Lecture 28b

Topological derivative-based topology optimization

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Recap

Topological derivative of a functional quantifies the sensitivity with respect to an infinitesimal domain perturbations such as a hole, an inclusion, a crack, a source term, etc.

Topological derivative is obtained from the first-order term in the topological asymptotic expansion of the performance functional. Topological derivative is also the limiting value of the shape derivative.

We use Eshelby's theorem to obtain the stress state inside the inclusion. The uniform Eshelby mapping provides fourth-order Polarization tensor that plays an important role in the topological derivative expression.

The topological derivative for structural mean compliance is given by

$$\boxed{TD(\hat{\boldsymbol{x}}) = \mathbb{P}_\gamma \boldsymbol{\sigma}(\boldsymbol{u}(\hat{\boldsymbol{x}})) \cdot \boldsymbol{\epsilon}(\boldsymbol{u}(\hat{\boldsymbol{x}}))}$$

Polarization tensor: $\mathbb{P}_\gamma = \frac{1-\gamma}{2(1+\gamma\alpha_2)}\left[(1+\alpha_2)\mathbb{I} + \frac{1-\gamma}{2(1+\gamma\alpha_1)}(\alpha_1 - \alpha_2)\boldsymbol{I} \otimes \boldsymbol{I}\right]$

Closed-form topological derivative

$$TD(\hat{\boldsymbol{x}}) = \frac{1-\gamma}{2(1+\gamma\alpha_2)}\left[(1+\alpha_2)\boldsymbol{\sigma}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big)\cdot\boldsymbol{\epsilon}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big) + \frac{1-\gamma}{2(1+\gamma\alpha_1)}(\alpha_1-\alpha_2)tr\left(\boldsymbol{\sigma}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big)\right)tr\left(\boldsymbol{\epsilon}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big)\right)\right]$$

Outline of the lecture

Algorithm 1: Topological derivative-based optimization in an open loop.

Algorithm 2: Pareto-optimal solutions with topological derivative (closed loop).

Algorithm 3: Topological derivative-based optimization algorithm combined with level-set domain representation.

What we will learn:

How topological derivative identify optimal location of holes.

How to obtain Pareto-optimality condition for material distribution and its physical significance.

How topological derivatives are combined with the level-set model.

TD in open-loop implementation

Closed-form topological derivative

$$TD(\widehat{\boldsymbol{x}}) = \frac{1-\gamma}{2(1+\gamma\alpha_2)}\left[(1+\alpha_2)\boldsymbol{\sigma}\big(\boldsymbol{u}(\widehat{\boldsymbol{x}})\big)\cdot\boldsymbol{\epsilon}\big(\boldsymbol{u}(\widehat{\boldsymbol{x}})\big) + \frac{1-\gamma}{2(1+\gamma\alpha_1)}(\alpha_1-\alpha_2)tr\left(\boldsymbol{\sigma}\big(\boldsymbol{u}(\widehat{\boldsymbol{x}})\big)\right)tr\left(\boldsymbol{\epsilon}\big(\boldsymbol{u}(\widehat{\boldsymbol{x}})\big)\right)\right]$$

