Lecture 9

Failure constraints in truss optimization & Simultaneous material selection and geometry design

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Stress and buckling constraints

Material and design indices

Novel material+geometry optimization

What we will learn:

Ashby's method of material selection

Dealing with strength and stability constraints in truss optimization

Capturing geometric details into a design index

Simultaneous optimization in material and geometry space

Examples that illustrate material+geometry optimization

Approaches to structural design

Usually…

A

Choose material(s)
and then
design the geometry

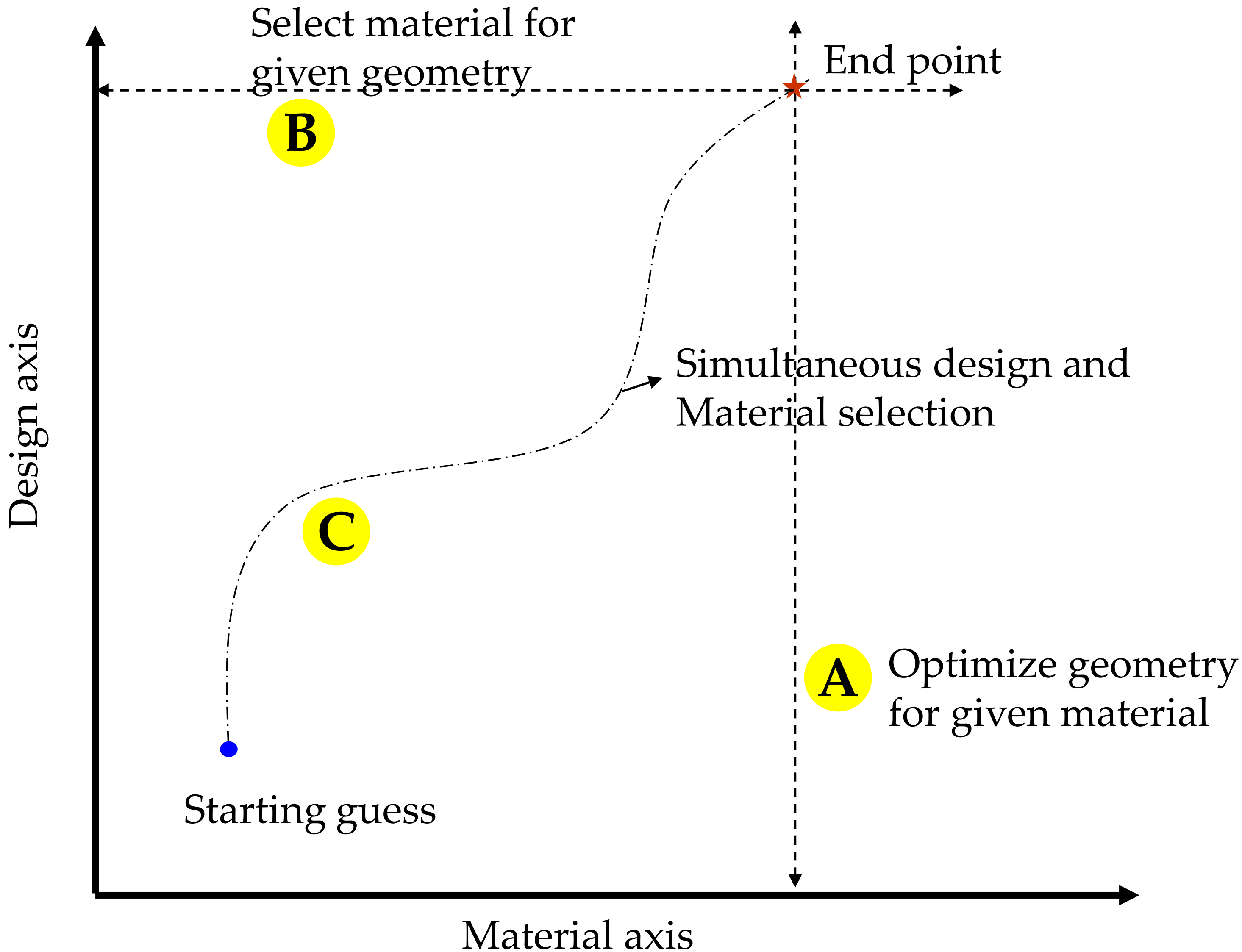
B

Choose geometry
and then
select material(s).

How about **C**?

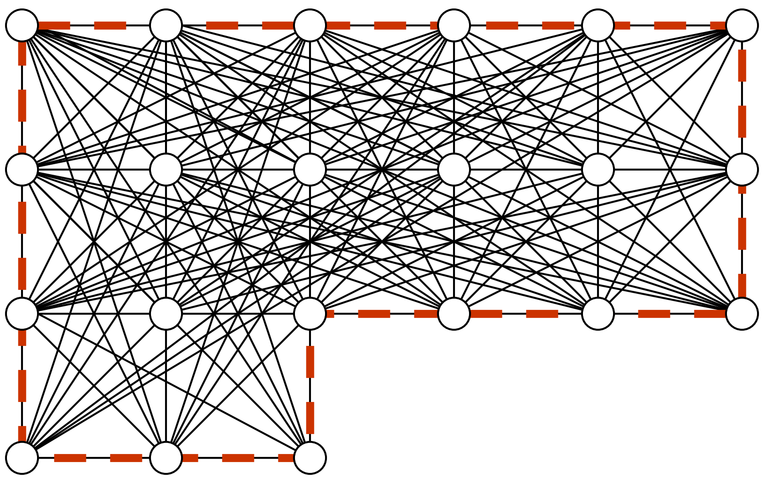
Simultaneously design the geometry and select material(s).

Geometry design vs. material selection

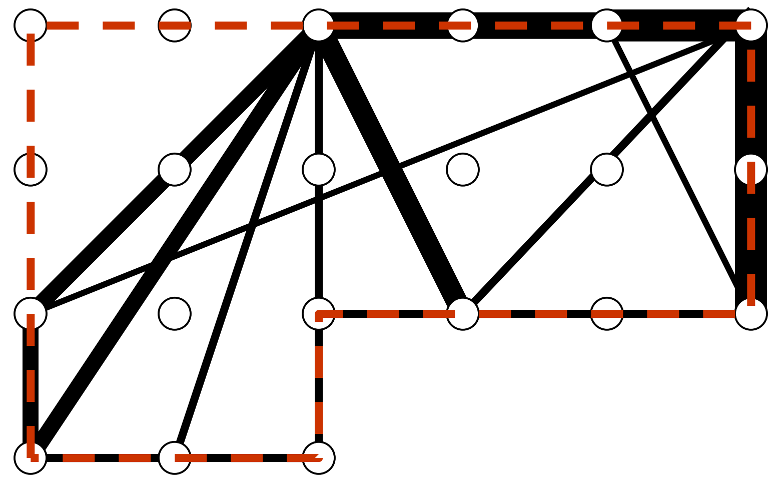


Truss topology optimization assuming a material

Ground structure



A possible solution

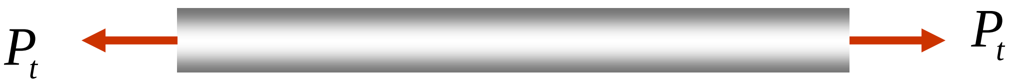


Associated with each truss element, define a c/s area variable. This leads to N optimization variables.

Each variable has lower (almost zero) and upper bounds.

Kirsch, U. (1989). Optmal Topologies of Structures. *Applied Mechanics Reviews* 42(8):233-239.

Material selection for the cheapest tensile truss members against failure



σ_y = yield or tensile failure strength of the material that is yet to be chosen.

To prevent failure, area of cross-section = $A_t = \dfrac{P_t}{\sigma_y}$

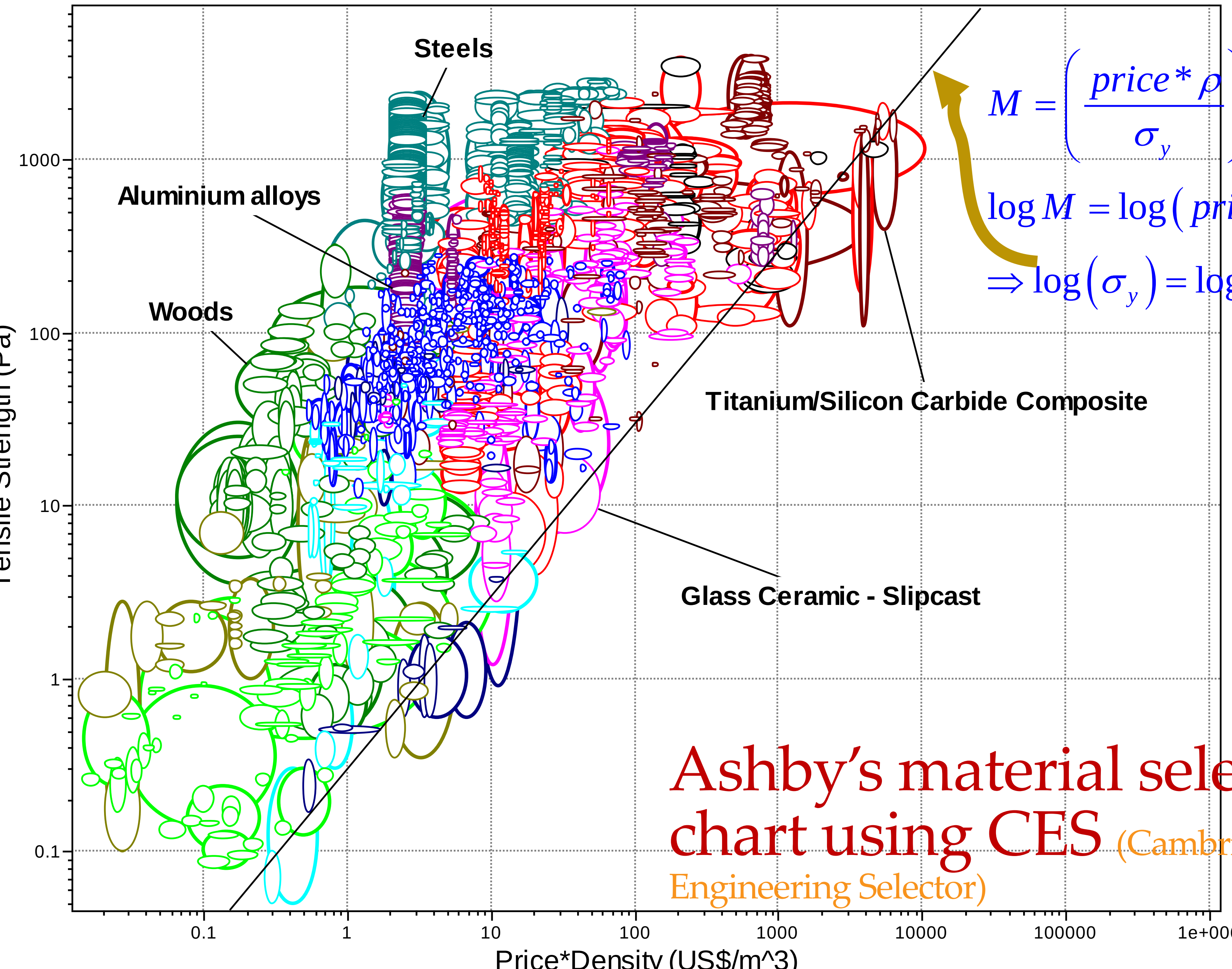
$$A_t = \frac{(FoS)P_t}{\sigma_y}$$

With the factor of safety (FoS)

$$\text{Cost} = price * \rho A_t l = price * \rho \frac{P_t}{\sigma_y} l = P_t l \left(\frac{price * \rho}{\sigma_y} \right)$$

Choose a material with lowest $\left(\dfrac{price * \rho}{\sigma_y} \right)$ ← Material index

The material with lowest Material index satisfies the strength criterion and is the cheapest for the component.



