Material derivative of the determinant of the Jacobian

When we denote the coordinates of a point in the original (material) domain as $\mathbf{X} = \{X, Y, Z\}^T$ and the corresponding point in the changed current (spatial) domain as $\mathbf{x}(\mathbf{X}, p) = \{x, y, z\}^T$ (where p is a parameter that changes the domain), the Jacobian $\mathbf{J}$ of transformation between the two domains is given as in

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{bmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{bmatrix} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} dX \\ dY \\ dZ \end{Bmatrix} \Rightarrow d\mathbf{x} = \mathbf{J}\, d\mathbf{X} \tag{1}$$

We will now compute the material derivative of $\mathbf{J}$ w.r.t. p, i.e., $\dot{\widehat{|\mathbf{J}|}} = \frac{d}{dp}\left(|\mathbf{J}(\mathbf{X})|\right)$. Note that we put an over-dot when we want to indicate that we are taking the material derivative. On the other hand, we use a prime to mean the spatial derivative. That is, $|\mathbf{J}|' \frac{d}{dp}\left(|\mathbf{J}(\mathbf{x})|\right)$.

<u>1D case</u>

$$dx = \frac{\partial x}{\partial X} dX \Rightarrow J = \frac{\partial x}{\partial X} \tag{2}$$

Now, by noting that spatial and parameter derivatives commute, we write

$$\frac{d}{dp}\left(\frac{\partial x}{\partial X}\right) = \frac{\partial}{\partial X}\left(\frac{dx}{dp}\right) \tag{3}$$

By using the chain rule in view of $x(X, p)$, we have

$$\frac{\partial}{\partial X}\left(\frac{dx}{dp}\right) = \frac{\partial}{\partial x}\left(\frac{dx}{dp}\right)\frac{\partial x}{\partial X} \tag{4}$$

By denoting the velocity of the domain as $V_x = \frac{dx}{dp}$, we write

$$\frac{d}{dp}\left(\frac{\partial x}{\partial X}\right) = \frac{\partial}{\partial x}\left(\frac{dx}{dp}\right)\frac{\partial x}{\partial X} = \frac{\partial V_x}{\partial x}\frac{\partial x}{\partial X} \tag{5a}$$

$$\Rightarrow \frac{d}{dp}\left(\frac{\partial x}{\partial X}\right) = \frac{\partial V_x}{\partial x} J = J\frac{\partial V_x}{\partial x} \tag{5b}$$

<u>2D case</u>

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \partial x/\partial X & \partial x/\partial Y \\ \partial y/\partial X & \partial y/\partial Y \end{bmatrix} \begin{Bmatrix} dX \\ dY \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} dX \\ dY \end{Bmatrix} \Rightarrow d\mathbf{x} = \mathbf{J}\, d\mathbf{X} \tag{6}$$

Now, by expanding the determinant, we write

$$\frac{d}{dp}\left(\begin{vmatrix} \partial x/\partial X & \partial x/\partial Y \\ \partial y/\partial X & \partial y/\partial Y \end{vmatrix} \right) = \frac{d}{dp}\left(\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} - \frac{\partial y}{\partial X}\frac{\partial x}{\partial Y} \right) \tag{7}$$

By taking the derivative using the product rule, we have

$$\frac{d}{dp}(|\mathbf{J}|) = \frac{d}{dp}\left(\frac{\partial x}{\partial X}\right)\frac{\partial y}{\partial Y} + \frac{d}{dp}\left(\frac{\partial y}{\partial Y}\right)\frac{\partial x}{\partial X} - \frac{d}{dp}\left(\frac{\partial y}{\partial X}\right)\frac{\partial x}{\partial Y} - \frac{d}{dp}\left(\frac{\partial x}{\partial Y}\right)\frac{\partial y}{\partial X} \tag{8}$$

As we did in Eqs. (3-5) of the 1D case, for the 2D case (where each quantity is a function of x and y) we write:

$$\frac{d}{dp}\left(\frac{\partial x}{\partial X}\right) = \frac{\partial}{\partial X}\left(\frac{dx}{dp}\right) = \frac{\partial V_x}{\partial X} = \frac{\partial V_x}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial X} \tag{9a}$$

$$\frac{d}{dp}\left(\frac{\partial y}{\partial Y}\right) = \frac{\partial}{\partial Y}\left(\frac{dy}{dp}\right) = \frac{\partial V_y}{\partial Y} = \frac{\partial V_y}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial Y} \tag{9b}$$

$$\frac{d}{dp}\left(\frac{\partial y}{\partial X}\right) = \frac{\partial}{\partial X}\left(\frac{dy}{dp}\right) = \frac{\partial V_y}{\partial X} = \frac{\partial V_y}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial X} \tag{9c}$$

$$\frac{d}{dp}\left(\frac{\partial x}{\partial Y}\right) = \frac{\partial}{\partial Y}\left(\frac{dx}{dp}\right) = \frac{\partial V_x}{\partial Y} = \frac{\partial V_x}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial Y} \tag{9d}$$

