ANALYTICALLY SOLVING BAR OPTIMIZATION PROBLEMS

BY
RAHUL PRATAP SINGH
SR NUMBER: 17531
ME (MTECH)
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PROBLEM 1

Minimize the strain energy for given volume of material under given loading.

$$\underset{A(x)}{Min}\ SE = \int \frac{1}{2} EAu'^2 dx$$

subjected to

$$\lambda(x) : \left(EAu'\right)' + P = 0$$

$$\Lambda : \int A dx - V^* \leq 0$$

$$Data : L, P(x), E, V^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[\frac{1}{2} EAu'^2 + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(A - \frac{V^*}{L} \right) \right] dx$$

$$\hat{L} = \frac{1}{2} EAu'^2 + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(A - \frac{V^*}{L} \right)$$

STEP 2: VARIATION W.R.T DEGINE VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0$$ Euler Lagrange Equation

$$\Rightarrow \left(\frac{1}{2} EAu'^2 + \lambda Eu'' + \Lambda \right) - \left(\lambda Eu' \right)' = 0$$

$$\Rightarrow \boxed{\Lambda = Eu'\lambda' - \frac{1}{2} EAu'^2}$$ Design Equation

STEP 3: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$ Euler Lagrange Equation

$$\Rightarrow 0 - \left(EAu' + \lambda EA'\right)' - \left(\lambda EA\right)'' = 0$$

$$\Rightarrow \boxed{\left(EA\lambda'\right) + P = 0}$$ Adjoint Equation

STEP 4: COLLECTING ALL EQUATIONS

$$\Lambda = Eu'\lambda' - \frac{1}{2}EAu'^2$$ Design Equation-------(1)

$$\left(EA\lambda'\right) + P = 0$$ Adjoint Equation-----(2)

$$\left(EAu'\right)' + P = 0$$ Governing Equation---(3)

$$\Lambda\left[\int\left(A - \frac{V^*}{L}\right)dx\right] = 0$$ Complementarity Criteria--(4)

$$\int Adx - V^* \leq 0$$ Feasibility Criteria-----(5)

STEP 5: OPTIMALITY CRITERIA

Comparing eq (2) & (3), we get

$\lambda = u$

Equation (1) becomes

$$\boxed{\Lambda = \frac{1}{2}Eu'^2} \leftarrow \text{Optimality Criteria} \qquad \text{----}(6)$$

Clearly, $\Lambda \geq 0$

$\Rightarrow$ Volume constraint is active

$$\therefore \int A dx - V^* = 0 \qquad \text{----}(7)$$

From (6), we get

$$u' = \pm\sqrt{\frac{2\Lambda}{E}} \qquad \text{----}(8)$$

STEP 6: BOUNDARY CONDITIONS

$$\hat{L} = \frac{1}{2} EAu'^{2} + \lambda\left\{\left(EAu'\right)' + P\right\} + \Lambda\left(A - \frac{V^{*}}{L}\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow \left(\lambda E u'\right)\delta A\Big|_0^L = 0 \Rightarrow \boxed{\left(uu'\right)\delta A_0^L = 0}$$ (BCI)

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left[\left(EAu' + \lambda EA'\right) - \left(\lambda EA\right)'\right]\delta u\Big|_0^L = 0 \Rightarrow \boxed{0\delta u\big|_0^L = 0}$$ (Automatically Satisfied) (BCII)

$$\left.\left(\frac{\partial \hat{L}}{\partial u''}\right)\delta u'\right|_0^L = 0 \Rightarrow (\lambda EA)\delta u'\Big|_0^L \Rightarrow \boxed{(Au)\delta u'\big|_0^L = 0}$$ (BCIII)

STEP 7: SOLVING FOR DESIGN VARIABLE

From (2) & (8), we get

$$P+\left(EA\left(\pm\sqrt{\frac{2\Lambda}{E}}\right)\right)'=0$$

$$\Rightarrow \boxed{A(x)=\pm\int\frac{P}{\sqrt{2E\Lambda}}dx+C} \qquad \text{-----}(9)$$

From (7) & (9), we get Λ as

$$\int_0^L\left\{\pm\int\frac{P}{\sqrt{2E\Lambda}}dx+C\right\}=V^* \qquad \text{-----}(10)$$

CONCLUSION

For minimum SE,

$$A(x)=\pm\int\frac{P}{\sqrt{2E\Lambda}}dx+C$$

where Λ & C are given by eq (10) & BCs respectively

PROBLEM 2

Minimize volume of material to be used with an upper bound constraint on strain energy under given loading.

$$\underset{A(x)}{Min}\ \mathrm{V} = \int A dx$$

subjected to

$$\lambda(x) : \left(EAu'\right)' + P = 0$$

$$\Lambda : \int \frac{1}{2} EAu'^2\, dx - SE^* \leq 0$$

$$Data : L, P(x), E, SE^*$$

STEP 1: WRITING LAGRANGIAN

$$L=\int\left[A+\lambda\left\{\left(EAu^{'}\right)^{'}+P\right\}+\Lambda\left(\frac{1}{2}EAu^{'2}-\frac{SE^{*}}{L}\right)\right]dx$$

$$\hat{L}=A+\lambda\left\{\left(EAu^{'}\right)^{'}+P\right\}+\Lambda\left(\frac{1}{2}EAu^{'2}-\frac{SE^{*}}{L}\right)$$

STEP 2: VARIATION W.R.T DEGINE VARIABLE A(x)

$$\delta_{A}\hat{L}=0\Rightarrow\frac{\partial\hat{L}}{\partial A}-\left(\frac{\partial\hat{L}}{\partial A^{'}}\right)^{'}=0$$

(Euler Lagrange Equation)

$$\Rightarrow\left(1+\lambda Eu^{''}+\frac{\Lambda}{2}Eu^{'2}\right)-\left(\lambda Eu^{'}\right)^{'}=0$$

$$\Rightarrow\boxed{\Lambda=\frac{2}{Eu^{'2}}(Eu^{'}\lambda^{'}-1)}$$

(Design Equation)

STEP 3: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$

$$\Rightarrow 0 - \left(\Lambda E A u' + \lambda E A'\right)' + \left(\lambda E A\right)'' = 0$$

(Using governing eqution in between)

$$\Rightarrow \boxed{\left(EA\frac{\lambda'}{\Lambda}\right)' + P = 0} \quad \leftarrow \quad \text{(Adjoint Equation)}$$

STEP 4: Collecting all Equations

$$\Lambda = \frac{2}{Eu'^2}(Eu'\lambda' - 1)$$ Design Equation-------(1)

$$\left(EA\frac{\lambda'}{\Lambda}\right)' + P = 0$$ Adjoint Equation------(2)

$$\left(EAu'\right)' + P = 0$$ Governing Equation---(3)

$$\Lambda\left[\int\left(\frac{1}{2}EAu'^2 - \frac{SE^*}{L}\right)dx\right] = 0$$ Complementarity Criteria--(4)

$$\int \frac{1}{2}EAu'^2\,dx - SE^* \le 0$$ Feasibility Criteria-----(5)