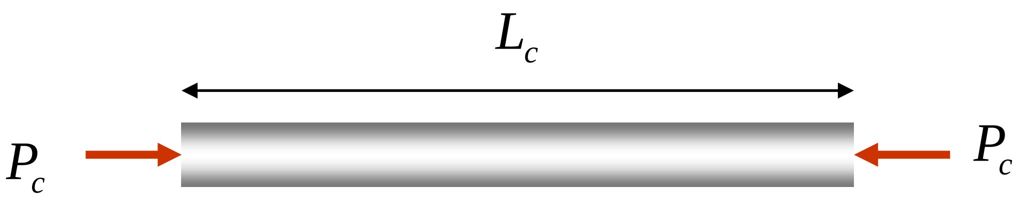
$$M = \left(\frac{price * \rho}{\sigma_y} \right)$$

$$\log M = \log(price * \rho) - \log(\sigma_y)$$

$$\Rightarrow \log(\sigma_y) = \log(price * \rho) - \log M$$

Ashby's material selection chart using CES (Cambridge Engineering Selector)

Material selection for the cheapest compression members against (buckling) failure



E = Young's modulus of the material that is yet to be chosen.

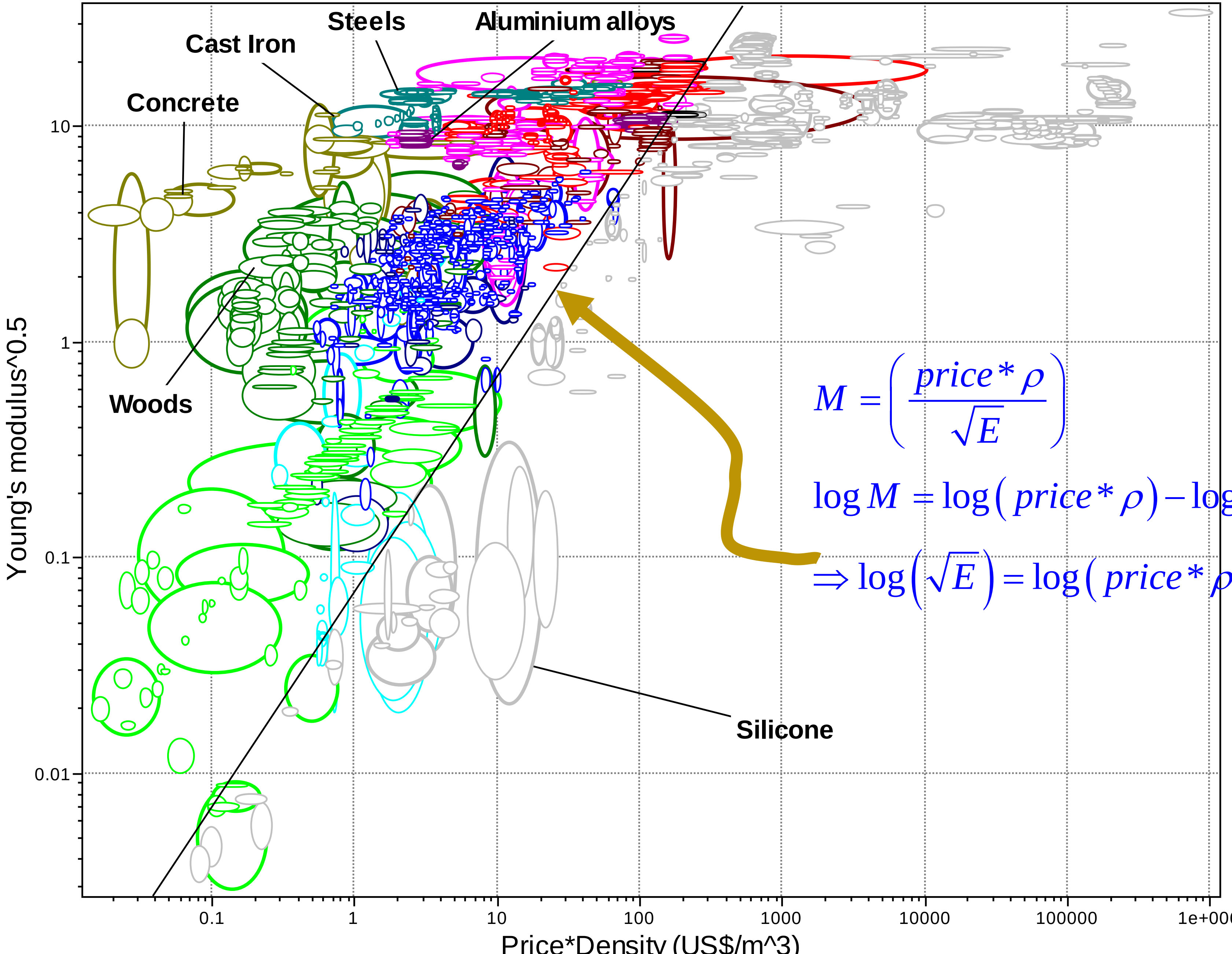
To prevent failure, area of cross-section = $A_c = \sqrt{\dfrac{12 L_c^2 P_c}{\pi^2 E}}$

$$A_c = \sqrt{\frac{12 L_c^2 P_c (FoS)}{\pi^2 E}}$$

With the factor of safety (FoS)

Cost = $price * \rho A_c L_c = price * \rho L_c \sqrt{\dfrac{12 L_c^2 P_c}{\pi^2 E}} = \sqrt{\dfrac{12 L_c^4 P_c}{\pi^2}} \left(\dfrac{price * \rho}{\sqrt{E}} \right)$

Choose a material with lowest, $\left(\dfrac{price * \rho}{\sqrt{E}} \right)$ ← Material index



$$M = \left(\frac{price * \rho}{\sqrt{E}} \right)$$

$$\log M = \log\left(price * \rho\right) - \log\left(\sqrt{E}\right)$$

$$\Rightarrow \log\left(\sqrt{E}\right) = \log\left(price * \rho\right) + \log M$$

Material selection for the entire truss

One option—choose two best materials, one for the tensile members and one for the compression members.
This is not attractive from the manufacturability viewpoint.

Second option—choose one material that optimizes the tensile and compression members.
But how?

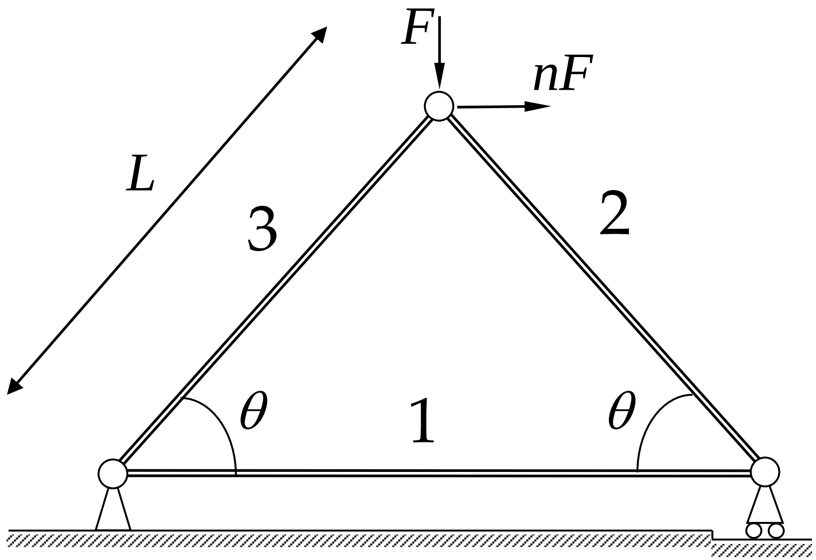
Related work

- Lightest, fail-safe truss subjected to equal stress constraints in tension and compression done by Dorn *et al*. (1964)
- Achtziger showed that this problem can be solved by taking different stress constraints in tension and compression. (1996). But does not address buckling.
- Stolpe and Svanberg (2004) mathematically proved that at most two materials (one for tension and another for compression) are required when solving this problem.
- Achtziger (1999) has given guidelines for solving truss topology optimization problem using buckling as constraint. But not strength constraint.

The content of this lecture is from:

- Ananthasuresh, G. K. and M. F. Ashby, "Concurrent Design and Material Selection for Trusses," Proceedings of the Workshop on Optimal Design of Materials and Structures, Ecole Polytechnique, Palaiseau, France, Nov. 26-28, 2003.
- Rakshit, S. and Ananthasuresh, G. K., "Simultaneous material selection and geometry design of statically determinate trusses using continuous optimization," *Structural and Multidisciplinary Optimization*, 35 (2008), pp. 55-68, DOI 10.1007/s00158-007-0116-4.

