Lecture 19c

Topology optimization of 2D frames for stiffness

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Posing and solving the topology optimization of 2D frames in which design variables are cross-section dimensions of beam elements.

Maximizing stiffness for given volume.

What we will learn:

How to apply the six steps to identify the optimality criterion and use it in the numerical method.

How to ensure that we get a realistic solution when non-intuitive displacement is desired for given applied force.

Problem 12

Minimize the volume of material of a beam (statically determinate or indeterminate) for a deflection constraint in its span with an upper bound on the strain energy.

$$\underset{A(x)}{Min}\ V = \int_0^L A\,dx$$

Subject to

$$\lambda(x): \quad \left(E\alpha A w''\right)'' - q = 0$$

$$\lambda_d(x): \quad \left(E\alpha A v''\right)'' - q_d = 0$$

$$\Lambda: \quad \int_0^L E\alpha A w'' v''\,dx - \Delta^* = 0$$

$$\Gamma: \quad \int_0^L \frac{1}{2} E\alpha A w''^2\,dx - SE^* = 0$$

$$Data: L, q(x), q_d(x), \alpha = t^2/12, E, \Delta^*, SE^*$$

Problem

Minimize strain energy of a frame for given volume of material.

$$\underset{\mathbf{b}}{Min}\ SE = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \quad \mathbf{K}\mathbf{u} - \mathbf{f} = \mathbf{0}$$

$$\Lambda: \quad \mathbf{l}^T\mathbf{a} - V^* \le 0$$

$$Data: \mathbf{K}(\mathbf{l}, \mathbf{b}, d, E), \mathbf{l}, \mathbf{f}, V^*, d$$

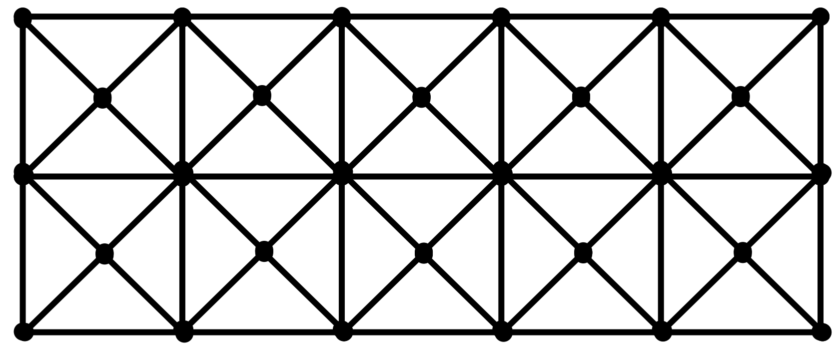
n Number of nodes

Each node has three DoF: x and y displacements and rotation aout the z-axis.

$\mathbf{K}_{3n\times 3n}$ Global stiffness matrix before applying boundary conditions

$\mathbf{u}_{3n\times 1}$ Complete state variable vector of DoF, three per node

A frame with beam elements



We can take as many beam elements as needed

$$I_i = \frac{b_i d^3}{12}$$

We assume here that only b (breadth) of beam elements in the frame is variable with d (depth) held constant and the same for all beam elements.

$$\mathbf{b}^T = \{b_1 \quad b_2 \quad \cdots \quad b_i \quad \cdots \quad b_N\}$$

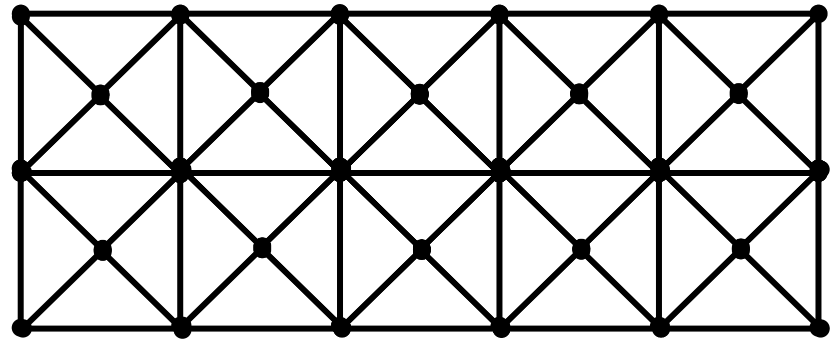
Design variables have lower and upper bounds