STEP 5: OPTIMALITY CRITERIA

Comparing eq (2) & (3), we get

$\lambda = \Lambda u$

Equation (1) becomes

$$\boxed{\Lambda = \frac{2}{Eu'^2}} \leftarrow \text{Optimality Criteria} \qquad ----(6)$$

Clearly, $\Lambda \geq 0$

$\Rightarrow$ Strain Energy constraint is active

$$\therefore \int \frac{1}{2} EAu'^2 \, dx - SE^* = 0 \qquad ----(7)$$

From (6), we get

$$u' = \pm\sqrt{\frac{2}{E\Lambda}} \qquad ----(8)$$

STEP 6: BOUNDARY CONDITIONS

$$\hat{L} = A + \lambda\left\{\left(EAu'\right)' + P\right\} + \Lambda\left(\frac{1}{2}EAu'^2 - \frac{SE^*}{L}\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow \left.\left(\lambda E u'\right)\delta A\right|_0^L = 0 \Rightarrow \boxed{\left(uu'\right)\delta A_0^L = 0} \quad \text{(BCI)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left.\left[\left(\Lambda EAu' + \lambda EA'\right) - \left(\lambda EA\right)'\right]\delta u\right|_0^L = 0 \Rightarrow \boxed{\left.0\delta u\right|_0^L = 0} \text{ (Automatically Satisfied)} \quad \text{(BCII)}$$

$$\left.\left(\frac{\partial \hat{L}}{\partial u''}\right)\delta u'\right|_0^L = 0 \Rightarrow \left.(\lambda EA)\delta u'\right|_0^L \Rightarrow \boxed{\left.(Au)\delta u'\right|_0^L = 0} \quad \text{(BCIII)}$$

STEP 7: SOLVING FOR DESIGN VARIBLE

From (2) & (8), we get

$$P + \left(EA\left(\pm\sqrt{\frac{2}{E\Lambda}} \right) \right)' = 0$$

$$\Rightarrow \boxed{A(x) = \pm\int \frac{P}{E}\sqrt{\frac{\Lambda}{2}}dx + C} \qquad \text{-----}(9)$$

From (7) & (9), we get Λ as

$$\int_0^L \left\{ \int \frac{1}{2}E\left(\pm\int \frac{P}{E}\sqrt{\frac{\Lambda}{2}}dx + C \right)dx + C \right\} = SE^* \qquad \text{-----}(10)$$

CONCLUSION

For minimum Volume,

$$A(x) = \pm\int \frac{P}{E}\sqrt{\frac{\Lambda}{2}}dx + C$$

where Λ & C are given by eq (10) & BCs respectively

PROBLEM 3

Minimize volume of material to be used with an upper bound constraint on mean compliance under given loading.

$$\underset{A(x)}{Min}\ \mathrm{V} = \int A dx$$

subjected to

$$\lambda(x) : \left(EAu'\right)' + P = 0$$

$$\Lambda : \int P u dx - MC^{*} \leq 0$$

$$Data : L, P(x), E, MC^{*}$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[A + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(Pu - \frac{MC^*}{L} \right) \right] dx$$

$$\hat{L} = A + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(Pu - \frac{MC^*}{L} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0 \qquad \text{(Euler Lagrange Equation)}$$

$$\Rightarrow \left(1 + \lambda E u'' \right) - \left(\lambda E u' \right)' = 0$$

$$\Rightarrow \boxed{E u' \lambda' = 1} \qquad \text{(Design Equation)}$$

STEP 3: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$

$$\Rightarrow \Lambda P - \left(\lambda E A'\right)' + \left(\lambda E A\right)'' = 0$$

(Using governing eqution in between)

$$\Rightarrow \boxed{\left(EA\frac{\lambda'}{\Lambda}\right)' + P = 0} \quad \leftarrow \quad \text{(Adjoint Equation)}$$

STEP 4: COLLECTING ALL EQUATION

$Eu'\lambda' = 1$ Design Equation-------(1)

$\left(EA\frac{\lambda'}{\Lambda}\right)' + P = 0$ Adjoint Equation-----(2)

$\left(EAu'\right)' + P = 0$ Governing Equation---(3)

$\Lambda\left[\int\left(Pu - \frac{MC^*}{L}\right)dx\right] = 0$ Complementarity Criteria--(4)

$\int_0^L Pudx - SE^* \le 0$ Feasibility Criteria-----(5)

STEP 5: OPTIMALITY CRITERIA

Comparing eq (2) & (3), we get

$\lambda = \Lambda u$

Equation (1) becomes

$$\boxed{\Lambda = \frac{2}{Eu'^2}} \leftarrow \text{Optimality Criteria} \qquad \text{----}(6)$$

Clearly, $\Lambda \geq 0$

$\Rightarrow$ Mean Compliance constraint is active

$$\therefore \int Pudx - SE^* = 0 \qquad \text{----}(7)$$

From (6), we get

$$u' = \pm\sqrt{\frac{2}{E\Lambda}} \quad \Rightarrow \boxed{u = \int \pm\sqrt{\frac{2}{E\Lambda}}dx + C_1} \qquad \text{----}(8)$$

STEP 6: BOUNDARY CONDITIONS

$$\hat{L} = A + \lambda\left\{\left(EAu'\right)' + P\right\} + \Lambda\left(Pu - \frac{MC^*}{L}\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow \left.\left(\lambda E u'\right)\delta A\right|_0^L = 0 \Rightarrow \boxed{\left(uu'\right)\delta A_0^L = 0} \quad \text{(BCI)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left.\left[\left(\lambda E A'\right) - \left(\lambda E A\right)'\right]\delta u\right|_0^L = 0 \Rightarrow \boxed{\left.\left(Au'\right)\delta u\right|_0^L = 0} \quad \text{(BCII)}$$

$$\left.\left(\frac{\partial \hat{L}}{\partial u''}\right)\delta u'\right|_0^L = 0 \Rightarrow \left.(\lambda E A)\delta u'\right|_0^L \Rightarrow \boxed{\left.(Au)\delta u'\right|_0^L = 0} \quad \text{(BCIII)}$$

STEP 7: SOLVING FOR DESIGN VARIABLE

From (2) & (8), we get

$$P+\left(EA\left(\pm\sqrt{\frac{1}{E\Lambda}}\right)\right)'=0$$

$$\Rightarrow \boxed{A(x)=\pm\int P\sqrt{\frac{\Lambda}{E}}dx+C} \qquad \text{------}(9)$$

From (7) & (8), we get Λ as

$$\int_0^L\left\{P\left(\int\pm\sqrt{\frac{2}{E\Lambda}}dx+C_1\right)dx\right\}=MC^* \qquad \text{-----}(10)$$