A simple truss



Internal forces

$$P_1 = \frac{F(1+n\tan\theta)}{2\tan\theta}$$

$$P_2 = -\frac{F(1+n\tan\theta)}{2\sin\theta}$$

$$P_3 = -\frac{F(1-n\tan\theta)}{2\sin\theta}$$

$$R_l = \frac{F(1-n\tan\theta)}{2}; \quad R_r = \frac{F(1+n\tan\theta)}{2}$$

Vertical reaction forces

Areas of c/s against failure

$$A_t = \frac{P_t}{\sigma_y}; \quad A_c = \sqrt{\frac{12L_c^2 P_c}{\pi^2 E}}$$

Basis for a single material selection for a truss

$$\text{Mass} = \quad m = \left\{ \sum_{i=1}^{N_t} (P_{t_i} L_{t_i}) \right\} \frac{\rho}{\sigma_y} + \left\{ \sum_{j=1}^{N_c} \frac{L_{j_c}^2}{\pi} \sqrt{12 \left| P_{j_c} \right|} \right\} \frac{\rho}{E^{1/2}} = \psi_t \frac{\rho}{\sigma_y} + \psi_c \frac{\rho}{E^{1/2}}$$

N_t = number of tensile members

N_c = number of compression members

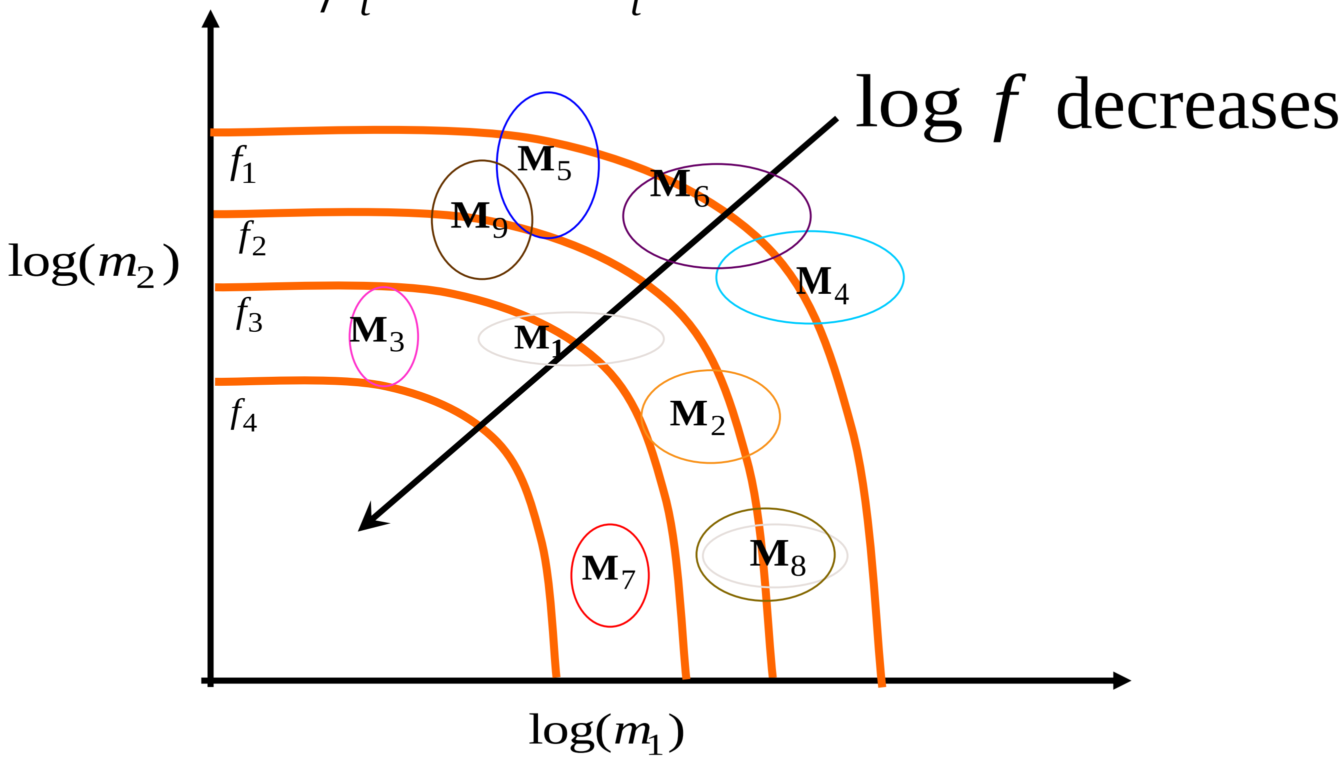
Minimum mass depends on the weighted sum of two material indices.

Design index = $\gamma = \dfrac{\psi_c}{\psi_t}$ ← Depends only on design of geometry and the loading

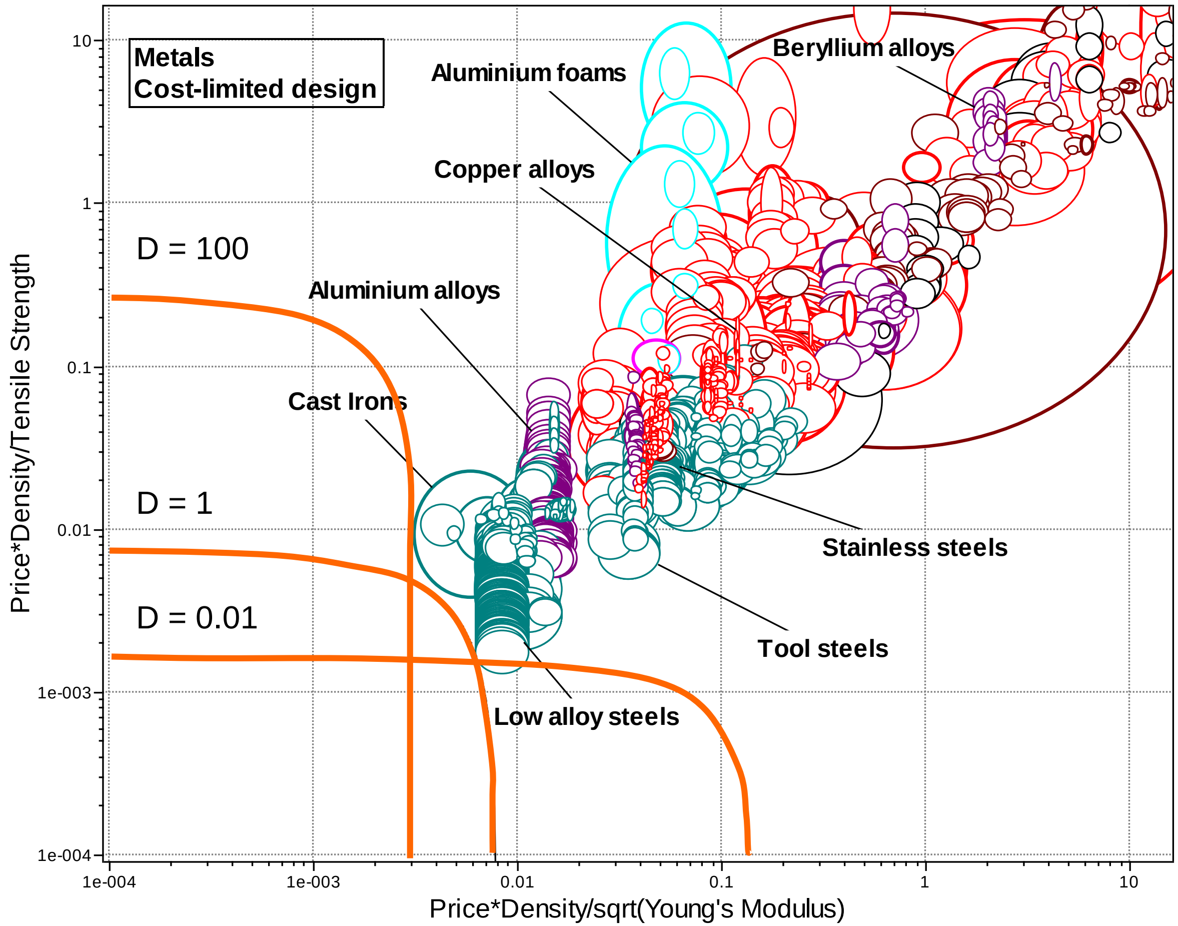
$$mass = \rho A L = \sum_{i=1}^{N_t} (\rho_t A_t L_t) + \sum_{i=1}^{N_c} (\rho_c A_c L_c)$$

$$mass = \left\{ \sum_{i=1}^{N_t} (P_{t_i} L_{t_i}) \right\} \frac{\rho}{\sigma_y} + \left\{ \sum_{i=1}^{N_c} \frac{L_{j_c}}{\pi} \sqrt{12 | P_{j_c} |} \right\} \frac{\rho}{E^{1/2}} = \psi_t \frac{\rho}{\sigma_y} + \psi_c \frac{\rho}{E^{1/2}}$$

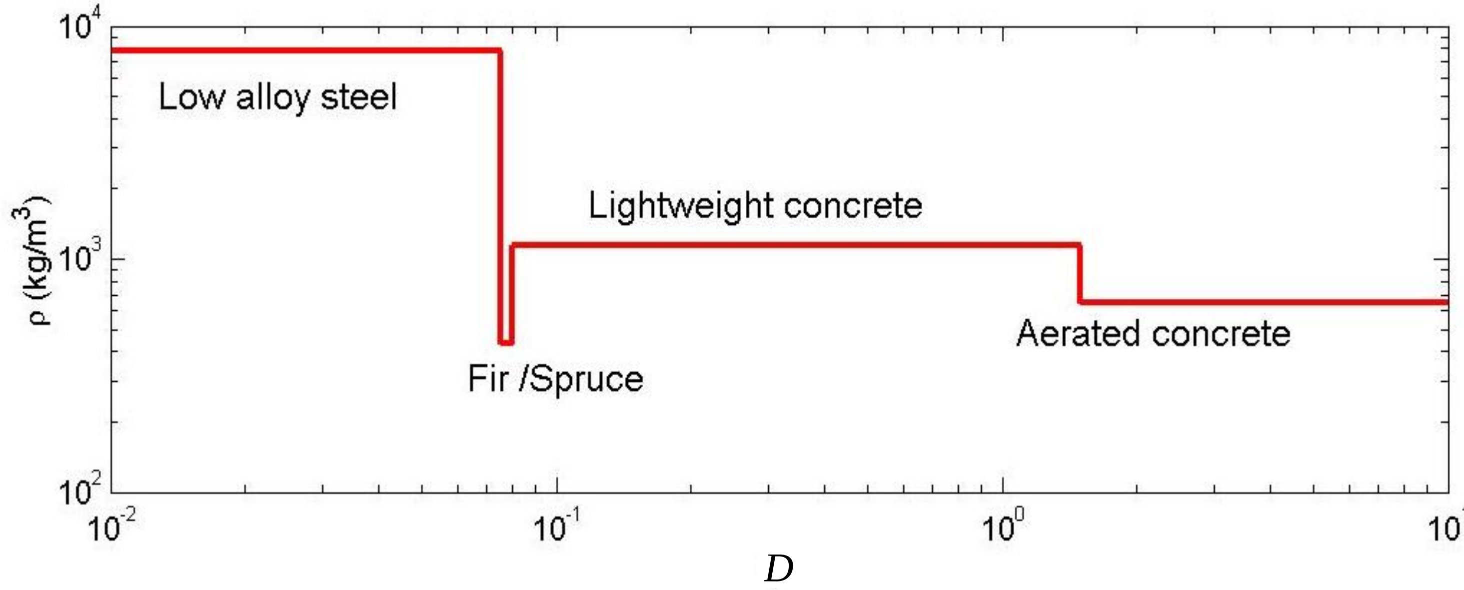
$$f = \frac{\psi_c}{\psi_t} \frac{\rho}{E^{1/2}} + \frac{\rho}{S_t} = D m_1 + m_2$$