By substituting the expansions in Eqs. 9(a-d) into Eq. (8), we get

$$\begin{aligned}
\frac{d}{dp}(|\mathbf{J}|) &= \left(\frac{\partial V_x}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial X}\right)\frac{\partial y}{\partial Y} + \left(\frac{\partial V_y}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial Y}\right)\frac{\partial x}{\partial X} \\
&\quad - \left(\frac{\partial V_y}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial X}\right)\frac{\partial x}{\partial Y} - \left(\frac{\partial V_x}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial V_x}{\partial y}\frac{\partial y}{\partial Y}\right)\frac{\partial y}{\partial X} \\
&= \left(\frac{\partial V_x}{\partial x}\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} + \cancel{\frac{\partial V_x}{\partial y}\frac{\partial y}{\partial X}\frac{\partial y}{\partial Y}}\right) + \left(\cancel{\frac{\partial V_y}{\partial x}\frac{\partial x}{\partial Y}\frac{\partial x}{\partial X}} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial Y}\frac{\partial x}{\partial X}\right) \\
&\quad - \left(\cancel{\frac{\partial V_y}{\partial x}\frac{\partial x}{\partial X}\frac{\partial x}{\partial Y}} + \frac{\partial V_y}{\partial y}\frac{\partial y}{\partial X}\frac{\partial x}{\partial Y}\right) - \left(\frac{\partial V_x}{\partial x}\frac{\partial x}{\partial Y}\frac{\partial y}{\partial X} + \cancel{\frac{\partial V_x}{\partial y}\frac{\partial y}{\partial Y}\frac{\partial y}{\partial X}}\right)
\end{aligned} \tag{10}$$

$$\Rightarrow \frac{d}{dp}\left(|\mathbf{J}|\right) = \left(\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} - \frac{\partial y}{\partial X}\frac{\partial x}{\partial Y}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right)$$

$$\Rightarrow \frac{d}{dp}\left(|\mathbf{J}|\right) = \dot{\widehat{|\mathbf{J}|}} = |\mathbf{J}|\left(\nabla \cdot \mathbf{V}\right) \qquad (11)$$

3D case

As it becomes tedious to write the long expressions, we use the summation notation while following Eqs. (7-11).

We begin with writing the determinant of the Jacobian in a convenient indicial notation.

$$|\mathbf{J}| = \left|\begin{bmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{bmatrix}\right| = \in_{ijk} J_{1i} J_{2j} J_{3k} \qquad (12)$$

where $\in_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ are in 123 permutation order} \\ 0 & \text{if any two or all of } ijk \text{ are the same} \\ -1 & \text{if } ijk \text{ are in 321 permutation order} \end{cases}$

How do we reconcile with Eq. (12)? The answer is by direct calculation.

First, let us note that $\in_{ijk}$ is 1 for 123, 231, 312 and it is (-1) for 321, 213, 321. For all others, it is zero. Therefore, we have

$$\left(\in_{ijk} J_{1i} J_{2j} J_{3k}\right) = J_{11}J_{22}J_{33} + J_{12}J_{23}J_{31} + J_{13}J_{21}J_{32} - J_{13}J_{22}J_{31} - J_{12}J_{21}J_{33} - J_{11}J_{23}J_{32}$$

$$= \frac{\partial x}{\partial X}\frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z} + \frac{\partial x}{\partial Y}\frac{\partial y}{\partial Z}\frac{\partial z}{\partial X} + \frac{\partial x}{\partial Z}\frac{\partial y}{\partial X}\frac{\partial z}{\partial Y} - \frac{\partial x}{\partial Z}\frac{\partial y}{\partial Y}\frac{\partial z}{\partial X} - \frac{\partial x}{\partial Y}\frac{\partial y}{\partial X}\frac{\partial z}{\partial Z} - \frac{\partial x}{\partial X}\frac{\partial y}{\partial Z}\frac{\partial z}{\partial Y}$$