CONCLUSION

For minimum Volume,

$$A(x)=\pm\int\frac{P}{E}\sqrt{\frac{\Lambda}{2}}dx+C$$

where Λ & C are given by eq (10) & BCs respectively

PROBLEM 4

Minimize the strain energy, expressed in terms of internal forces in a bar, subject to the volume constraint.

$$\underset{A(x)}{Min}\ \mathrm{SE} = \int \frac{P^2}{2AE} dx$$

subjected to

$$\Lambda : \int A dx - V^* \leq 0$$

$$Data : L, P(x), E, V^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[\frac{P^2}{2AE} + \Lambda \left(A - \frac{V^*}{L} \right) \right] dx$$

$$\hat{L} = \frac{P^2}{2AE} + \Lambda \left(A - \frac{V^*}{L} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x) & SOLUTION

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0 \qquad \text{(Euler Lagrange Equation)}$$

$$\Rightarrow \boxed{\Lambda = \frac{P^2}{2EA^2}} \text{ or } A(x) = \frac{P}{\sqrt{2E\Lambda}} \qquad \text{(Design Equation)}$$

Clearly $\Lambda \geq 0 \Rightarrow$ Volume constraint is active

$$\Rightarrow \int A dx - V^* = 0$$

$$\Rightarrow \int \sqrt{\frac{P}{2E\Lambda}} dx = V^* \qquad \leftarrow \text{This eq give } \Lambda$$

PROBLEM 5

A min-max formulation for the stiffest bar for given volume of material

$$\underset{u(x)}{Max}\ \underset{A(x)}{\text{Min}}\ \text{PE} = \int \left(\frac{1}{2} EAu'^2 - Pu \right) dx$$

subjected to

$$\Lambda : \int A dx - V^* \leq 0$$

$$Data : L, P(x), E, V^*$$

Problem can be restated as

$$\underset{A(x)}{Min}\ (-\beta)$$

subjected to

$$\Gamma := \beta - \int_0^L \left(\frac{1}{2} EAu'^2 - Pu \right) dx$$

$$\Lambda : \int A dx - V^* \leq 0$$

$$Data : L, P(x), E, V^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[-\beta + \Lambda \left(\frac{\beta}{L} - \frac{1}{2} E A u'^2 + P u \right) + \Lambda \left(A - \frac{V^*}{L} \right) \right] dx$$

$$\hat{L} = -\beta + \Lambda \left(\frac{\beta}{L} - \frac{1}{2} E A u'^2 + P u \right) + \Lambda \left(A - \frac{V^*}{L} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0 \qquad \text{(Euler Lagrange Equation)}$$

$$\Rightarrow \boxed{\Lambda = \frac{\Gamma E u'^2}{2}} \qquad \text{(Design Equation)}$$

STEP 3a: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$

$$\Rightarrow \Gamma P - \Gamma\left(-EAu'\right)' = 0$$

$$\Rightarrow P + \left(EAu'\right)' = 0 \qquad \leftarrow \quad \text{Adjoint Eqution}$$

STEP 3b: VARIATION W.R.T STATE VARIABLE β(x)

$$\delta_\beta \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial \beta} - \left(\frac{\partial \hat{L}}{\partial \beta'}\right)' + \left(\frac{\partial \hat{L}}{\partial \beta''}\right)'' = 0$$

$$\Rightarrow \Gamma = L \qquad \leftarrow \quad \text{Adjoint Eqution}$$

STEP 4: COLLECTING ALL EQUATION

$$\Lambda = \frac{\Gamma E u'^2}{2}$$ Design Equation-------(1)

$$\Gamma = L$$ Adjoint Equation (w.r.t β)-----(2)

$$P + \left(EAu'\right)' = 0$$ Adjoint Equation (w.r.t u) ---(3)

$$\Gamma\left[\int\left(\frac{\beta}{L} - \frac{1}{2}EAu'^2 + Pu\right)dx\right] = 0$$ Complementarity Criteria (w.r.t β) ---(4a)

$$\Lambda\left[\int_0^L Adx - V^*\right] = 0$$ Complementarity Criteria (w.r.t volume) -(4b)

$$\beta - \int_0^L\left(\frac{1}{2}EAu'^2 - Pu\right)dx \le 0$$ Feasibility Criteria-----(5a)

$$\int Adx - V^* \le 0$$ Feasibility Criteria-----(5b)

STEP 5: OPTIMALITY CRITERIA

Comparing eq (1) & (2), we get

$$\boxed{\Gamma = L}$$

$$\boxed{\Lambda = \frac{Eu'^2 L}{2}} \leftarrow \text{Optimality Criteria} \qquad \text{----}(6)$$

Clearly, $\Gamma, \Lambda \geq 0$

$\Rightarrow$ Both constraints are active

$$\therefore \; \beta = \int \left(\frac{1}{2} EAu'^2 + Pu \right) dx = 0$$

$$\& \int A dx - V^* = 0 \qquad \text{----}(7)$$

From (6), we get

$$u' = \pm \sqrt{\frac{2\Lambda}{EL}} \Rightarrow \boxed{u = \int \pm \sqrt{\frac{2\Lambda}{EL}} dx + C_1} \qquad \text{----}(8)$$

STEP 6: BOUNDARY CONDITIONS

$$\hat{L} = -\beta + \Lambda\left(\frac{\beta}{L} - \frac{1}{2}EAu'^2 + Pu\right) + \Lambda\left(A - \frac{V^*}{L}\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow (0)\,\delta A\big|_0^L = 0 \text{ (Automatically Satisfied)} \quad \text{(BCI)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left(\Gamma EAu'\right)\delta u\big|_0^L = 0 \Rightarrow \boxed{\left(Au'\right)\delta u\big|_0^L = 0} \quad \text{(BCII)}$$

STEP 7: SOLVING FOR DESIGN VARIABLE

From (2) & (8), we get

$$P+\left(EA\left(\pm\sqrt{\frac{2\Lambda}{EL}}\right)\right)'=0$$

$$\Rightarrow \boxed{A(x)=\pm\sqrt{\frac{L}{2\Lambda E}}\int Pdx+C} \qquad \text{-----}(9)$$

From (7) & (8), we get Λ as

$$\int_0^L\left(\pm\sqrt{\frac{L}{2\Lambda E}}\int Pdx+C\right)dx=V^* \qquad \text{-----}(10)$$

CONCLUTION

For minimum Volume,

$$A(x)=\pm\sqrt{\frac{L}{2\Lambda E}}\int Pdx+C$$

where Λ & C are given by eq (10) & BCs respectively

PROBLEM 6

Minimize the mean compliance for given volume of material with the governing equation in the weak form.