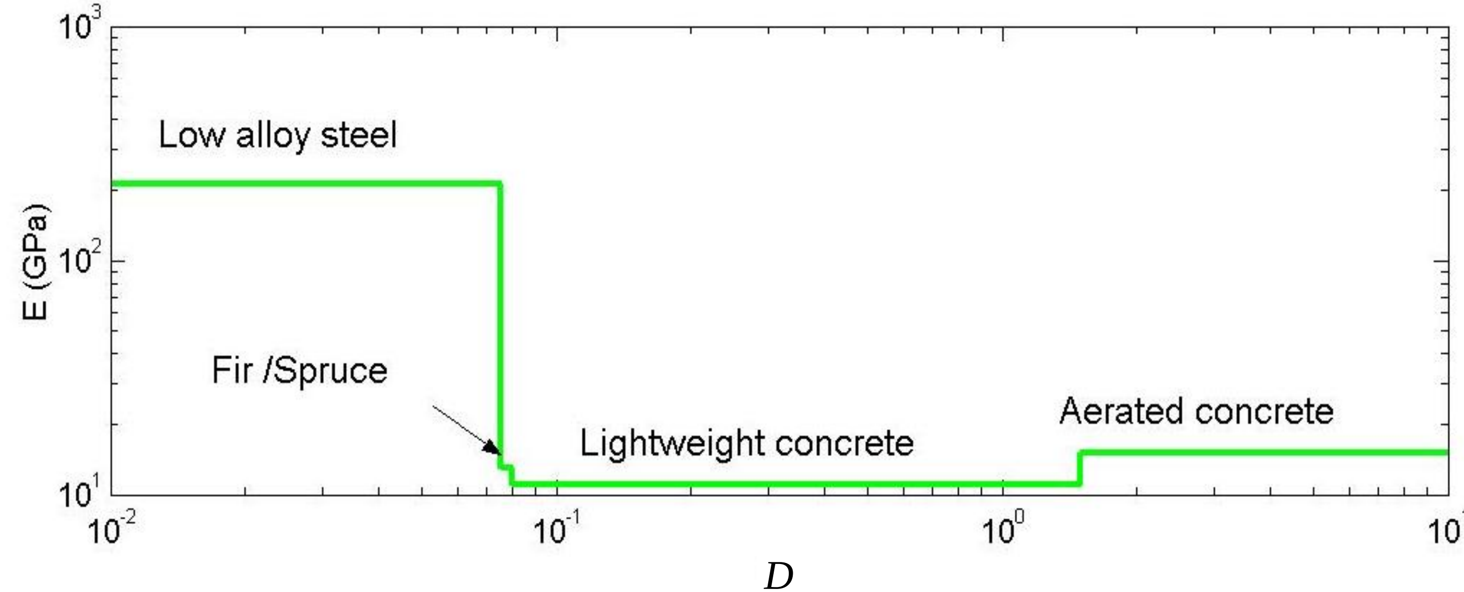


Cheapest single material for the entire truss



CES


ρ (kg/m3)
Low alloy steel
Lightweight concrete
Fir /Spruce
Aerated concrete
D


E (GPa)
Low alloy steel
Fir /Spruce
Lightweight concrete
Aerated concrete
D

Statement of the problem for simultaneous geometry and material optimization

Minimize Strain Energy
Geometry variables

$$Min_x : \int_\Omega \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{U} d\Omega$$

Subject to

Static elastic equilibrium equation $\quad$ $\boldsymbol{Ku} = \boldsymbol{F}$

Satisfying

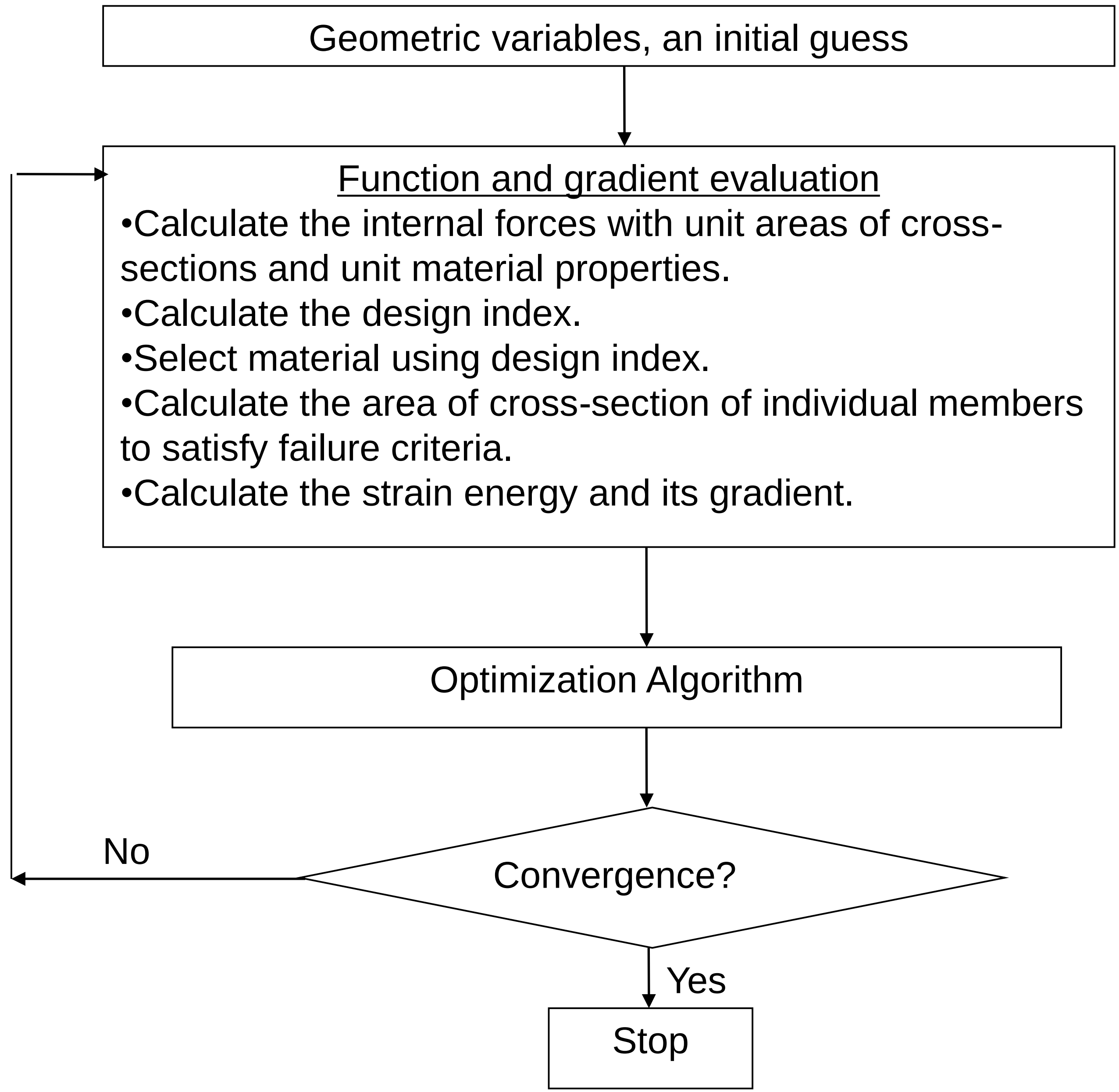
The failure criteria for tensile and compression members