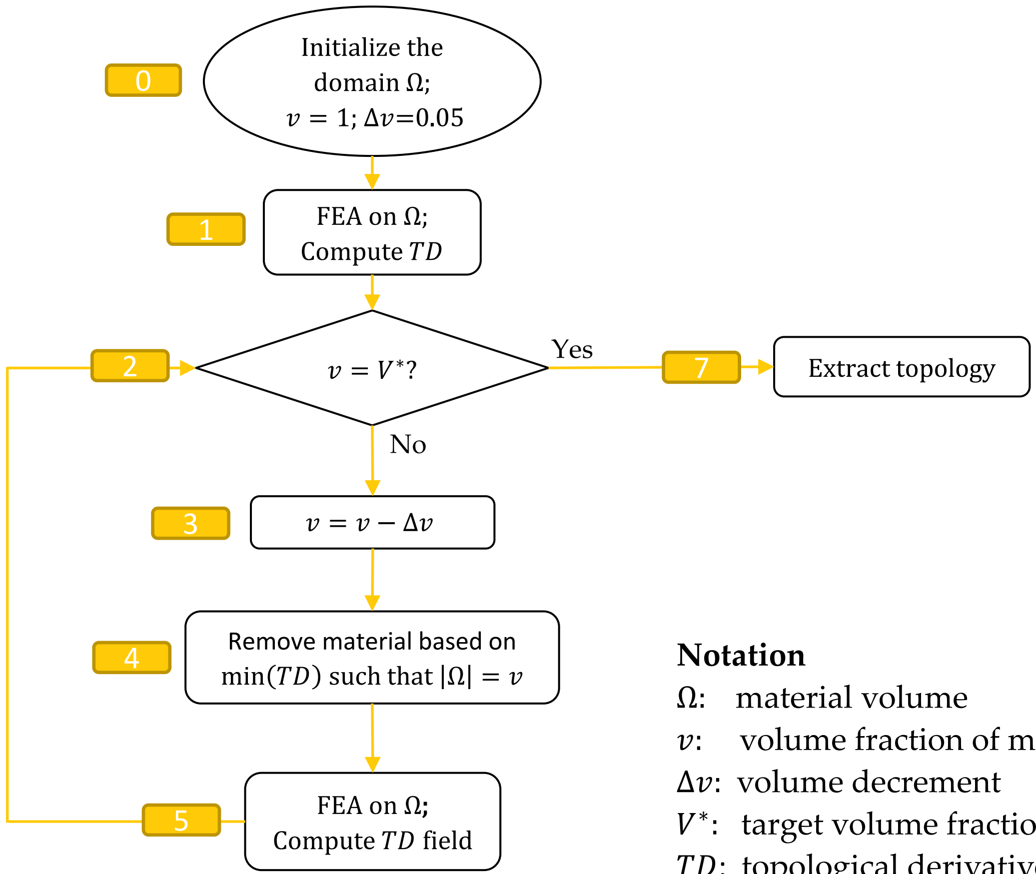
In the open loop approach, material is removed bit by bit; once material is removed, it is removed forever. In other words, we do not allow the material to come back, once it is removed..

Here, we only use the topological derivative expression for creating voids, i.e., $TD_{I\to V}$. On substituting $\gamma \to 0$, we get the topological derivative for creating holes/singularities in the domain.

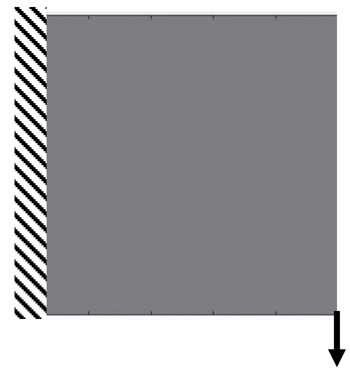
$$TD_{I\to V}(\widehat{\boldsymbol{x}}) = \frac{2}{1+\nu}\boldsymbol{\sigma}(\widehat{\boldsymbol{x}})\cdot\boldsymbol{\epsilon}(\widehat{\boldsymbol{x}}) - \frac{1-3\nu}{2(1-\nu^2)}tr\big(\boldsymbol{\sigma}(\widehat{\boldsymbol{x}})\big)tr\big(\boldsymbol{\epsilon}(\widehat{\boldsymbol{x}})\big)$$

The limiting value of the contrast parameter provides the topological derivative for creating voids.
We do not use topological derivative for adding back material in the void region while implementing open-loop optimization algorithm.

Algorithm 1: TD-based open loop



$$TD(\hat{\boldsymbol{x}}) = \mathbb{P}_0 \boldsymbol{\sigma}(\hat{\boldsymbol{x}}) \cdot \boldsymbol{\epsilon}(\hat{\boldsymbol{x}})$$



Notation

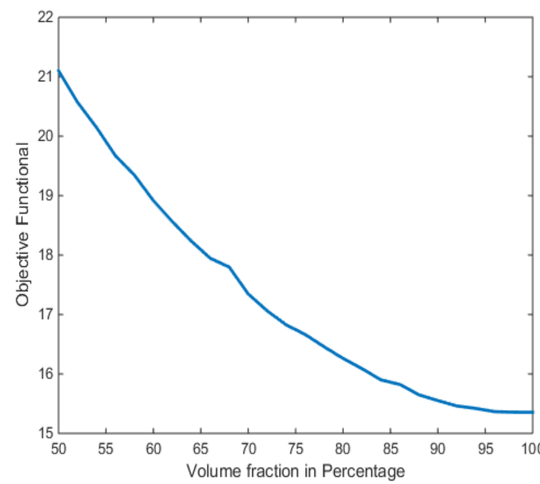
Ω: material volume

v: volume fraction of material

Δv: volume decrement

V^*: target volume fraction

TD: topological derivative



Voids are created in the region where topological derivative is minimum.