$$\underset{A(x)}{Min}\ \mathrm{MC} = \int Pu\,dx$$

subjected to

$$\Gamma : \int_0^L \left(EAu'v' - Pv \right) dx = 0$$

$$\Lambda : \int A\,dx - V^* \leq 0$$

$$Data : L, P(x), E, V^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[Pu + \Gamma\left(EAu'v' - Pv\right) + \Lambda\left(A - \frac{V^*}{L}\right) \right] dx$$

$$\hat{L} = Pu + \Gamma\left(EAu'v' - Pv\right) + \Lambda\left(A - \frac{V^*}{L}\right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'}\right)' = 0$$ Euler –Lagrange Equation

$$\Rightarrow \left(\Gamma E u' v'\right) + \Lambda = 0$$

$$\Rightarrow \boxed{\Lambda = -Eu'\lambda'}$$ (where $\lambda = \Gamma v$) Design Equation

STEP 3a: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$ Euler Lagrange Equation

$$\Rightarrow P - \left(\Gamma E A v'\right)' = 0$$

$$\Rightarrow \boxed{P + \left\{EA(-\lambda)'\right\} = 0}$$ Adjoint Equation

STEP 3b: VARIATION W.R.T v

$$\delta_v \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial v} - \left(\frac{\partial \hat{L}}{\partial v'}\right)' + \left(\frac{\partial \hat{L}}{\partial v''}\right)'' = 0$$ Euler Lagrange Equation

$$\Rightarrow P + \left(EAu'\right)' = 0$$ (Governing Equation)

STEP 4: COLLECTING ALL EQUATIONS

$$\Lambda = -Eu'\lambda'$$

Design Equation-------(1)

$$\{EA(-\lambda')\}' + P = 0$$

Adjoint Equation-----(2)

$$(EAu')' + P = 0$$

Governing Equation---(3)

$$\Lambda\left[\int\left(A - \frac{V^*}{L}\right)dx\right] = 0$$

Complementarity Criteria--(4)

$$\int Adx - V^* \leq 0$$

Feasibility Criteria-----(5)

STEP 5: OPTIMALITY CRITERIA

Comparing eq (2) & (3), we get

$\lambda = -u$

Equation (1) becomes

$\boxed{\Lambda = Eu'^2}$ $\leftarrow$ Optimality Criteria ----(6)

Clearly, $\Lambda \geq 0$

$\Rightarrow$ Volume constraint is active

$\therefore \int A dx - V^* = 0$ ----(7)

From (6), we get

$u' = \pm\sqrt{\frac{\Lambda}{E}}$ ---(8)

STEP 6: BOUNDARY CONDITIONS

$$\hat{L} = Pu + \Gamma\left(EAu'v' - Pv\right) + \Lambda\left(A - \frac{V^*}{L}\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow (0)\,\delta A\Big|_0^L = 0 \Rightarrow \text{Automatically satisfied (BCI)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'}\right\}\delta u\right|_0^L = 0 \Rightarrow \left[\left(\Gamma EAv'\right)'\right]\delta u\Big|_0^L = 0 \Rightarrow \boxed{\left(Av'\right)\delta u\Big|_0^L = 0} \text{ (Automatically Satisfied) (BCII)}$$

STEP 7: SOLVING FOR DESIGN VARIABLE

From (3) & (8), we get

$$P+\left(EA\left(\pm\sqrt{\frac{\Lambda}{E}}\right)\right)'=0$$

$$\Rightarrow \boxed{A(x)=\pm\int\frac{P}{\sqrt{E\Lambda}}dx+C} \qquad \text{-----}(9)$$

From (7) & (9), we get Λ as

$$\int_0^L\left\{\pm\int\frac{P}{\sqrt{E\Lambda}}dx+C\right\}=V^* \qquad \text{-----}(10)$$

CONCLUSION

For minimum SE,

$$A(x)=\pm\int\frac{P}{\sqrt{E\Lambda}}dx+C$$

where Λ & C are given by eq (10) & BCs respectively

PROBLEM 7

Minimize the volume of a statically determinate bar with a deflection constraint at a point somewhere on its axis. Internal force, P, due to some applied load, and that due to a unit virtual load at the point of interest are given.

$$\underset{A(x)}{Min}\ \mathrm{V} = \int A dx$$

subjected to

$$\Lambda : \int \frac{P P_d}{AE} dx - \Delta^* = 0$$

$$Data : L, P(x), P_d(x), E, \Delta^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[A + \Lambda \left(\frac{PP_d}{AE} - \frac{\Delta^*}{L} \right) \right] dx$$

$$\hat{L} = A + \Lambda \left(\frac{PP_d}{AE} - \frac{\Delta^*}{L} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0$$

Euler –Lagrange Equation

$$\Rightarrow 1 - \left(\Lambda \frac{PP_d}{A^2} \right) = 0$$

$$\Rightarrow \boxed{\Lambda = \frac{A^2 E}{PP_d}} \quad \text{or} \quad \boxed{A = \sqrt{\frac{\Lambda PP_d}{E}}}$$

Design Equation

STEP 4: COLLECTING ALL EQUATIONS

$$\Lambda = \frac{A^2 E}{PP_d}$$ Design Equation

$$\int \left(\frac{PP_d}{AE} - \frac{\Delta^*}{L} \right) dx = 0$$ Equality Constraint

STEP 5: SOLVING FOR DESIGN VARIABLE

$$\frac{PP_d}{AE} = \frac{A}{\Lambda}$$

$\therefore Eq(2)$ become

$$\Lambda = \frac{1}{\Delta^*} \int_0^L A dx$$

Numerical Steps

a) Choose Λ

b)Calculate $A = \sqrt{\frac{\Lambda PP_d}{E}}$ from optimality criteria

c)$\Lambda_{new} = \frac{1}{\Delta^*} \int A dx$

d) Repeat steps a) to c)

PROBLEM 8

Minimize the volume of a bar (indeterminate or determinate) for a deflection constraint at a point.

$$\underset{A(x)}{Min}\ \mathrm{V} = \int A dx$$

subjected to

$$\lambda(x) : \left(EAu'\right)' + P = 0$$

$$\lambda_d(x) : \left(EAu_d'\right)' + P_d = 0$$

$$\Lambda : \int EAu' u_d' dx - \Delta^* = 0$$

$$Data : L, P(x), P_d(x), E, \Delta^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[A + \lambda \left\{ \left(EAu' \right)' + P \right\} + \lambda_d \left\{ \left(EAu_d' \right)' + P_d \right\} + \Lambda \left(EAu'u_d' - \frac{\Delta^*}{L} \right) \right] dx$$

$$\hat{L} = A + \lambda \left\{ \left(EAu' \right)' + P \right\} + \lambda_d \left\{ \left(EAu_d' \right)' + P_d \right\} + \Lambda \left(EAu'u_d' - \frac{\Delta^*}{L} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0$$

Euler –Lagrange Equation

$$\Rightarrow \left(1 + \lambda E u'' + \lambda_d E u_d'' + \Lambda E u' u_d' \right) - \left(\lambda E u' + \lambda_d E u_d' \right)' = 0$$

$$\Rightarrow \boxed{\Lambda E u' u_d' = \lambda' E u' + \lambda_d' E u_d' - 1}$$