$$S_t = \frac{P_t}{A_t}$$

$$P_c = \frac{\pi^2 E A_c^2}{12 L_c^2}$$

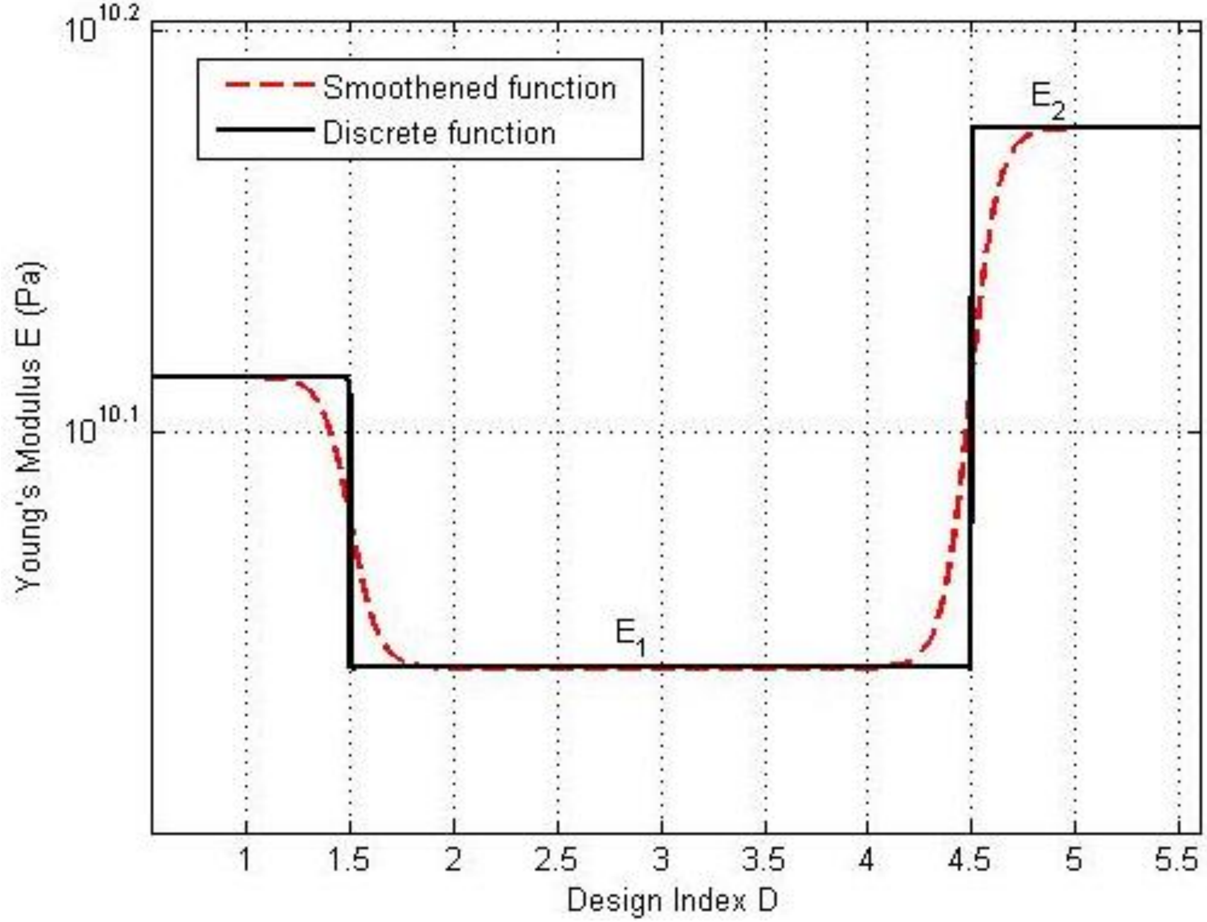
To give:
- Loads, boundary conditions, size of the domain
- Areas of cross-section, geometry and material.

Algorithm


Geometric variables, an initial guess
Function and gradient evaluation
•Calculate the internal forces with unit areas of cross-sections and unit material properties.
•Calculate the design index.
•Select material using design index.
•Calculate the area of cross-section of individual members to satisfy failure criteria.
•Calculate the strain energy and its gradient.
Optimization Algorithm
Convergence?
No
Yes
Stop

Dealing with non-smoothness

$$E = E_1 + \sum_{i=2}^{N_m} \frac{E_i - E_{i-1}}{1 - e^{-s(D_i - D_{i-1})}}$$



Sensitivity Analysis

$$\frac{dSE}{dx} = \frac{1}{2}\mathbf{u}^T\left(\frac{\partial \mathbf{K}}{\partial x} + \frac{\partial \mathbf{K}}{\partial E}\frac{dE}{dx} + \frac{\partial \mathbf{K}}{\partial \mathbf{A}}\frac{d\mathbf{A}}{dx}\right)\mathbf{u} + \mathbf{u}^T\mathbf{K}\frac{d\mathbf{u}}{dx}$$

$$\frac{dE}{dx} = \left(\frac{\partial E}{\partial D}\right)\left(\frac{dD}{dx}\right)$$

where, $\frac{dD}{dx} = \frac{\psi_t \frac{d\psi_c}{dx} - \psi_c \frac{d\psi_t}{dx}}{\psi_t^2}$

For tensile members

$$\frac{dA_i}{dx} = \frac{dS_t}{dx}P_{t_i} + S_t\frac{dP_{t_i}}{dx}$$

For compressive members

$$\frac{dA_i}{dx} = \sqrt{12\pi}\left\{\sqrt{\frac{P_{c_i}}{E}}\frac{dL_{c_i}}{dx} + \frac{L_{c_i}}{2\sqrt{EP_{c_i}}}\frac{dP_{c_i}}{dx} - \frac{L_{c_i}\sqrt{P_{c_i}}}{2E^{1.5}}\frac{dE}{dx}\right\}$$

We use element equilibrium equation $\frac{EA_i}{L_i}\left(u_{i_1} - u_{i_2}\right) = P_i$ to calculate

$$\frac{d\boldsymbol{P}}{dx}$$

$$\begin{bmatrix}\boldsymbol{C} & \boldsymbol{B}\end{bmatrix}\begin{Bmatrix}\dfrac{d\boldsymbol{P}}{dx} \\ \dfrac{d\boldsymbol{u}}{dx}\end{Bmatrix} = \{\boldsymbol{g}\}$$

This gives m equations, where

m is the number of members in the truss

We use the global equilibrium equation $\boldsymbol{Ku} = \boldsymbol{F}$ to calculate the rest $2n$ equations where n is the number of nodes in the truss to get

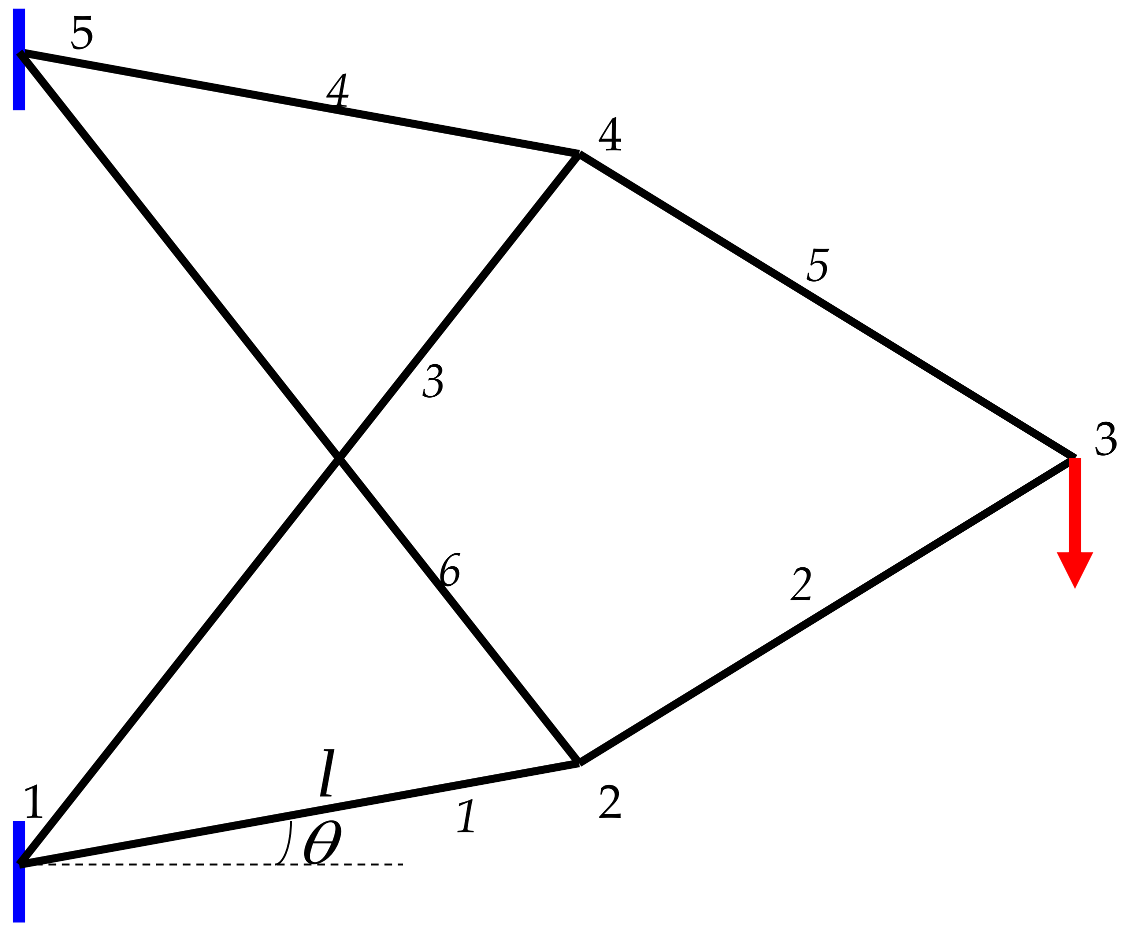
$$\begin{bmatrix} C & B \end{bmatrix} \begin{Bmatrix} \dfrac{d\boldsymbol{P}}{dx} \\ \dfrac{d\boldsymbol{u}}{dx} \end{Bmatrix} = \left\{ -\frac{\partial \boldsymbol{K}}{\partial x}\boldsymbol{u} - \frac{\partial f}{\partial x}\frac{\partial \boldsymbol{K}}{\partial \boldsymbol{A}}\boldsymbol{u} \right\}$$

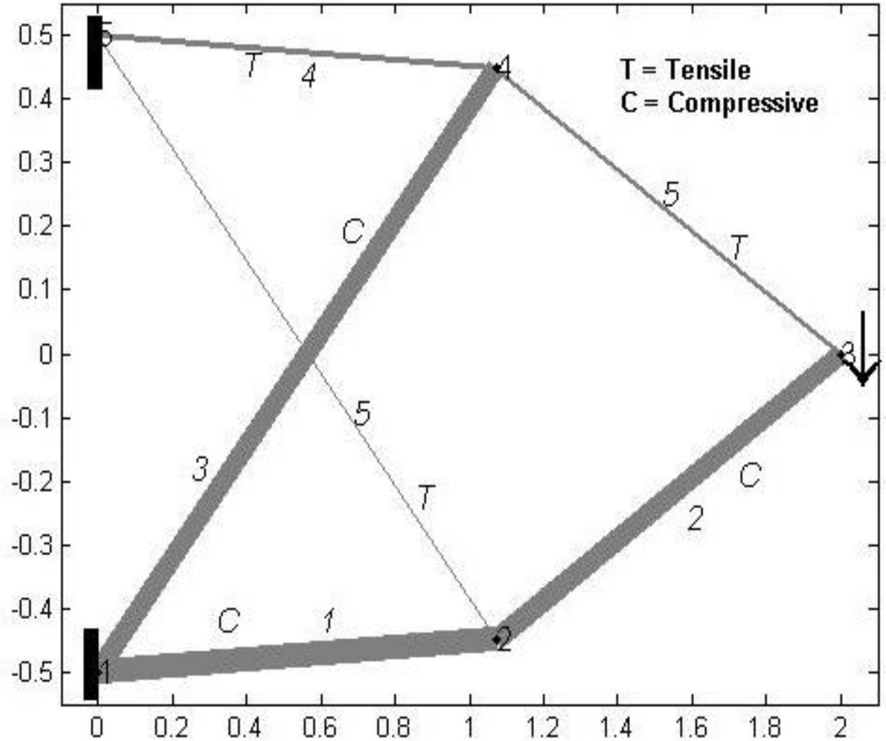
Now assemble

$$\begin{bmatrix} \boldsymbol{C} & \boldsymbol{B} \\ \boldsymbol{Q} & \boldsymbol{K} \end{bmatrix} \begin{Bmatrix} \dfrac{d\boldsymbol{P}}{dx} \\ \dfrac{d\boldsymbol{u}}{dx} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{g} \\ \boldsymbol{h} \end{Bmatrix}$$