TD in closed-loop implementation

The topological derivative for structural mean compliance is given by

$$TD(\hat{\boldsymbol{x}}) = \mathbb{P}_\gamma \boldsymbol{\sigma}(\boldsymbol{u}(\hat{\boldsymbol{x}})) \cdot \boldsymbol{\epsilon}(\boldsymbol{u}(\hat{\boldsymbol{x}}))$$

Polarization tensor: $\mathbb{P}_\gamma = \frac{1-\gamma}{2(1+\gamma\alpha_2)}\left[(1+\alpha_2)\mathbb{I} + \frac{1-\gamma}{2(1+\gamma\alpha_1)}(\alpha_1 - \alpha_2)\boldsymbol{I} \otimes \boldsymbol{I}\right]$

In the topological derivative-based close-loop optimization approach, material is removed bit by bit and we keep on checking iteratively, by adding material back again until convergence. In other words, we allow the material to come back in the void region.

Here, we use topological derivative expressions for creating voids in the material region ($TD_{I\to V}$) as well as adding back material in the void locations ($TD_{V\to I}$).

On substituting $\gamma \to 0$, we obtain topological derivative for creating voids

Isotropic → Void → $TD_{I\to V}(\hat{\boldsymbol{x}}) = \frac{2}{1+\nu}\boldsymbol{\sigma}(\hat{\boldsymbol{x}}) \cdot \boldsymbol{\epsilon}(\hat{\boldsymbol{x}}) - \frac{1-3\nu}{2(1-\nu^2)} tr(\boldsymbol{\sigma}(\hat{\boldsymbol{x}})) tr(\boldsymbol{\epsilon}(\hat{\boldsymbol{x}}))$

On substituting $\gamma \to \infty$, topological derivative for adding back material

Void → Isotropic → $TD_{V\to I}(\hat{\boldsymbol{x}}) = -\frac{2}{3-\nu}\boldsymbol{\sigma}(\hat{\boldsymbol{x}}) \cdot \boldsymbol{\epsilon}(\hat{\boldsymbol{x}}) - \frac{1-3\nu}{2(1+\nu)(3-\nu)} tr(\boldsymbol{\sigma}(\hat{\boldsymbol{x}})) tr(\boldsymbol{\epsilon}(\hat{\boldsymbol{x}}))$

TD-based close-loop optimization

Step 0

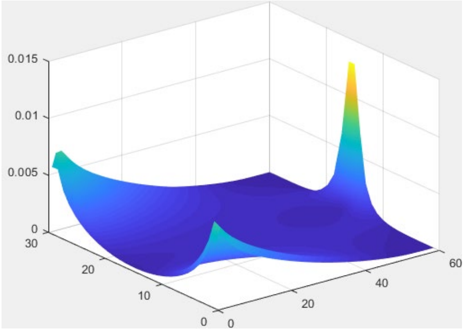
The design is initialized by setting isotropic material volume (Ω) equal to the design space (D) which means that the volume fraction of the material (v) in the design space is one. The fraction of material to be replaced with voids is set as $\Delta v = 0.05$.

$$\Omega = D;\ v = 1;$$
$$\Delta v = 0.05$$

Step 1

Finite element analysis is carried over the material volume and topological sensitivity field is computed. The Gaussian filter is used to smoothen the sensitivity field.

FEA on Ω;
Compute TD



Step 2

We check that whether the volume fraction of the material v has attained the desired volume fraction V^* or not.

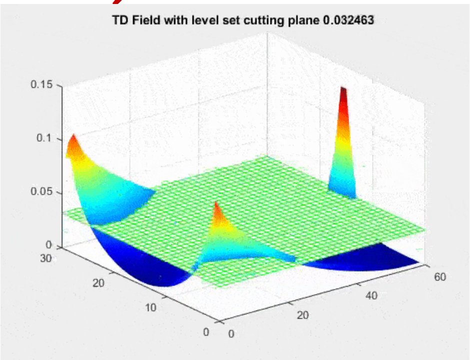
$$v = V^*?$$

Step 3

If the condition in **Step 2** is not satisfied, then material is further removed by modifying the material volume fraction v.

$$v = v - \Delta v$$

TD-based optimization (Cont.)