Design Equation

STEP 3a: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$

$$\Rightarrow 0 - \left(\Lambda EAu_d' + \lambda EA'\right)' + \left(\lambda EA\right)'' = 0$$

$$\Rightarrow \left(\lambda' EA - \Lambda EAu_d'\right)' = 0 \quad \leftarrow \quad (\text{Adjoint Equation w.r.t } u(x))$$

STEP 3b: VARIATION W.R.T u_d

$$\delta_{u_d} \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u_d} - \left(\frac{\partial \hat{L}}{\partial u_d'}\right)' + \left(\frac{\partial \hat{L}}{\partial u_d''}\right)'' = 0$$

$$\Rightarrow 0 - \left(\Lambda EAu' + \lambda_d EA'\right)' + \left(\lambda_d EA\right)'' = 0$$

$$\Rightarrow \left(\lambda_d' EA' - \Lambda EAu'\right)' = 0 \quad \leftarrow \quad (\text{Adjoint Equation w.r.t } u(x)$$

STEP 4: COLLECTING ALL EQUATIONS

$\Lambda E u' u_d' = \lambda' E u' + \lambda_d' E u_d' -1$ Design Equation-------(1)

$\left(\lambda' EA - \Lambda EA u_d'\right)' = 0$ Adjoint Equation------(2a)

$\left(\lambda_d' EA' - \Lambda EA u'\right)' = 0$ Adjoint Equation------(2b)

$\left(EAu'\right)' + P = 0$ Governing Equation---(3a)

$\left(EAu_d'\right)' + P_d = 0$ Governing Equation---(3a)

$\int \left(EAu' u_d' - \frac{\Delta^*}{L} \right) dx = 0$ Equality Constraint---(4)

STEP 5: OPTIMALITY CRITERIA

Comparing eq $(2a)$ & $(3b)$, we get

$$\left(\frac{\lambda' EA}{\Lambda}\right)' = \left(EAu_d'\right)' \Rightarrow \left(\frac{\lambda' EA}{\Lambda}\right)' + P_d = 0$$

$$\Rightarrow \boxed{\lambda = \Lambda u_d}$$

Similiarly,

Comparing eq $(2b)$ & $(3a)$, we get

$$\boxed{\lambda_d = \Lambda u}$$

Putting $\lambda = \Lambda u_d$ & $\lambda_d = \Lambda u$ in equation (1), we get

$$\boxed{\Lambda E u' u_d' = 1} \leftarrow \text{Optimality Criteria} \qquad ----(6)$$

From (6) & (4), we get

$$E u' u_d' = \frac{1}{\Lambda} \Rightarrow \Lambda = \frac{1}{\Delta^*} \int_0^L A dx \qquad ----(8)$$

STEP 6: BOUNDARY CONDITIONS

$$\hat{L} = A + \lambda\left\{\left(EAu'\right)' + P\right\} + \lambda_d\left\{\left(EAu_d'\right)' + P_d\right\} + \Lambda\left(EAu'u_d' - \frac{\Delta^*}{L}\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow \boxed{\left.\left(uu_d' + u_d u'\right)\delta A\right|_0^L} = 0 \qquad \text{(BCI)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left.\left[\left(\lambda EA' + \Lambda EAu_d'\right)' - \left(\lambda EA\right)'\right]\delta u\right|_0^L = 0 \Rightarrow \boxed{\left.(0)\,\delta u\right|_0^L = 0} \text{ (Automatically Satisfied)} \quad \text{(BCII)}$$

$$\left.\frac{\partial \hat{L}}{\partial u''}\delta u'\right|_0^L = 0 \Rightarrow \boxed{\left.\left(Au_d\right)'\delta u'\right|_0^L} = 0 \qquad \text{(BCIII)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u_d'} - \left(\frac{\partial \hat{L}}{\partial u_d''}\right)'\right\}\delta u_d\right|_0^L = 0 \Rightarrow \boxed{\left.(0)\,\delta u_d\right|_0^L = 0} \text{ (Automatically Satisfied similiar to previous condition)} \quad \text{(BCIV)}$$

$$\left.\frac{\partial \hat{L}}{\partial u_d''}\delta u'\right|_0^L = 0 \Rightarrow \boxed{\left.\left(Au\right)'\delta u_d'\right|_0^L = 0} \qquad \text{(BCV)}$$

STEP 7: SOLVING FOR DESIGN VARIABLE

Numerical Steps

a) Choose initial Area profile $\mathrm{A}(x)$

b) Calculate $\Lambda = \frac{1}{\Delta^*}\int_0^L A dx$ from equation (8)

c)Calculate $\mathrm{u}(x)$ & $\mathrm{u}_d(x)$ using FEM

d)$\mathrm{A}_{k+1} = \left(\Lambda \mathrm{E} \mathrm{A} \mathrm{u}' \mathrm{u}_d'\right) \mathrm{A}_k$

d) Repeat steps b) to d)

PROBLEM 9

Minimize the volume of a bar (indeterminate or determinate) for a deflection constraint at a point and strain energy constraint.

$$\underset{A(x)}{Min}\ \mathrm{V} = \int A dx$$

subjected to

$$\lambda(x) : \left(EAu'\right)' + P = 0$$

$$\lambda_d(x) : \left(EAu_d'\right)' + P_d = 0$$

$$\Lambda : \int EAu'u_d' dx - \Delta^* = 0$$

$$\Gamma : \int_0^L \frac{1}{2} EAu'^2 dx - SE^* = 0$$

$$Data : L, P(x), P_d\left(x\right), E, \Delta^*, SE^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[A + \lambda \left\{ \left(EAu' \right)' + P \right\} + \lambda_d \left\{ \left(EAu_d' \right)' + P_d \right\} + \Lambda \left(EAu' u_d' - \frac{\Delta^*}{L} \right) + \Gamma \left(\frac{1}{2} EAu'^2 - \frac{SE^*}{L} \right) \right] dx$$

$$\hat{L} = A + \lambda \left\{ \left(EAu' \right)' + P \right\} + \lambda_d \left\{ \left(EAu_d' \right)' + P_d \right\} + \Lambda \left(EAu' u_d' - \frac{\Delta^*}{L} \right) + \Gamma \left(\frac{1}{2} EAu'^2 - \frac{SE^*}{L} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0$$

Euler –Lagrange Equation

$$\Rightarrow \left(1 + \lambda E u'' + \lambda_d E u_d'' + \Lambda E u' u_d' + \Gamma \frac{E u'}{2} \right) - \left(\lambda E u' + \lambda_d E u_d' \right)' = 0$$

$$\Rightarrow \boxed{\Lambda E u' u_d' + \frac{\Gamma E u'^2}{2} = \lambda' E u' + \lambda_d' E u_d' \; -1}$$