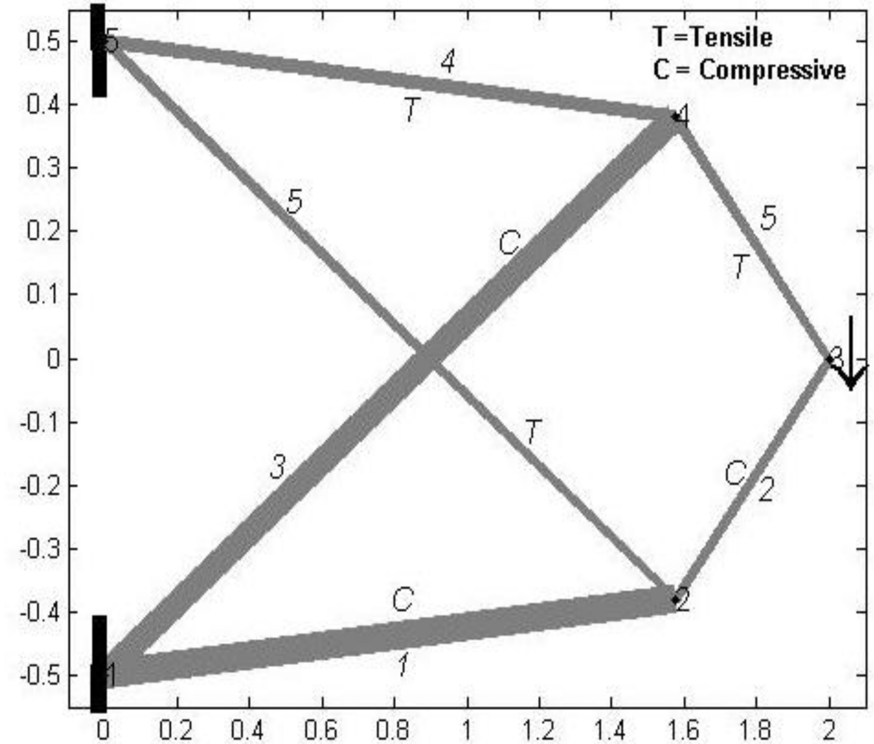
Results

Example 1



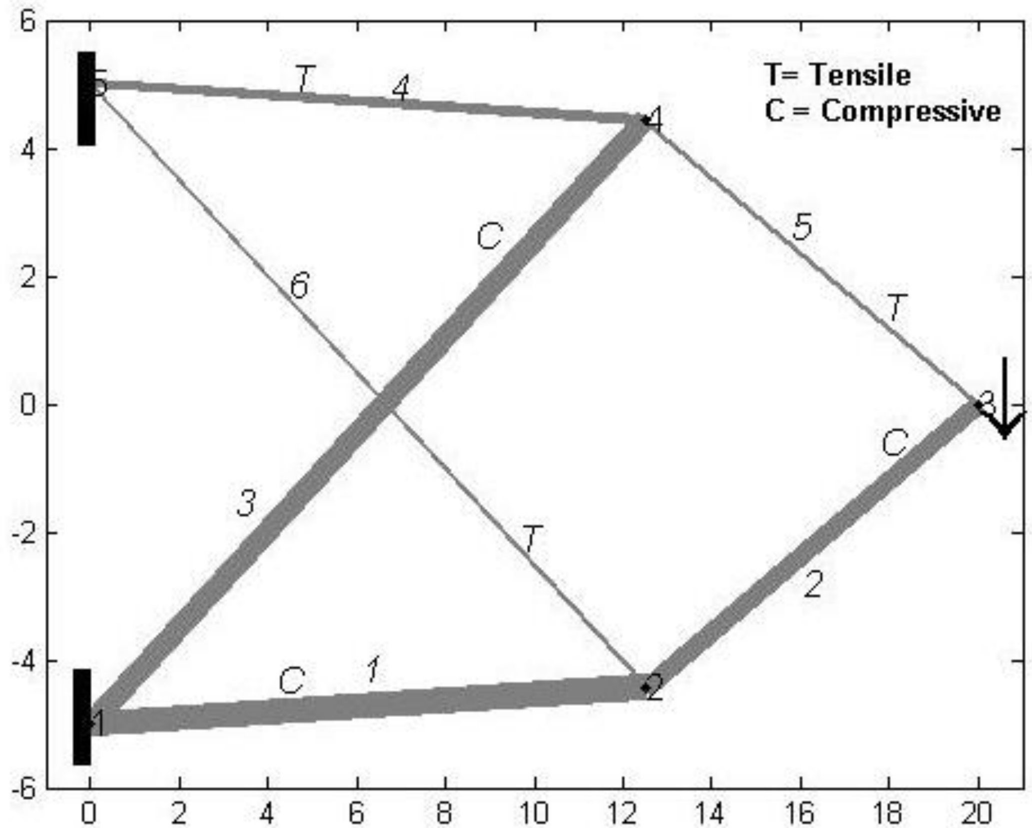


Optimal layout of structure for force = 1000 N. Selected best material is low alloy steel. The material cost of the truss is $1.287 = Rs 59.54. Strain energy calculated is 18.7325 Joules



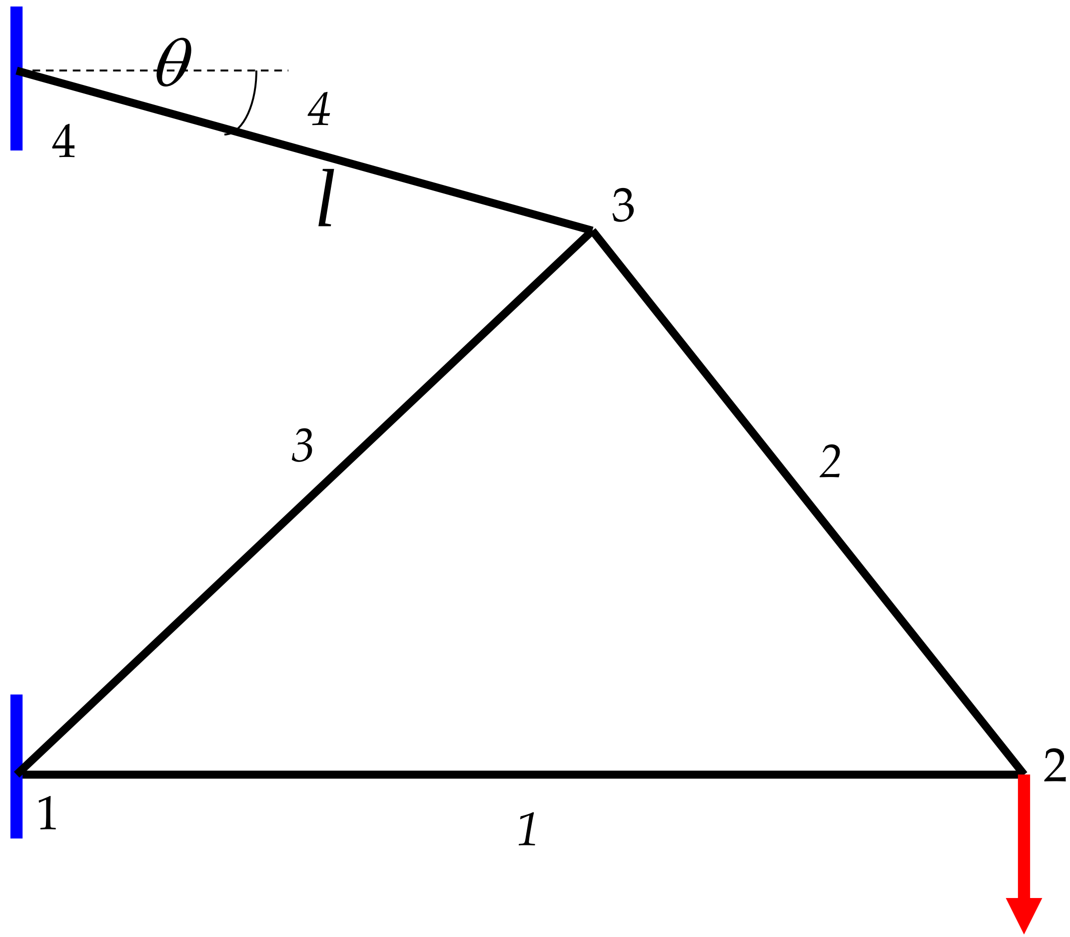
Optimal layout for structure with force = 90 N. The best material is lightweight concrete.

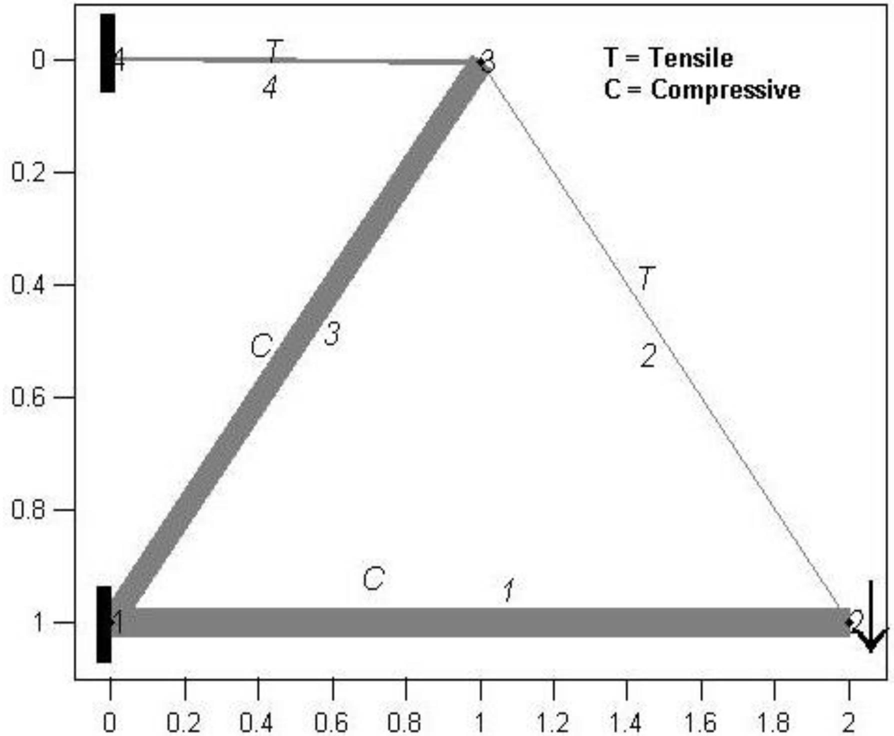
The cost of the material is $0.0357 = Rs 1.65. Strain energy is 0.034 Joules.