Step 4

Next, the level-set value τ is computed by adjusting the level-set plane in the topological sensitivity field. In this step, bisection method is adopted to obtain the level-set value between the maximum and minimum values of the field and a fixed-point iteration scheme is used to arrive at a design such that the material volume is equal to v.

Find τ such that $|\Omega| = v$

Step 5

Follow **Step 1** to perform finite element analysis and obtain filtered topological sensitivity field for the material volume.

FEA on Ω; Compute TD

Step 6

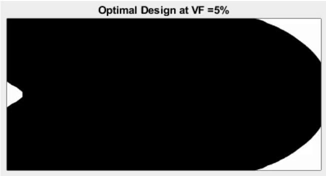
The Pareto-optimality condition for interchanging material and voids is given by

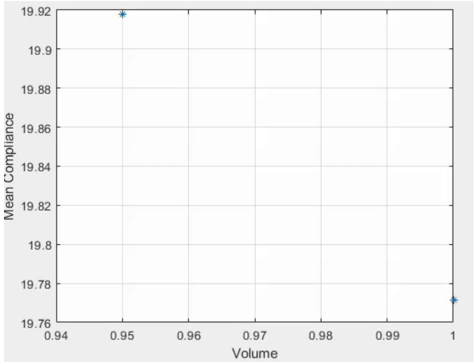
$$\min(TD_{I \to V}) + \min(TD_{V \to I}) \geq 0$$

If the condition follows, move to **Step 2**. If not, then go to **Step 4** and search for the Pareto-optimal design at the same volume fraction.

Step 7

If the condition in **Step 2** is satisfied, then topology is extracted in the form of iso-surface. In this process we also trace the Pareto curve of performance functional and volume fraction of the material.





Pareto-optimality condition

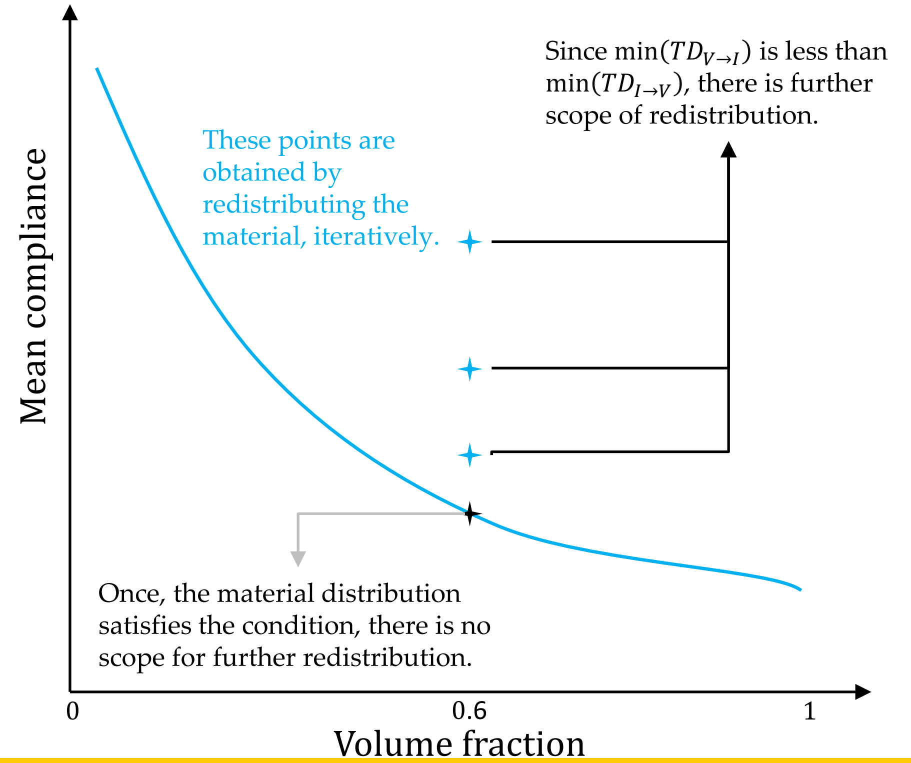
Topological asymptotic expansion: $\psi(\Omega_\varepsilon) - \psi(\Omega) = f(\varepsilon)TD(\hat{\boldsymbol{x}}) + o(f(\varepsilon))$

If the performance functional is mean compliance (measure of strain energy and stiffness), then it will increase when voids are created, that is, $\psi(\Omega_\varepsilon) - \psi(\Omega) > 0$. Therefore, $TD_{I\to V} > 0$.