$$b_{\min} \le b_i \le b_{\max}$$

How optimized topology comes about…

$$\mathbf{b}^T = \{ b_1 \quad b_2 \quad \cdots \quad b_i \quad \cdots \quad b_N \}$$

Design variables have lower and upper bounds

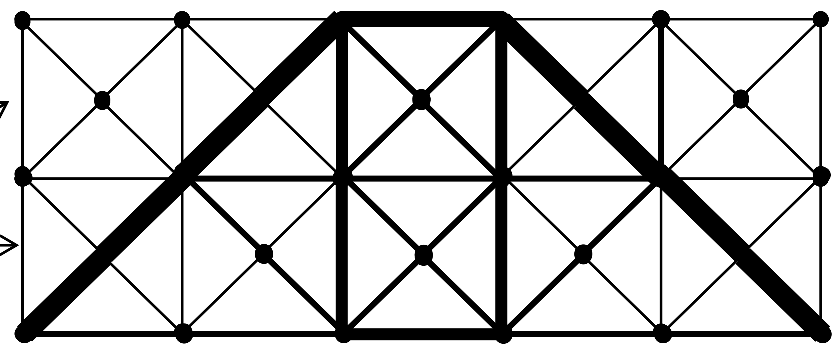


$$b_{\min} \le b_i \le b_{\max}$$

We make the lower bound small enough to not cause the stiffness matrix singular (if one or more elements get that value during optimization) but make it as small as possible so that those elements do not contribute to stiffness.

If optimized values of "b"s reach the lower bound (which is nearly zero), then the corresponding beams elements "disappear" in the design leaving the optimized topology.

Optimized topology of the frame wherein many elements (thin lines) have reached the lower bound on *b*.



Steps in the solution procedure

Step 1: Write the Lagrangian

Step 2: Take derivative of the Lagrangian w.r.t. the design variable and equate to zero to get the design equation.

Step 3: re-arrange the terms in the design equation to avoid computing the derivative of the state variables and thereby get the adjoint equation(s).

Step 4: Collect all the equations, including the governing equation(s), complementarity condition(s), resource constraints, etc.

Step 5: Obtain the optimality criterion by substituting adjoint and equilibrium equations into the design equation, when it is possible.

Step 6: Use the optimality criteria method to solve the equations numerically.

Solution

$$\underset{\mathbf{b}}{Min}\ SE = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \qquad \mathbf{K}\mathbf{u} - \mathbf{f} = 0$$

$$\Lambda: \qquad \mathbf{l}^T\mathbf{a} - V^* \le 0$$

$$Data: \mathbf{K}(\mathbf{l}, \mathbf{b}, d, E), \mathbf{l}, \mathbf{f}, V^*, d$$

Step 1 Writing the Lagrangian

$$L = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} + \boldsymbol{\lambda}^T\left(\mathbf{K}\mathbf{u} - \mathbf{f}\right) + \Lambda\left(\mathbf{l}^T\mathbf{a} - V^*\right)$$

Design equation

Step 2 Taking the derivative of the Lagrangian w.r.t. to the design variable

$$L = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} + \boldsymbol{\lambda}^T\left(\mathbf{K}\mathbf{u} - \mathbf{f}\right) + \Lambda\left(\mathbf{l}^T\mathbf{a} - V^*\right)$$

$$\frac{\partial L}{\partial b_i} = \frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u} + \mathbf{u}^T\mathbf{K}\frac{\partial \mathbf{u}}{\partial b_i} + \boldsymbol{\lambda}^T\left(\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u} + \mathbf{K}\frac{\partial \mathbf{u}}{\partial b_i} - \cancel{\frac{\partial \mathbf{f}}{\partial b_i}}\right) + \Lambda\left(l_i\right) = 0$$