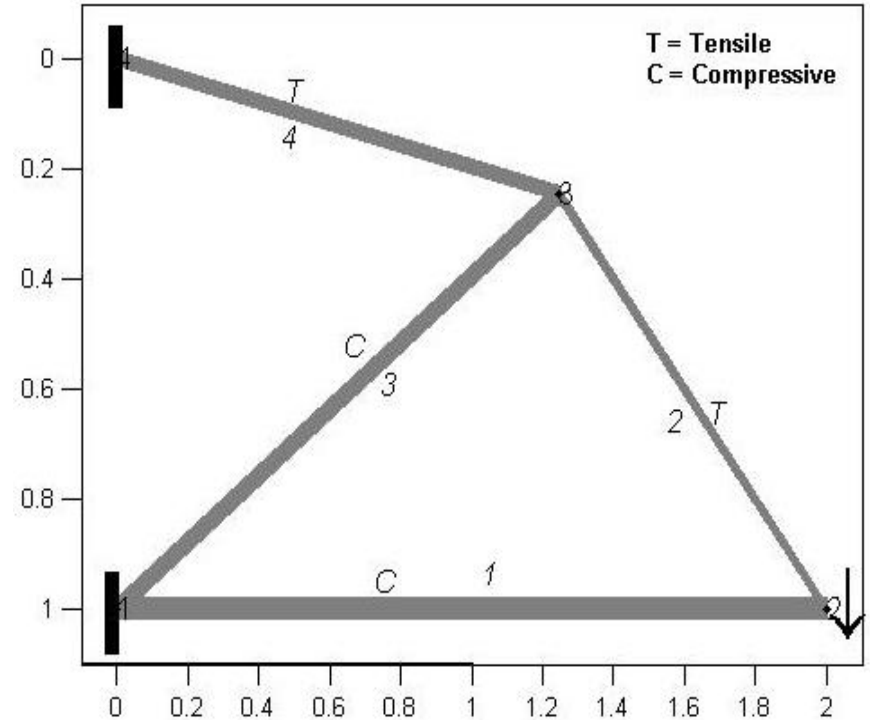
Optimum geometry corresponding to a structure that is ten times as big as the initial structure. The applied force F = 1000 N. The best material comes out to be lightweight concrete. The cost of the material is $7.659 = Rs 354.3 and the strain energy is 2.935 Joules.

Example 2

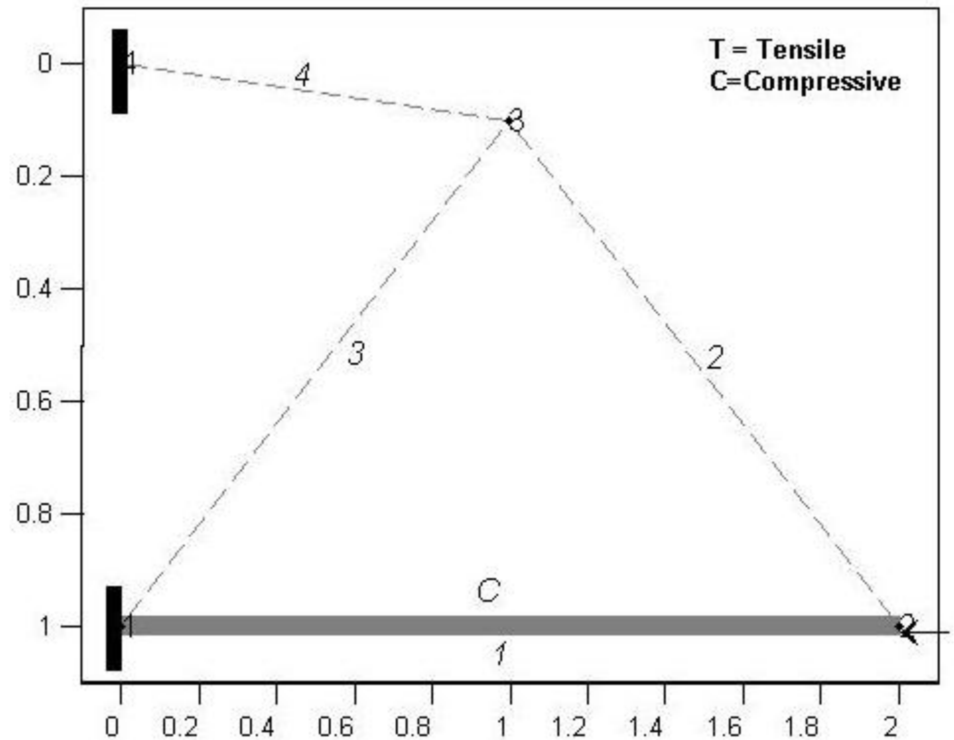




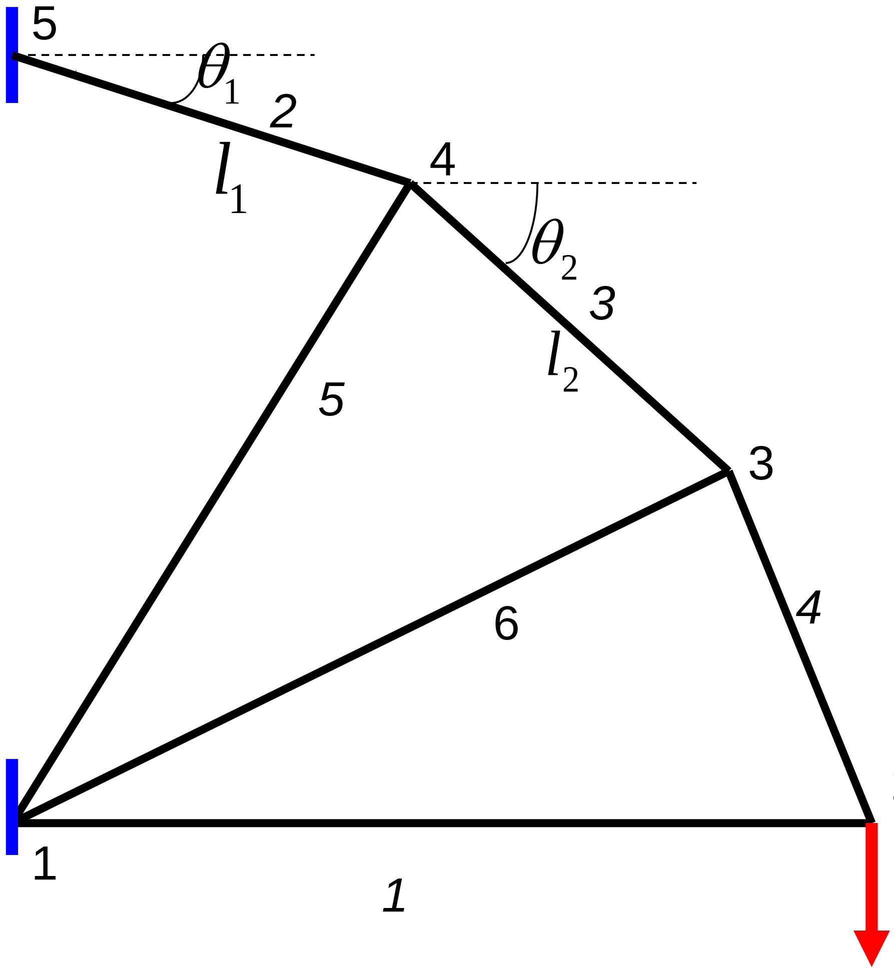
Optimal layout for the structure with force = 1000N. Best single material is low alloy steel. The material cost is $ 1.947 = Rs 90.07 and strain energy 20.0125 Joules.

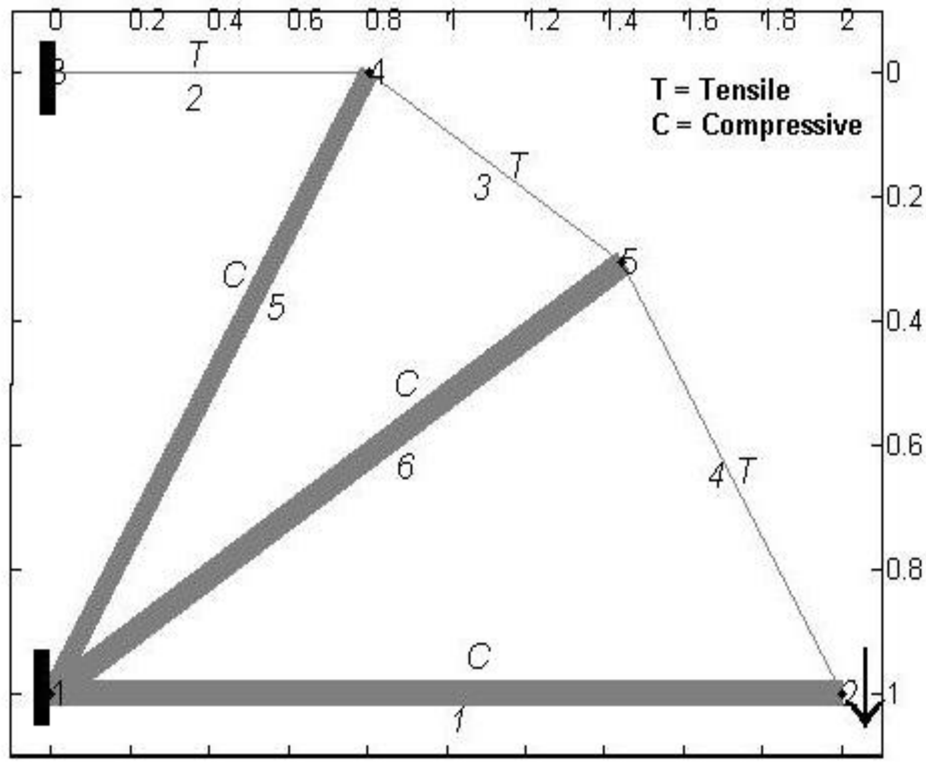


Optimal structure for force = 100N
Selected material is lightweight concrete. The material cost is $ 0.0397 = Rs 1.84 and strain energy is 0.0368 Joules.