$$= \frac{\partial x}{\partial X}\left(\frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z} - \frac{\partial y}{\partial Z}\frac{\partial z}{\partial Y}\right) + \frac{\partial x}{\partial Y}\left(\frac{\partial y}{\partial Z}\frac{\partial z}{\partial X} - \frac{\partial y}{\partial X}\frac{\partial z}{\partial Z}\right) + \frac{\partial x}{\partial Z}\left(\frac{\partial y}{\partial X}\frac{\partial z}{\partial Y} - \frac{\partial y}{\partial Y}\frac{\partial z}{\partial X}\right)$$

$$= \frac{\partial x}{\partial X}\left(\frac{\partial y}{\partial Y}\frac{\partial z}{\partial Z} - \frac{\partial y}{\partial Z}\frac{\partial z}{\partial Y}\right) - \frac{\partial x}{\partial Y}\left(\frac{\partial y}{\partial X}\frac{\partial z}{\partial Z} - \frac{\partial y}{\partial Z}\frac{\partial z}{\partial X}\right) + \frac{\partial x}{\partial Z}\left(\frac{\partial y}{\partial X}\frac{\partial z}{\partial Y} - \frac{\partial y}{\partial Y}\frac{\partial z}{\partial X}\right)$$

The last line is nothing but the determinant of $|\mathbf{J}| = \left|\begin{bmatrix} \partial x/\partial X & \partial x/\partial Y & \partial x/\partial Z \\ \partial y/\partial X & \partial y/\partial Y & \partial y/\partial Z \\ \partial z/\partial X & \partial z/\partial Y & \partial z/\partial Z \end{bmatrix}\right|$.

Now, we write the material derivative of the determinant of the Jacobian as

$$\dot{\overline{|\mathbf{J}|}} = \in_{ijk} \left(\dot{J}_{1i} J_{2j} J_{3k} + J_{1i} \dot{J}_{2j} J_{3k} + J_{1i} J_{2j} \dot{J}_{3k} \right) \tag{13}$$

Let us consider $\dot{J}_{1i}$ first.

$$\dot{J}_{1i} = \frac{dJ_{1i}}{dp} = \frac{d}{dp}\left(\frac{\partial x_1}{\partial X_i} \right) = \frac{\partial}{\partial X_i}\left(\frac{dx_1}{dp} \right) \tag{14}$$

where the last step is due to the commutative property of spatial and parameter sensitivities. By denoting $V_1 = \left(\frac{dx_1}{dp} \right)$ and noting that it depends on $\{x, y, z\}^T$, which in turn depends on X_i , we re-expand the last term of Eq. (14) as follows.

$$\dot{J}_{1i} = \frac{\partial V_1}{\partial X_i} = \left(\frac{\partial V_1}{\partial x_l} \right) \frac{\partial x_l}{\partial X_i} = V_{1,l}\, J_{li} \tag{15}$$

Note that $V_1 = V_x = \left(\frac{dx_1}{dp} \right) = \frac{dx}{dp}$ because the subscripts 1, 2, and 3 or x, y, and z re interchangeably used depending upon the convenience of writing this down. You have to have our wits about you when you work with indicial notation.

Now, by using Eq. (15), we expand the first term of Eq. (13) as

$$\begin{aligned} \in_{ijk} \left(\dot{J}_{1i} J_{2j} J_{3k} \right) &= \in_{ijk} V_{1,l}\, J_{li} J_{2j} J_{3k} \\ &= \in_{ijk} V_{1,1} J_{1i} J_{2j} J_{3k} + \in_{ijk} V_{1,2} J_{2i} J_{2j} J_{k3} + \in_{ijk} V_{1,3} J_{3i} J_{2j} J_{3k} \end{aligned} \tag{15a}$$

By virtue of the compact representation of the determinant of a matrix given in Eq. (12), the first term in the preceding equation can be recognized as $|\mathbf{J}| V_{1,1}$. Since the J-indices in the second and third terms are repeated, we can recognize them as determinants of matrices that have identical rows, and hence they are zero. Therefore, we can re-write Eq. (13) as

$$\dot{\overline{|\mathbf{J}|}} = \frac{\in_{ijk}}{6} \left(\dot{J}_{1i} J_{2j} J_{3k} + J_{1i} \dot{J}_{2j} J_{3k} + J_{1i} J_{2j} \dot{J}_{3k} \right) = |\mathbf{J}| \left(V_{1,1} + V_{2,2} + V_{3,3} \right) = |\mathbf{J}| \left(\nabla_{\mathbf{x}} \cdot \mathbf{V} \right) \tag{16}$$