= 0 if loads are not dependent on the design variables

Design equation

Step 3 Re-arrange the design equation to separate out sensivities of state variables.

$$\frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u} + \underbrace{\left(\mathbf{u}^T\mathbf{K} + \boldsymbol{\lambda}^T\mathbf{K}\right)}_{\text{Equate this to zero vector to avoid computing } \frac{\partial \mathbf{u}}{\partial b_i}}\frac{\partial \mathbf{u}}{\partial b_i} + \boldsymbol{\lambda}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u} + \Lambda\left(l_i\right) = 0$$

Adjoint equation and all equations

Step 3 Adjoint equation

$$\left(\mathbf{u}^T\mathbf{K} + \boldsymbol{\lambda}^T\mathbf{K}\right) = \mathbf{0}$$

Step 4 Collect all equations

Design equation
$$\frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u} + \boldsymbol{\lambda}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u} + \Lambda\left(l_i\right) = 0$$

Adjoint equation
$$\left(\mathbf{u}^T\mathbf{K} + \boldsymbol{\lambda}^T\mathbf{K}\right) = \mathbf{0} \Rightarrow \boldsymbol{\lambda} = -\mathbf{u}$$

Feasibility equation
$$\mathbf{l}^T\mathbf{a} - V^* \le 0$$

Complementarity condition
$$\Lambda\left(\mathbf{l}^T\mathbf{a} - V^*\right) = 0; \quad \Lambda \ge 0$$

Optimality criterion

Step 5 Substitute the solution to the adjoint variable into the design equation.

$$\frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u}+\boldsymbol{\lambda}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u}+\Lambda\left(l_i\right)=0$$

$$\left(\mathbf{u}^T\mathbf{K}+\boldsymbol{\lambda}^T\mathbf{K}\right)=\mathbf{0}\Rightarrow\boldsymbol{\lambda}=-\mathbf{u}$$

$$\frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u}-\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u}+\Lambda\left(l_i\right)=0$$

$$\Rightarrow\frac{1}{2l_i}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u}=\Lambda$$

Optimality criterion

$$\frac{1}{2l_i}\mathbf{u}^T\frac{\partial \mathbf{K}}{\partial b_i}\mathbf{u}=\Lambda$$

Since $\dfrac{\partial \mathbf{K}_i}{\partial b_i}=\dfrac{K_i}{b_i}$

$$\frac{\frac{1}{2}\mathbf{u}_i{}^T\mathbf{K}_i\mathbf{u}_i}{l_i b_i d}=\frac{\Lambda}{d}$$

So, the strain energy density is constant for all beam elements.

Numerical solution

Step 6 Use the optimality criteria method to find "b"s in outer and inner loops.

Initial guess for $\mathbf{b}, \Lambda$

Update $b_i^{(k+1)} = \dfrac{\mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i}{2\Lambda l_i}$

Check if b_i has exceeded bounds and equate to the bounds if they did.
Update Λ until b_i does not exceed bounds anymore.

Inner loop

$k = k + 1$

Continue until $\mathbf{b}^{(k+1)} = \mathbf{b}^{(k)}$

Outer loop

k

$\dfrac{\mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i}{2\Lambda l_i b_i} = 1$ or $b_i = b_{\min}$ *or* $b_{\max}$

What we need to achieve for all elements, $i = 1, 2, \cdots, N$

The end note

Topology optimization of frames for stiffness

- Observe how we used size optimization of individual beam elements in a "super structure" to give the topology of the frame.
- We follow six steps to solve the discretized (or finite-variable optimization) problem.
- Identify the optimality criterion.
- Interpret the optimality criterion.
- Iterative numerical solution, when it is needed, remains the same.

Thanks