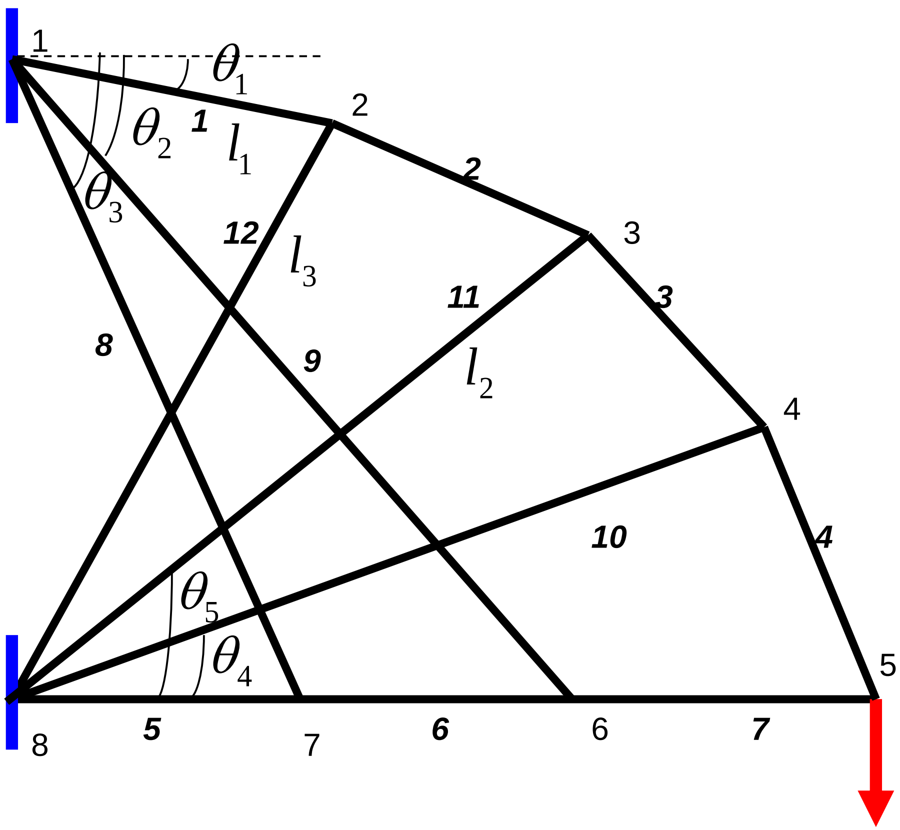


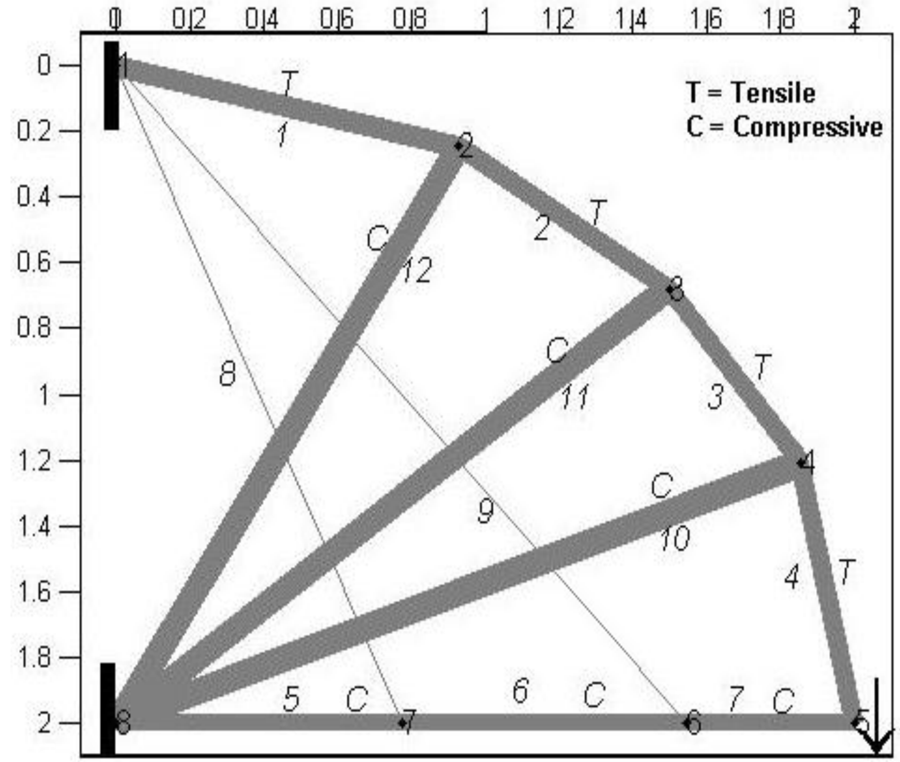
Optimal structure when only a horizontal load = 1000 N is applied. The internal forces in the dashed members are zero and their cross sections have reached the lower limit. Hence such members may be safely removed from the parent structure. Best material is aerated concrete. The material cost is \$0.0378 = Rs 1.75 and strain energy is 0.11708 Joules.





Optimal layout for force = 1000 N.
The best material is low alloy steel.
The material cost is \$2.229 = Rs 103.113
and the strain energy is 19.415 Joules.

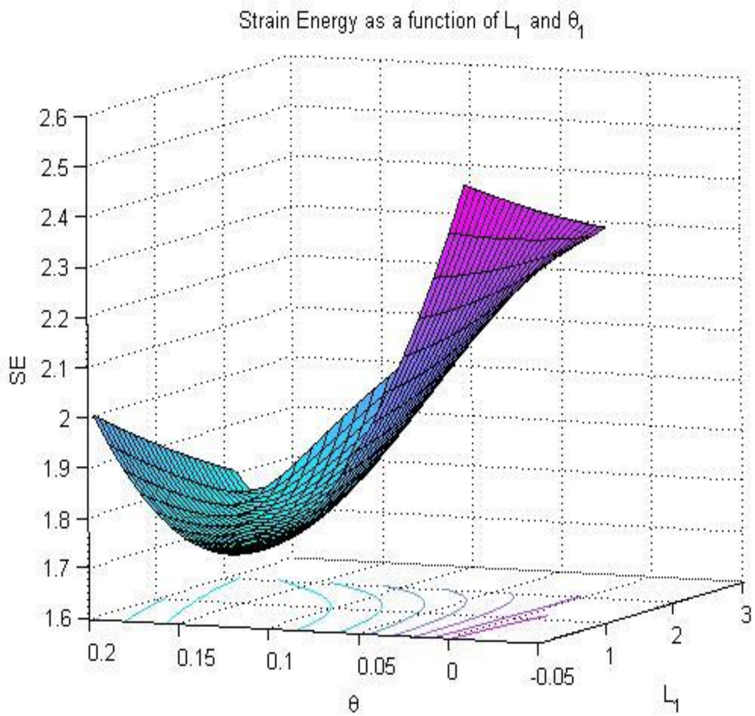




Optimal layout for structure with force = 100 N. The best material is lightweight concrete. The material cost is $ 0.0445 = Rs 2.086 and the strain energy is 0.0251 Joules.

- CONVERGENCE PROBLEMS:

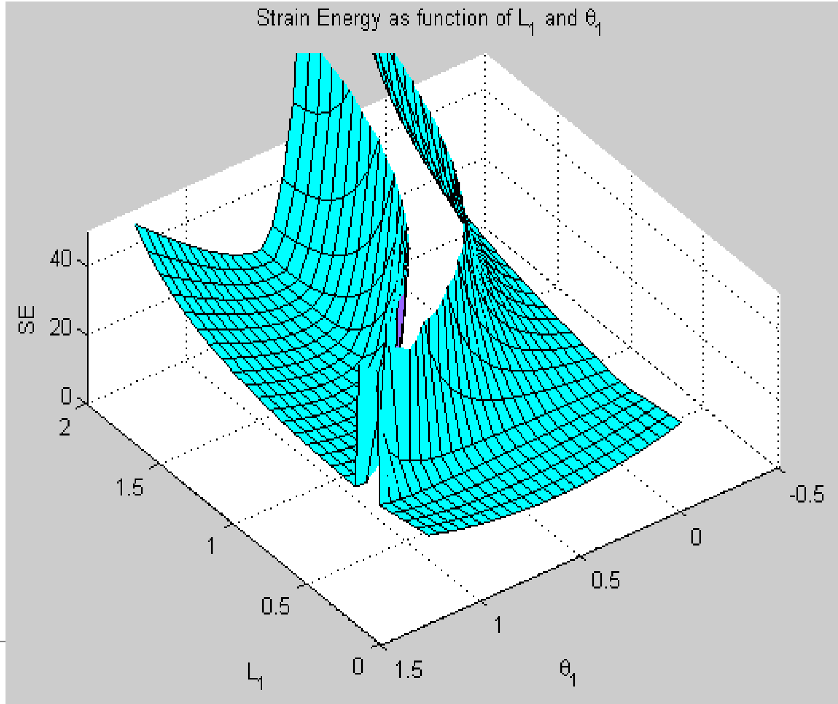
- STRAIN ENERGY IS A DISCONTINUOUS FUNCTION OF DESIGN VARIABLES



The Strain Energy as a smooth function in the range

$0 \le L_1 \le 2\,\mathrm{m}$ and $0 \le \theta_1 \le 0.2\,\mathrm{rad}$

As seen the function is smooth.



The Strain Energy in the range

$0 \le L_1 \le 2\,\mathrm{m}$ and $0 \le \theta_1 \le \frac{\pi}{2}$ rad

As seen the function is nonsmooth. Nonsmoothness occurs when there is transition from tensile to compressive members.

The end note

Truss optimization

- Geometry optimization for given material vs. Material selection for given geometry
- Ashby's method of material selection
- Strength (stress) and stability (buckling) constraints
- Design index—one number that captures all geometry variables
- **Optimization of geometry and material simultaneously**
- Observe how we traversed the discrete material axis using differentiable model

Thanks