Design Equation

STEP 3a: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'} \right)' + \left(\frac{\partial \hat{L}}{\partial u''} \right)'' = 0$$

$$\Rightarrow 0 - \left(\Lambda E A u_d' + \lambda E A' + \Gamma E A u' \right)' + \left(\lambda E A \right)'' = 0$$

$$\Rightarrow \quad \left(\Lambda E A u_d' + \Gamma E A u' \right)' = \left(\lambda' \mathrm{EA} \right)' \quad \leftarrow \quad \text{Adjoint Equation w.r.t u}(x)$$

STEP 3b: VARIATION W.R.T u_d

$$\delta_{u_d} \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u_d} - \left(\frac{\partial \hat{L}}{\partial u_d'} \right)' + \left(\frac{\partial \hat{L}}{\partial u_d''} \right)'' = 0$$

$$\Rightarrow 0 - \left(\Lambda E A u' + \lambda_d E A' \right)' + \left(\lambda_d E A \right)'' = 0$$

$$\Rightarrow \quad \left(\Lambda E A u' \right)' = \left(\lambda_d' E A' \right)' \quad \leftarrow \quad \text{Adjoint Equation w.r.t } \mathrm{u}_d (x)$$

STEP 4: COLLECTING ALL EQUATIONS

$$\Lambda E u' u_d' + \frac{\Gamma E u'^2}{2} = \lambda' E u' + \lambda_d' E u_d' \ -1$$ Design Equation------(1)

$$\left(\lambda' E A\right)' = \left(\Lambda E A u_d' + \Gamma E A u'\right)'$$ Adjoint Equation-----(2a)

$$\left(\lambda_d' E A'\right)' = \left(\Lambda E A u'\right)'$$ Adjoint Equation-----(2b)

$$\left(E A u'\right)' + P = 0$$ Governing Equation---(3a)

$$\left(E A u_d'\right)' + P_d = 0$$ Governing Equation---(3a)

$$\int \left(E A u' u_d' - \frac{\Delta^*}{L} \right) dx = 0$$ Equality Constraint---(4a)

$$\int \left(\frac{1}{2} E A u'^2 - \frac{SE^*}{L} \right) dx = 0$$ Equality Constraint---(4b)

STEP 5: OPTIMALITY CRITERIA

Comparing eq $(2b)$ & $(3a)$, we get

$$\left(\frac{\lambda_d' EA}{\Lambda}\right)' = \left(EAu_d'\right)' \Rightarrow \left(\frac{\lambda_d' EA}{\Lambda}\right)' + P_d = 0$$

$$\Rightarrow \boxed{\lambda_d = \Lambda u}$$

Similiarly,

Comparing eq $(2a)$ & $(3b)$, we get

$$\left(\frac{\lambda' EA}{\Lambda}\right)' = \left(EAu_d'\right)' + \left(\frac{\Gamma EAu'}{\Lambda}\right)'$$

$$\Rightarrow \left[\left(\frac{\lambda' - \Gamma u'}{\Lambda}\right)EA\right]' = -P_d \Rightarrow \left[\left(\frac{\lambda' - \Gamma u'}{\Lambda}\right)EA\right]' + P_d$$

$$\Rightarrow \boxed{\lambda = \Lambda u_d + \Gamma u}$$

Putting $\lambda = \left(\Lambda u_d + \Gamma u\right)$ & $\lambda_d = \Lambda u$ in equation (1), we get

$$\boxed{\Lambda E u' u_d' + \frac{\Gamma E u'^2}{2} = 1} \leftarrow \text{Optimality Criteria} \qquad ----(6)$$

Multiply b/s by A & integrate, we get

$$\int_0^L \Lambda E u' u_d' dx + \int_0^1 \frac{\Gamma E u'^2}{2} = \int_0^L A dx$$

$$\Rightarrow \boxed{\Gamma SE^* + \Lambda \Delta^* = \int_0^L A dx} \qquad ----(7)$$

STEP 6: BOUNDARY CONDITIONS

$$\hat{L} = A + \lambda\left\{\left(EAu'\right)' + P\right\} + \lambda_d\left\{\left(EAu_d'\right)' + P_d\right\} + \Lambda\left(EAu'u_d' - \frac{\Delta^*}{L}\right) + \Gamma\left(\frac{1}{2}EAu'^2 - \frac{SE^*}{L}\right)$$

$$\left.\frac{\partial\hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow \boxed{\left.\left(\Lambda Eu'u_d + \Lambda Euu_d' + \Gamma Euu'\right)\delta A\right|_0^L} = 0 \qquad \text{(BCI)}$$

$$\left.\left\{\frac{\partial\hat{L}}{\partial u'} - \left(\frac{\partial\hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left.\left[\left(\lambda EA' + \Lambda EAu_d' + \Gamma EAu'\right)' - \left(\lambda EA\right)'\right]\delta u\right|_0^L = 0 \Rightarrow \boxed{\left.(0)\,\delta u\right|_0^L = 0} \text{ (Automatically Satisfied)} \qquad \text{(BCII)}$$

$$\left.\frac{\partial\hat{L}}{\partial u''}\delta u'\right|_0^L = 0 \Rightarrow \boxed{\left.\left(\Lambda EAu_d + \Gamma EAu\right)'\delta u'\right|_0^L} = 0 \qquad \text{(BCIII)}$$

$$\left.\left\{\frac{\partial\hat{L}}{\partial u_d'} - \left(\frac{\partial\hat{L}}{\partial u_d''}\right)'\right\}\delta u_d\right|_0^L = 0 \Rightarrow \boxed{\left.(0)\,\delta u_d\right|_0^L = 0} \text{ (Automatically Satisfied similiar to previous condition)} \qquad \text{(BCIV)}$$

$$\left.\frac{\partial\hat{L}}{\partial u_d''}\delta u'\right|_0^L = 0 \Rightarrow \boxed{\left.(Au)\,\delta u_d'\right|_0^L = 0} \qquad \text{(BCV)}$$

PROBLEM 10

Minimize the mean compliance of a bar for given volume and upper and lower bounds on the displacement

$$\underset{A(x)}{Min}\ \mathrm{MC} = \int pudx$$

subjected to

$$\lambda(x) : \left(EAu'\right)' + P = 0$$

$$\Lambda : \int_0^L Adx' - V^* = 0$$

$$\mu_u\left(x\right) : u - u_u \leq 0$$

$$Data : L, P(x), E, V^*, u_u$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[pu + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(A - \frac{V^*}{L} \right) + \mu_u \left(u - u_u \right) \right] dx$$

$$\hat{L} = pu + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(A - \frac{V^*}{L} \right) + \mu_u \left(u - u_u \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0$$

Euler –Lagrange Equation

$$\Rightarrow \left(\lambda E u'' + \Lambda \right) - \left(\lambda E u' \right)' = 0$$

$$\Rightarrow \boxed{\Lambda = \lambda' E u'}$$

Design Equation

STEP 3: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$

$$\Rightarrow (P + \mu_u) - (\lambda E A')' + (\lambda E A)'' = 0$$

$$\Rightarrow \text{P}+(\text{EA}\lambda')+\mu_u=0 \quad \leftarrow \quad \text{Adjoint Equation w.r.t } \text{u}(x)$$