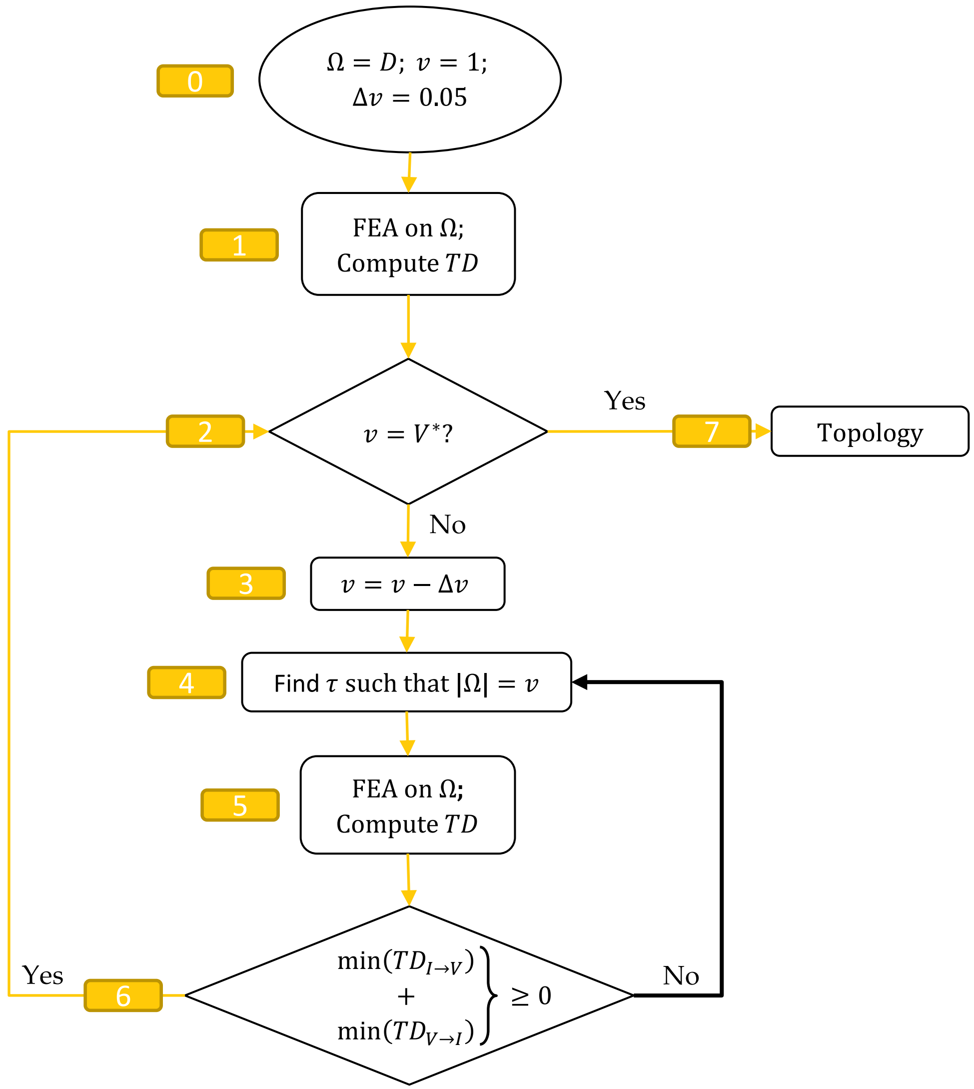
On the contrary, the performance functional will decrease on adding material in the void locations, that is, $\psi(\Omega_\varepsilon) - \psi(\Omega) < 0$. Therefore, $TD_{V\to I} < 0$.