Material derivative of the Jacobian and its other forms

Sometimes we may need to take the material derivative of the Jacobian or its transpose, inverse, inverse of the transpose, or the transpose of the inverse. The indicial notation makes it easy to obtain them.

$$\dot{\mathbf{J}} = \frac{d\,\dot{\mathbf{J}}}{dp} = ?$$

Consider:

$$\dot{J}_{ij} = \frac{dJ_{ij}}{dp} = \frac{d}{dp}\left(\frac{\partial x_i}{\partial X_j}\right) = \frac{\partial}{\partial X_j}\left(\frac{dx_i}{dp}\right) = \frac{\partial V_i}{\partial X_j} = V_{i,j} \tag{17a}$$

$$\Rightarrow \dot{\mathbf{J}} = \begin{bmatrix} \frac{dJ_{11}}{dp} & \frac{dJ_{12}}{dp} & \frac{dJ_{13}}{dp} \\ \frac{dJ_{21}}{dp} & \frac{dJ_{22}}{dp} & \frac{dJ_{23}}{dp} \\ \frac{dJ_{31}}{dp} & \frac{dJ_{32}}{dp} & \frac{dJ_{33}}{dp} \end{bmatrix} = \begin{bmatrix} \frac{\partial V_1}{\partial X} & \frac{\partial V_1}{\partial Y} & \frac{\partial V_1}{\partial Z} \\ \frac{\partial V_2}{\partial X} & \frac{\partial V_2}{\partial Y} & \frac{\partial V_2}{\partial Y} \\ \frac{\partial V_3}{\partial X} & \frac{\partial V_3}{\partial Y} & \frac{\partial V_3}{\partial Z} \end{bmatrix} = \nabla_{\mathbf{X}}\mathbf{V} \tag{17b}$$

Since $\dot{J}_{ij} = V_{\mathrm{j,i}}$, we have

$$\dot{\mathbf{J}}^T = \left(\nabla_{\mathbf{X}}\mathbf{V}\right)^T = \nabla_{\mathbf{X}}\mathbf{V}^T \tag{18}$$

For $\overset{\bullet}{\overline{\mathbf{J}^{-1}}}$, consider the identity;

$$\mathbf{J}\mathbf{J}^{-1} = \mathbf{I} \tag{19}$$

By taking the material derivative of the preceding equation, we get

$$\begin{aligned} &\dot{\mathbf{J}}\,\mathbf{J}^{-1} + \mathbf{J}\,\overset{\bullet}{\overline{\mathbf{J}^{-1}}} = \mathbf{0} \\ &\Rightarrow \overset{\bullet}{\overline{\mathbf{J}^{-1}}} = -\mathbf{J}^{-1}\,\dot{\mathbf{J}}\,\mathbf{J}^{-1} = -\mathbf{J}^{-1}\left(\nabla_{\mathbf{X}}\mathbf{V}\right)\mathbf{J}^{-1} \end{aligned} \tag{20}$$

Furthermore, it may be recognized or verified by direct calculation that

$$\mathbf{J}^{-T} = \left(\mathbf{J}^{-1}\right)^T = \left(\mathbf{J}^T\right)^{-1} \tag{21}$$

Thus,

$$\overset{\bullet}{\overline{\left(\mathbf{J}^{-1}\right)^T}} = \overset{\bullet}{\overline{\left(\mathbf{J}^T\right)^{-1}}} = -\mathbf{J}^{-T}\left(\nabla_{\mathbf{X}}\mathbf{V}^T\right)\mathbf{J}^{-T} \tag{22}$$

With what we have so far, we are set to take the material derivative of domain integrals. Before that, let us apply the formulae we have derived to an example that is amenable for verification by analytical calculation.

Example

Let $\left\{\begin{array}{l} x = X(1+p) \\ y = Y(1+p^2) \\ z = Z\left(1+2p\right) \end{array}\right\}$. Find $\dot{\mathbf{J}}$, $\dot{\overline{\mathbf{J}^{-1}}}$, and $\dot{\overline{|\mathbf{J}|}}$.

Solution

We have $\mathbf{J} = \begin{bmatrix} (1+p) & 0 & 0 \\ 0 & (1+p^2) & 0 \\ 0 & 0 & (1+2p) \end{bmatrix}$ and $|\mathbf{J}| = (1+p)(1+p^2)(1+2p)$.