STEP 4: COLLECTING ALL EQUATIONS

$\Lambda = Eu'\lambda'$ Design Equation-------(1)

$P + \{EA\lambda'\}' + \mu_u = 0$ Adjoint Equation-----(2)

$P + (EAu')' = 0$ Governing Equation---(3)

$\Lambda\left[\int\left(A - \frac{V^*}{L}\right)dx\right] = 0;\ \Lambda \geq 0$ Complementarity Criteria--(4a)

$\mu_u(u - u_u) = 0;\ \mu_u \geq 0$ Complementarity Criteria--(4a)

$\int Adx - V^* \leq 0$ Feasibility Criteria-----(5)

CASE a): $u<u_u$

$\mu_u = 0$

$\Rightarrow$ Comparing Adjoint & Governing Eq, we get

$\boxed{\lambda = u}$

Eq 1 become

$\boxed{\Lambda = Eu'^2}$ $\leftarrow$ optimality criteria

$\Rightarrow u' = \pm\sqrt{\frac{\Lambda}{E}}$

BOUNDARY CONDITIONS

$$\hat{L} = pu + \lambda\left\{\left(EAu'\right)' + P\right\} + \Lambda\left(A - \frac{V^*}{L}\right) + \mu_u\left(u - u_u\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow \left(\lambda E u'\right)\delta A\Big|_0^L = 0 \Rightarrow \boxed{\left(uu'\right)\delta A_0^L = 0} \quad \text{(BCI)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left[\left(\lambda E A'\right) - \left(\lambda E A\right)'\right]\delta u\Big|_0^L = 0 \Rightarrow \boxed{\left(Au'\right)\delta u\Big|_0^L = 0} \quad \text{(BCII)}$$

$$\left.\left(\frac{\partial \hat{L}}{\partial u''}\right)\delta u'\right|_0^L = 0 \Rightarrow (\lambda E A)\delta u'\Big|_0^L \Rightarrow \boxed{(Au)\delta u'\Big|_0^L = 0} \quad \text{(BCIII)}$$

CASE b: $u=u_u$

Consider when volume constraint is inactive

$\Rightarrow \Lambda=0$

$\therefore \lambda' u' = 0$

$\Rightarrow \lambda' = 0$ or $u' = 0$

If $u' = 0 \Rightarrow P = 0$ (not possible everywhere)

$\therefore \lambda' = 0$

$\Rightarrow \boxed{\mu_u = -P}$ (from eq 2)

$\therefore$ Min MC possible only if $P<0$

If $\mu_u = -P = 0$ then $u<u_u \Rightarrow$ given by case a

If volume constraint is active then only solution is when $u=u_u$ occur at end.

PROBLEM 11

Minimize the mean compliance of a bar for given volume with stress constraints.

$$\underset{A(x)}{Min}\ \mathrm{MC} = \int pudx$$

subjected to

$$\lambda(x) : \left(EAu'\right)' + P = 0$$

$$\Lambda : \int_0^L Adx - V^* = 0$$

$$\mu_t(x) : Eu' - \sigma_t \le 0$$

$$\mu_c(x) : \sigma_t - Eu' \le 0$$

$$Data : L, P(x), E, V^*, \sigma_t, \sigma_c$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[pu + \lambda \left\{ \left(EAu^{'} \right)^{'} + P \right\} + \Lambda \left(A - \frac{V^{*}}{L} \right) + \mu_{t} \left(Eu^{'} - \sigma_{t} \right) + \mu_{c} \left(\sigma_{c} - Eu^{'} \right) \right] dx$$

$$\hat{L} = pu + \lambda \left\{ \left(EAu^{'} \right)^{'} + P \right\} + \Lambda \left(A - \frac{V^{*}}{L} \right) + \mu_{t} \left(Eu^{'} - \sigma_{t} \right) + \mu_{c} \left(\sigma_{c} - Eu^{'} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_{A} \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A^{'}} \right)^{'} = 0$$ Euler –Lagrange Equation

$$\Rightarrow \left(\lambda E u^{''} + \Lambda \right) - \left(\lambda E u^{'} \right)^{'} = 0$$

$$\Rightarrow \boxed{\Lambda = \lambda^{'} E u^{'}}$$ Design Equation

STEP 3: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'} \right)' + \left(\frac{\partial \hat{L}}{\partial u''} \right)'' = 0$$

$$\Rightarrow P - \left(\lambda E A' + \mu_t E - \mu_c E \right)' + \left(\lambda E A \right)'' = 0$$

$$\Rightarrow \text{P} + \left(\text{EA}\lambda' \right) + \left(\mu_c - \mu_t \right) \text{E} = 0 \quad \leftarrow \quad \text{Adjoint Equation w.r.t } \text{u}\left(x \right)$$

STEP 4: COLLECTING ALL EQUATIONS

$\Lambda = Eu'\lambda'$ Design Equation------(1)

$P + \{EA\lambda'\}' + (\mu_c - \mu_t)E = 0$ Adjoint Equation-----(2)

$P + (EAu')' = 0$ Governing Equation---(3)

$\Lambda\left[\int\left(A - \frac{V^*}{L}\right)dx\right] = 0;\ \Lambda \geq 0$ Complementarity Criteria--(4a)

$\mu_t(Eu' - \sigma_t) = 0;\ \mu_t \geq 0$ Complementarity Criteria--(4b)

$\mu_c(\sigma_c - Eu') = 0;\ \mu_c \geq 0$ Complementarity Criteria--(4c)

$\int Adx - V^* \leq 0$ Feasibility Criteria-----(5a)

$Eu' - \sigma_t \leq 0$ Feasibility Criteria-----(5b)

$\sigma_c - Eu' \leq 0$ Feasibility Criteria-----(5c)

CASE a): $\sigma_t = Eu'$ (upper limit of stress reached)

$$Eu' = \sigma_t \Rightarrow u' = \frac{\sigma_t}{E}$$

$$\Rightarrow A' = \frac{-P}{\sigma_t} \qquad \text{(using governing equation no } (3))$$

CASE b): $\sigma_c = Eu'$ (Lower limit of stress reached)

$$Eu' = \sigma_c \Rightarrow u' = \frac{\sigma_c}{E}$$

$$\Rightarrow A' = \frac{-P}{\sigma_c} \qquad \text{(using governing equation no } (3))$$

CASE c): $\sigma_c < \sigma < \sigma_t$ (Stress constraint are inactive)

$$\mu_c = \mu_t = 0$$

$\Rightarrow$ Adjoint eq becomes $\text{P} + \left(\text{EA}\lambda'\right)' = 0$

$$\Rightarrow \boxed{\lambda = u}$$

$$\Rightarrow \boxed{\Lambda = Eu'^2} \qquad \leftarrow \text{optimality criteria}$$

