{a}}{\text{Min}}\ MC = \sum_{i=1}^{n} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \quad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

Data: $\rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$

$$\underset{\mathbf{a}}{\text{Min}}\ MC = \mathbf{p}^T \mathbf{u} = \sum_{i=1}^{n} P_i u_i$$

Subject to

$$\mu: \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

Data: $\rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$

$$u_i = \left(\delta \mathbf{u}^T \mathbf{K}\right) \mathbf{u}^*$$

Contd.

$$\underset{\mathbf{a}}{\text{Min}}\; MC = \mathbf{p}^T\mathbf{u} = \sum_{i=1}^{n} P_i u_i$$

$$u_i = \left(\delta\mathbf{u}^T\mathbf{K}\right)\mathbf{u}^* = \sum_{j=1}^{N} t_j^{(i)} \frac{T_j l_j}{A_j E}$$

Subject to

$$\mu: \qquad \rho\mathbf{a}^T\mathbf{l} - W^* \le 0$$

$T_j =$ Internal force in j^{th} truss member due to applied real loads.

Data: $\rho, W^*, l_{i=1,2\ldots,N}, N, P_{i=1,2,\ldots,N}, E$

$$L = \sum_{i=1}^{n} P_i \left(\sum_{j=1}^{N} t_j^{(i)} \frac{T_j l_j}{A_j E} \right) + \mu \sum_{j=1}^{N} \rho A_j l_j$$

$t_j^{(i)} =$ Internal force in j^{th} truss member due to unit virtual load applied on i^{th} DoF.

$$\frac{\partial L}{\partial A_k} = -\sum_{i=1}^{n} P_i \frac{t_j^{(i)} T_k l_k}{A_k^2 E} + \rho\mu l_k = 0$$

This enables us to obtain an expression for each area of cross section in terms of data and μ

Dual problem

$$\underset{\mu}{\text{Max}} \; L(\mu)$$

Now, it is a one variable unconstrained maximization problem except that μ should be non-negative.

Note that we can do this for a statically determinate truss, however large it may be.

We have two methods now.

General algorithms with outer and inner loops to find cross section areas of any kind of truss.

We have the Matlab code for this wherein we begin with an exhaustive "ground structure" with all truss elements defined between every pair of points in a grid of vertices.

We should check whether the resulting optimal truss is statically determinate or no.

A specific dual formulation that reduces the size optimization of statically determinate trusses to a one-variable problem.

Here, we need to first check if the truss is statically determinate or not using the Maxwell's rule and then with the rank-deficiency of the equilibrium matrix.

The end note

Truss optimization

- Clayperon's theorem and its implication in stiffness optimization
- Maxwell's rule for static determinacy and Calladine's modification
- Degrees of freedom (DoF) and states of self stress (SoSS)
- Force equilibrium and displacement compatibility matrices
- **Rank deficiency of the force equilibrium gives the number of SoSS and that of displacement compatibility gives the number of DoF.**
- Dual problem for statically determinate trusses.

Thanks