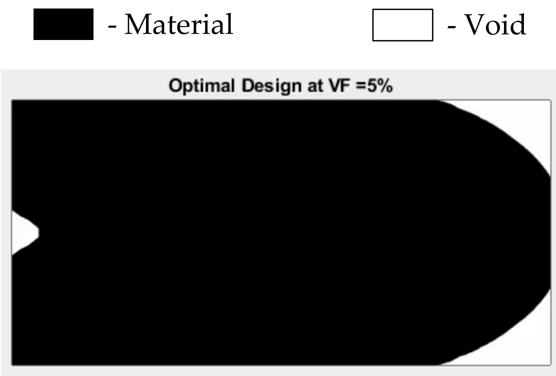
$$\underbrace{\min(TD_{I\to V})}_{\text{Always positive}} + \underbrace{\min(TD_{V\to I})}_{\text{Always negative}} \geq 0$$

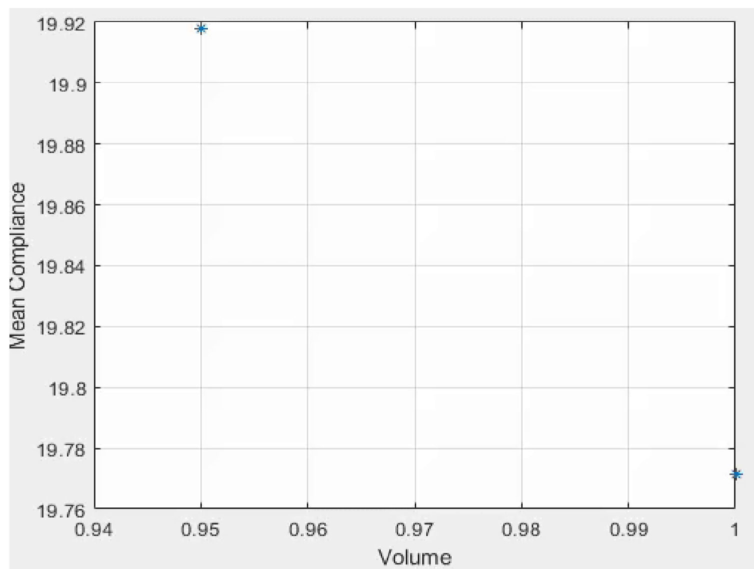
Pareto curve traces all the points that satisfy the Pareto-optimality condition. That is, once the algorithm attains the condition, there is no further redistribution possible. All the sub-optimal points lie above the Pareto curve.



Algorithm 2: TD-based close loop







Suresh, K., Structural and Multidisciplinary Optimization, 2010.

TD-based level-set approach

Topological derivative can be combined with conventional level-set approach to update the level-set function that represents the optimal domain.

In the level-set approach, a fictitious time t is considered along with the family of domains $\Omega(t)$. This family is represented by a level-set function $\psi(\boldsymbol{x}, t) \in D \times \mathbb{R} \mapsto \mathbb{R}$, defined as:

$$\begin{cases} \psi(\boldsymbol{x}, t) < 0 & if\ \boldsymbol{x} \in \Omega \\ \psi(\boldsymbol{x}, t) > 0 & if\ \boldsymbol{x} \in \mathrm{D}\backslash\bar{\Omega} \\ \psi(\boldsymbol{x}, t) = 0 & if\ \boldsymbol{x} \in \partial\Omega \end{cases}$$

In the topological derivative-based level-set optimization method, the level-set function $\psi(\boldsymbol{x}, t)$ is chosen as the design variable. Also, the topological derivatives for interchanging material are denoted in the form of a generalized function $g(\boldsymbol{x}, t)$ such that

$$g(\boldsymbol{x}, t) = \begin{cases} -TD_{I \to V} & if\ \boldsymbol{x} \in \Omega \\ -TD_{V \to I} & if\ \boldsymbol{x} \in \mathrm{D}\backslash\bar{\Omega} \end{cases}$$

$$\boxed{\begin{cases} g(\boldsymbol{x}, t) \le 0 & if\ \psi(\boldsymbol{x}, t) < 0 \\ g(\boldsymbol{x}, t) \ge 0 & if\ \psi(\boldsymbol{x}, t) > 0 \end{cases}}$$