BOUNDARY CONDITIONS

$$\hat{L} = pu + \lambda\left\{\left(EAu'\right)' + P\right\} + \Lambda\left(A - \frac{V^*}{L}\right) + \mu_t\left(Eu' - \sigma_t\right) + \mu_c\left(\sigma_c - Eu'\right)$$

$$\left.\frac{\partial \hat{L}}{\partial A'}\delta A\right|_0^L = 0 \Rightarrow \left(\lambda E u'\right)\delta A\Big|_0^L = 0 \Rightarrow \boxed{\left(uu'\right)\delta A_0^L = 0} \quad \text{(BCI)}$$

$$\left.\left\{\frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''}\right)'\right\}\delta u\right|_0^L = 0 \Rightarrow \left[\left(\lambda EA'\right) - \left(\lambda EA\right)'\right]\delta u\Big|_0^L = 0 \Rightarrow \boxed{\left(Au'\right)\delta u\Big|_0^L = 0} \quad \text{(BCII)}$$

$$\left.\left(\frac{\partial \hat{L}}{\partial u''}\right)\delta u'\right|_0^L = 0 \Rightarrow (\lambda EA)\delta u'\Big|_0^L \Rightarrow \boxed{(Au)\delta u'\Big|_0^L = 0} \quad \text{(BCIII)}$$

PROBLEM 12

Find the world load distribution for a bar of given geometry

$$\underset{A(x)}{Min}\ \left(\text{-MC}\right) = \int \left(-Pu\right)dx$$

subjected to

$$\lambda(x) : \left(EAu^{'}\right)^{'} + P = 0$$

$$\Lambda : \int_0^L Pdx - W^{*} = 0$$

$$Data : L, A(x), E, W^{*}$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[pu + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(P - \frac{W^*}{L} \right) \right] dx$$

$$\hat{L} = pu + \lambda \left\{ \left(EAu' \right)' + P \right\} + \Lambda \left(P - \frac{W^*}{L} \right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE P(x)

$$\delta_P \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial P} - \left(\frac{\partial \hat{L}}{\partial P'} \right)' = 0$$ Euler –Lagrange Equation

$$\Rightarrow -u + \lambda + \Lambda = 0$$

$$\Rightarrow \boxed{u = \lambda + \Lambda}$$ Design Equation

STEP 3: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$

$$\Rightarrow -P - \left(\lambda E A'\right)' + \left(\lambda E A\right)'' = 0$$

$$\Rightarrow \text{P} + \left(\text{EA}(-\lambda)'\right)' = 0 \quad \leftarrow \quad \text{Adjoint Equation w.r.t } \text{u}(x)$$

STEP 4: COLLECTING ALL EQUATIONS

$u = \lambda + \Lambda$ Design Equation------(1)

$P + \left\{EA(-\lambda)'\right\}' = 0$ Adjoint Equation-----(2)

$P + \left(EAu'\right)' = 0$ Governing Equation---(3)

$\Lambda\left[\int\left(P - \frac{W^*}{L}\right)dx\right] = 0;\ \Lambda \geq 0$ Complementarity Criteria--(4)

$\int P dx - W^* \leq 0$ Feasibility Criteria-----(5)

STEP 5: OPTIMALITY CRITERIA

Comparing eq (2) & (3), we get

$\lambda = -u$

Equation (1) becomes

$\boxed{\Lambda = 2u} \leftarrow$ Optimality Criteria ----(6)

$\Rightarrow u = \dfrac{\Lambda}{2}$ = constant $\Rightarrow u' = 0$

$\therefore$ Governing eq (3) become $\boxed{P=0}$

PROBLEM 13

Minimize a general objective function for given volume

$$\underset{A(x)}{Min}\ \mathrm{MSC} = \int Pu^2 dx$$

subjected to

$$\Gamma : \int_0^L \left(EAu'v' - Pv \right) dx = 0$$

$$\Lambda : \int A dx - V^* \leq 0$$

$$Data : L, P(x), E, V^*$$

STEP 1: WRITING LAGRANGIAN

$$L = \int \left[Pu^2 + \Gamma\left(EAu'v' - Pv\right) + \Lambda\left(A - \frac{V^*}{L}\right)\right] dx$$

$$\hat{L} = Pu^2 + \Gamma\left(EAu'v' - Pv\right) + \Lambda\left(A - \frac{V^*}{L}\right)$$

STEP 2: VARIATION W.R.T DESIGN VARIABLE A(x)

$$\delta_A \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'}\right)' = 0 \qquad \text{Euler –Lagrange Equation}$$

$$\Rightarrow \left(\Gamma E u'v'\right) + \Lambda = 0$$

$$\Rightarrow \boxed{\Lambda = -Eu'\lambda'} \qquad (\text{where } \lambda = \Gamma v) \qquad \text{Design Equation}$$

STEP 3a: VARIATION W.R.T STATE VARIABLE u(x)

$$\delta_u \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'}\right)' + \left(\frac{\partial \hat{L}}{\partial u''}\right)'' = 0$$

Euler Lagrange Equation

$$\Rightarrow 2Pu - \left(\Gamma E A v'\right)' = 0$$

$$\Rightarrow \boxed{2Pu + \left\{EA\left(-\lambda\right)'\right\}' = 0}$$

Adjoint Equation

STEP 3b: VARIATION W.R.T v

$$\delta_v \hat{L} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial v} - \left(\frac{\partial \hat{L}}{\partial v'}\right)' + \left(\frac{\partial \hat{L}}{\partial v''}\right)'' = 0$$

Euler Lagrange Equation

$$\Rightarrow P + \left(EAu'\right)' = 0$$

(Governing Equation)

STEP 4: COLLECTING ALL EQUATIONS

$\Lambda = -Eu'\lambda'$ Design Equation------(1)

$\left\{EA\left(-\lambda'\right)\right\}' + 2Pu = 0$ Adjoint Equation-----(2)

$\left(EAu'\right)' + P = 0$ Governing Equation---(3)

$\Lambda\left[\int\left(A - \frac{V^*}{L}\right)dx\right] = 0; \Lambda \geq 0$ Complementarity Criteria--(4)

$\int Adx - V^* \leq 0$ Feasibility Criteria-----(5)

STEP 5: OPTIMALITY CRITERIA

Note: Comparing eq (2) & (3), we get the sign of λ is opposite of u

$\Rightarrow$ Volume constraint is always active

$$\Rightarrow \boxed{\int_0^L V dx - V^* = 0} \qquad \text{----}(6)$$

Numerical Steps

a) Choose initial Λ & Area profile $\mathrm{A}(x)$ satisfying eq (6)

b) Calculate u from governing equation (3)

c)Calculate λ using adjoint equation (2).

d)$\mathrm{A}_{k+1} = \left(\dfrac{\Lambda}{E u' \lambda'} \right) \mathrm{A}_k$

e) Calculate Λ_{new} using A_{k+1} & eq (6)

f) Repeat steps b) to e)

THANK YOU