$\dot{\mathbf{J}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2p & 0 \\ 0 & 0 & 2 \end{bmatrix}$, which is the same as $\nabla_{\mathbf{x}}\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2p & 0 \\ 0 & 0 & 2 \end{bmatrix}$ because $\mathbf{V} = \left\{\begin{array}{c} dx/dp \\ dy/dp \\ dz/dp \end{array}\right\} = \left\{\begin{array}{c} X \\ 2Yp \\ 2Z \end{array}\right\}$.

$$\dot{\overline{\mathbf{J}^{-1}}} = \frac{d}{dp}\begin{bmatrix} \frac{1}{1+p} & 0 & 0 \\ 0 & \frac{1}{1+p^2} & 0 \\ 0 & 0 & \frac{1}{1+2p} \end{bmatrix} = \begin{bmatrix} -\frac{1}{(1+p)^2} & 0 & 0 \\ 0 & -\frac{2p}{\left(1+p^2\right)^2} & 0 \\ 0 & 0 & -\frac{2}{(1+2p)^2} \end{bmatrix}.$$

Let us see what Eq. (20) gives:

$$\dot{\overline{\mathbf{J}^{-1}}} = -\mathbf{J}^{-1}\left(\nabla_x \mathbf{V}\right)\mathbf{J}^{-1} = -\mathbf{J}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2p & 0 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{J}^{-1} = \begin{bmatrix} -\frac{1}{(1+p)^2} & 0 & 0 \\ 0 & -\frac{2p}{\left(1+p^2\right)^2} & 0 \\ 0 & 0 & -\frac{2}{(1+2p)^2} \end{bmatrix},$$

which is the same result we got directly.

Next, let us take the derivative of the determinant by direct calculation.

$$\frac{d}{dp}\left\{(1+p)(1+p^2)(1+2p)\right\} = (1+p^2)(1+2p) + (1+p)(2p)(1+2p) + 2(1+\mathrm{p})(1+\mathrm{p}^2).$$

Now, according to Eq. (16), $\dot{\overline{|\mathbf{J}|}} = |\mathbf{J}|\left(\nabla_{\mathbf{x}} \cdot \mathbf{V}\right) = (1+p)(1+p^2)(1+2p)\left(\frac{dV_1}{dx} + \frac{dV_2}{dy} + \frac{dV_3}{dz}\right)$

$$= (1+p)(1+p^2)(1+2p)\left\{\frac{dX}{dx} + \frac{d(2Yp)}{dy} + \frac{d(2Z)}{dz}\right\}$$

$$= (1+p)(1+p^2)(1+2p)\left\{\frac{d}{dx}\left(\frac{x}{1+p}\right) + \frac{d}{dy}\left(\frac{2py}{1+p^2}\right) + \frac{d}{dz}\left(\frac{2z}{1+2p}\right)\right\}$$

$$= (1+p)(1+p^2)(1+2p)\left\{\left(\frac{1}{1+p}\right) + \left(\frac{2p}{1+p^2}\right) + \left(\frac{2}{1+2p}\right)\right\}$$

$$= (1+p)(1+p^2)(1+2p)\frac{(1+p^2)(1+2p) + 2p(1+p)(1+2p) + 2(1+p)(1+p^2)}{(1+p)(1+p^2)(1+2p)}$$

$$= (1+p^2)(1+2p) + 2p(1+p)(1+2p) + 2(1+p)(1+p^2)$$

which is the same result obtained with direct calculation of the derivative. This example makes it clear that the derivative in $\left(\nabla_{\mathbf{x}} \cdot \mathbf{V}\right)$ to be taken in Eq. (16) is with respect to the spatial coordinates and not material coordinates. On the other hand, in Eqs. (17b), (18) and (22), the derivative is with respect to the material coordinates and not the spatial coordinates. Watch out when you use these formulae.

<u>Formulae to take note and possibly to commit to memory</u>

$$\dot{\overline{|\mathbf{J}|}} = |\mathbf{J}|\left(\nabla_{\mathbf{x}} \cdot \mathbf{V}\right)$$

$$\dot{\mathbf{J}} = \nabla_{\mathbf{X}}\mathbf{V}$$

$$\dot{\mathbf{J}}^T = \nabla_{\mathbf{X}}\mathbf{V}^T$$

$$\dot{\overline{\mathbf{J}^{-1}}} = -\mathbf{J}^{-1}\left(\nabla_X \mathbf{V}\right)\mathbf{J}^{-1}$$

$$\dot{\overline{\left(\mathbf{J}^{-1}\right)^T}} = \dot{\overline{\left(\mathbf{J}^T\right)^{-1}}} = -\mathbf{J}^{-T}\left(\nabla_{\mathbf{X}}\mathbf{V}^T\right)\mathbf{J}^{-T}$$