Lecture 7b

Truss optimization algorithm

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Size optimization (contd.)

Obeying the bounds on areas of cross section

What we will learn:

Modifying the algorithm to satisfy the bounds on the areas of cross section

Inner and outer loops in the algorithm

Size optimization of a truss

$$\underset{\mathbf{A}}{\text{Min}}\ MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\lambda: \qquad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \qquad \sum_{i=1}^{N} \rho A_i l_i - W^* \le 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \le 0$$

Data: $\rho, W^*, l_{i=1,2\ldots,N}, N, P_{i=1,2,\ldots,N}, E$

MC = mean compliance = work done by the external forces

MC is a measure of stiffness:
the lower the value of MC, the stiffer the truss.

K depends on areas of cross section of all truss elements and Young's modulus.

Necessary conditions

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \Rightarrow \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i = 0$$

$i = 1, 2, \cdots, N$ $\quad N$ equations

$\mathbf{K}\mathbf{u} = \mathbf{p}$ $\quad n$ equations

$\rho \mathbf{a}^T \mathbf{l} - W^* \le 0$

$\mu \left(\rho \mathbf{a}^T \mathbf{l} - W^* \right) = 0$ $\quad$ 1 equation

$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{p}$ $\quad n$ equations

$\mathbf{A}$ $\quad N$ unknowns

$\mathbf{u}$ $\quad n$ unknowns

$\boldsymbol{\lambda}$ $\quad n$ unknowns

μ $\quad 1$ unknown

(N+2n+1) equations in (N+2n+1) unknowns.

We also have two inequalities too: $\rho \mathbf{a}^T \mathbf{l} - W^* \le 0$ and $\mu \ge 0$

Algorithm

Step 0 Assume initial guess for all areas of cross section so that the weight constraint is satisfied.

$$A_i^{(o)}, i = 1, 2, \cdots, N$$

Step 1 Solve for the displacements using the loads.

$$\mathbf{Ku} = \mathbf{p}$$

Step 2 Solve $\left(\rho \mathbf{a}^T \mathbf{l} - W^*\right) = 0$ to find μ

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Step 2 and farther…

$$\left(\rho \mathbf{a}^T \mathbf{l} - W^*\right) = 0 \Rightarrow \sum_{i=1}^{N} \rho A_i l_i = W^*$$

$$\Rightarrow \sum_{i=1}^{N} \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)} \rho l_i = W^*$$

$$\Rightarrow \mu = \frac{\sum_{i=1}^{N} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{W^*}$$

Step 3 Update areas of cross section.

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Do you see any other issues here?

Go to Step 1 until convergence.

What if any A_is are out of bounds?

$$A_{lb} \le A_i \le A_{ub}$$ Practical bounds

Can areas of cross-section they become negative?

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \underbrace{\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right)}_{\text{Could this be negative?}} A_i^{(k)} \quad \text{where} \quad \mu = \underbrace{\frac{\sum_{i=1}^{N} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{W^*}}_{\text{Could this be negative?}}$$

Staying within bounds

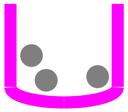
$$A_{lb} \le A_i \le A_{ub} \qquad A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

If $\quad A_i^{(k+1)} > A_{ub}\,,\quad A_i^{(k+1)} = A_{ub}$

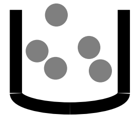
Else if $\quad A_i^{(k+1)} < A_{lb}\,,\quad A_i^{(k+1)} = A_{lb}$

Else $\quad$ leave $A_i^{(k+1)}$ as it is.

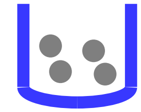
Three bins for areas of cross section


L

Lower bound


I

Intermediate


U

Upper bound

$A_i = A_{ub}$ $\qquad A_{lb} \le A_i \le A_{ub}$ $\qquad A_i = A_{ub}$

$$\sum_{i=1}^{N} \rho A_i l_i = W^* \Rightarrow \sum_{i \in L} A_{lb} \rho l_i + \sum_{i \in I} \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)} \rho l_i + \sum_{i \in U} A_{ub} \rho l_i = W^*$$

$$\Rightarrow \mu = \frac{\displaystyle\sum_{i=1}^{N} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{W^* - \displaystyle\sum_{i \in L} A_{lb} \rho l_i - \sum_{i \in U} A_{ub} \rho l_i}$$

Will this binning be complete in one go?
No, it is also iterative.

Let us modify the algorithm.

Step 0 $A_i^{(o)}, i = 1, 2, \cdots, N \qquad k = 0$

Step 1 Solve $\mathbf{Ku} = \mathbf{p}$ using $A_{i=1,2,\cdots,N}^{(k)}$

Step 2 Guess μ

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) A_i^{(k)}$$

Step 2a Check the bounds on $A_i^{(k+1)}s$ and update $\mu^{(i+1)}$

Step 2b

$$\mu^{(j+1)} = \frac{\sum_{i \in I}^{N}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) A_i^{(k)}}{W^* - \sum_{i \in L} A_{lb}\rho l_i - \sum_{i \in U} A_{ub}\rho l_i}$$

Step 3 Re-compute $A_i^{(k+1)} = \frac{1}{\mu^{(j+1)} \rho l_i}\left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i}\mathbf{u}\right) A_i^{(k)}$

Step 4 Go back to Step 2a until binning is unchanged.

Step 5 Go back to Step 1 until convergence.

Inner loop to satisfy the optimality condition for the entire problem.

$k = k + 1$

$j = j + 1$

Inner loop to satisfy the volume constraint and bounds on A_is.

Re-statement of the problem

$$\underset{\mathbf{A}}{\text{Min}}\ MC = \sum_{i=1}^{N} P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\boldsymbol{\lambda}: \qquad \mathbf{Ku} - \mathbf{p} = 0$$

$$\mu: \qquad \sum_{i=1}^{N} \rho A_i l_i - W^* \le 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \le 0$$

$$\left.\begin{array}{ll} \gamma_{ub}: & A_i - A_{ub} \le 0 \\ \gamma_{lb}: & A_{lb} - A_i \le 0 \end{array}\right\} i = 1, 2, \cdots, N$$

Data: $\rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$

We have $2N$ more inequality constraints. If we wish we can compute the corresponding Lagrange multipliers after the numerical algorithm is implemented and solved. How? Write the Lagrangian and the optimality conditions.

Re-stated optimality criterion

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \gamma_{ub_i} - \gamma_{lb_i} + \mu \rho l_i = 0$$

The end note

Truss optimization

- Checking the bounds on areas of cross section and binning them into three groups.
- Computing the value of Lagrange multiplier of the weight constraint in the iterative inner loop, to satisfy the constraint.
- Maintain the iterative outer loop as before, to satisfy the optimality criterion.
- **Complete algorithm for size optimization of trusses**
- You should implement it to fully understand the algorithm.

Thanks