Amstutz, S. and Andra, H., Journal of Computational Physics, 2006

Amstutz, S., Optimization Methods and Software, 2011

Sensitivity of the level-set function

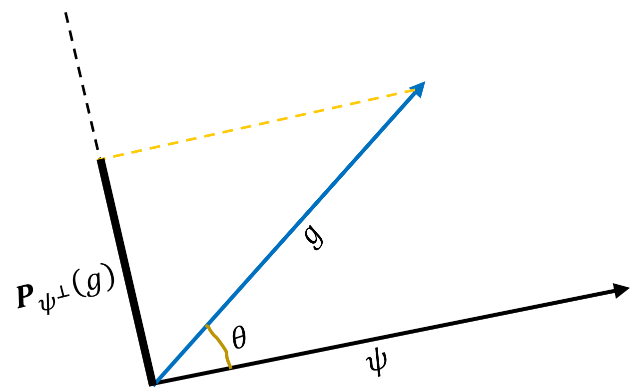
Instead of solving the governing equation of the level-set function $\psi(\boldsymbol{x}, t)$ in time, the following sensitivity equation is proposed:

$$\frac{\partial \psi}{\partial t} = \boldsymbol{P}_{\psi^{\perp}}(g)$$

$$\boldsymbol{P}_{\psi^{\perp}}(g) = g - \left[\frac{\langle g, \psi \rangle}{\|\psi\| \|g\|}\right] \psi$$

Here, $\boldsymbol{P}_{\psi^{\perp}}(g)$ is the orthogonal projector of g onto the orthogonal complement of ψ.

If $\boldsymbol{P}_{\psi^{\perp}}(g) = 0 \Rightarrow g = s\psi$ *or* g *and* ψ *are parallel.*



The time is discretized such that the variation of the topological gradient can be neglected in the interval $[t_i, t_{i+1}]$.

Level-set update equation

The sensitivity equation of the level-set function can be solved analytically in the interval $[t_i, t_{i+1}]$; there exists an angle $\xi_i \in [0, \theta_i]$, where θ_i is the angle between ψ_i and g_i, i.e.

$$\theta_i = \cos^{-1}\left[\frac{\langle g_i, \psi_i\rangle}{\|g_i\|\|\psi_i\|}\right]$$

And the sensitivity equation of the level-set function is written as

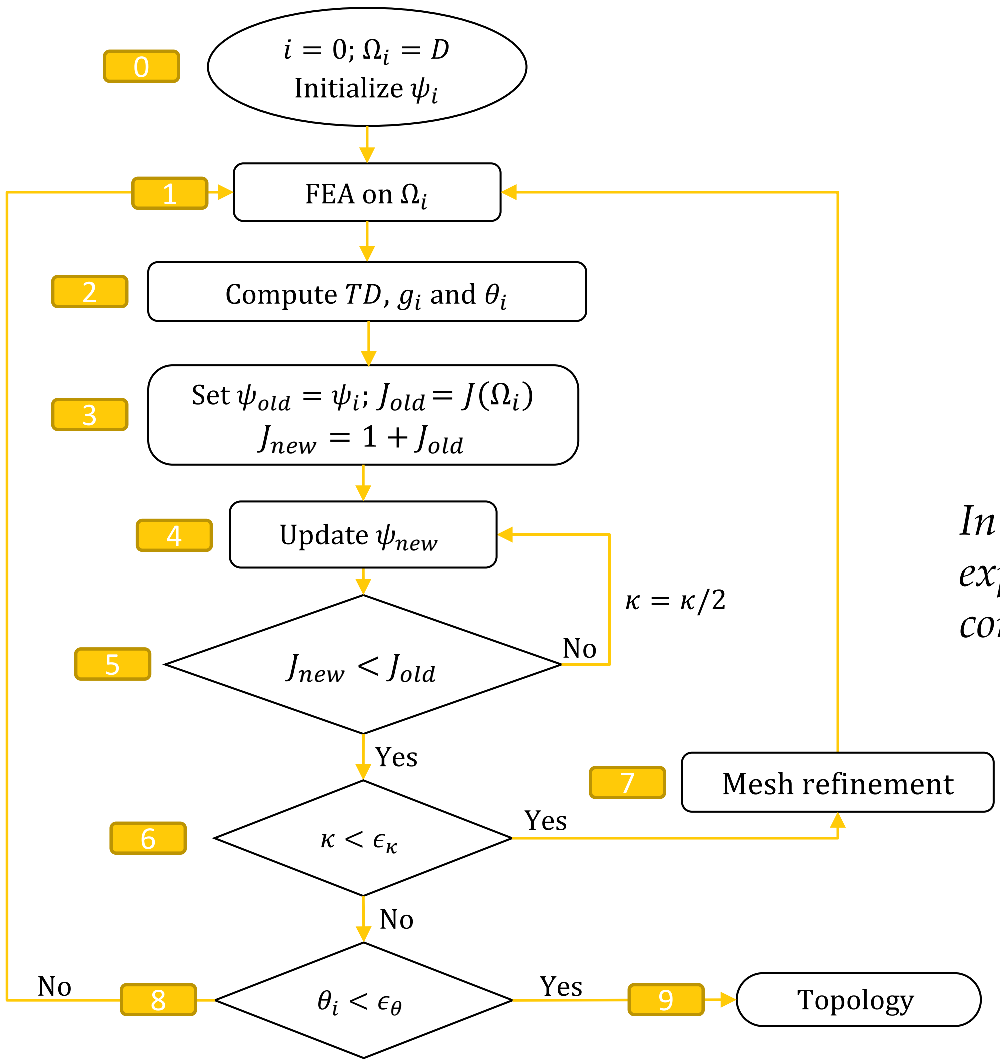
$$\psi_{i+1} = (\cos\xi_i)\psi_i + (\sin\xi_i)\frac{\boldsymbol{P}_{\psi_i^{\perp}}(g_i)}{\left\|\boldsymbol{P}_{\psi_i^{\perp}}(g_i)\right\|}$$

Here, $\psi_i = \psi(\boldsymbol{x}, t_i)$ and $g_i = g(\boldsymbol{x}, t_i)$. For numerical purpose, the parameter ξ_i is expressed as $\xi_i = \kappa_i\theta_i$ and the update equation is rearranged as:

$$\psi_{i+1} = \frac{1}{\sin\theta_i}\left[\sin((1-\kappa_i)\theta_i)\psi_i + \sin\kappa_i\theta_i\frac{g_i}{\|g_i\|}\right]$$

Topological derivative is used to update the level-set function. The angle θ between ψ and g changes in the updating step until it reaches 0 or close to 0.

Algorithm 3: TD-based level-set

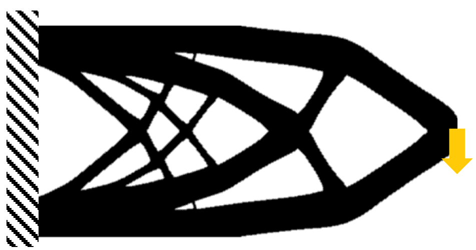


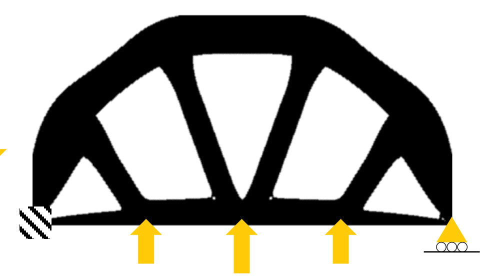
Notation

i: iteration number
Ω_i: material volume
ψ_i: level-set function
g_i: generalized TD function
J: performance functional
θ_i: angle between g_i and ψ_i
κ: line-search parameter
v: volume fraction
ϵ_κ: converge parameter for κ
ϵ_θ: converge parameter for θ

In this algorithm, the performance functional is expressed in the form of a Lagrangian associated to mean compliance and volume fraction, i.e.,

$$J = M.C. + penalty \times v$$





The end note

Topological derivative-based topology optimization

- In the topological derivative-based open-loop implementation, remove material based on minimum values of TD.
- The closed-loop implementation of topological derivatives in the optimization framework requires both $TD_{I\rightarrow V}$ and $TD_{V\rightarrow I}$.
- Interpret the Pareto-optimality condition for redistribution of the material.
- In the topological derivative-based level-set optimization algorithm, make the angle between the level-set function and topological derivative function as small as possible.
- Update the level-set function using the topological derivative combined with trigonometric functions.

Thanks