Lecture 12

Euler-Lagrange Equations
and their Extension to
Multiple Functions and Multiple Derivatives
in the integrand of the Functional

ME 260, Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

G. K. Ananthasuresh

Professor, Mechanical Engineering, Indian Institute of Science, Bengaluru

suresh@iisc.ac.in

Outline of the lecture

Review of functional, vector spaces, Gateaux variation

Euler-Lagrange equations

Boundary conditions

Multiple functions

Multiple derivatives

What we will learn:

First variation + integration by parts + fundamental lemma = Euler-Lagrange equations

How to derive boundary conditions (essential and natural)

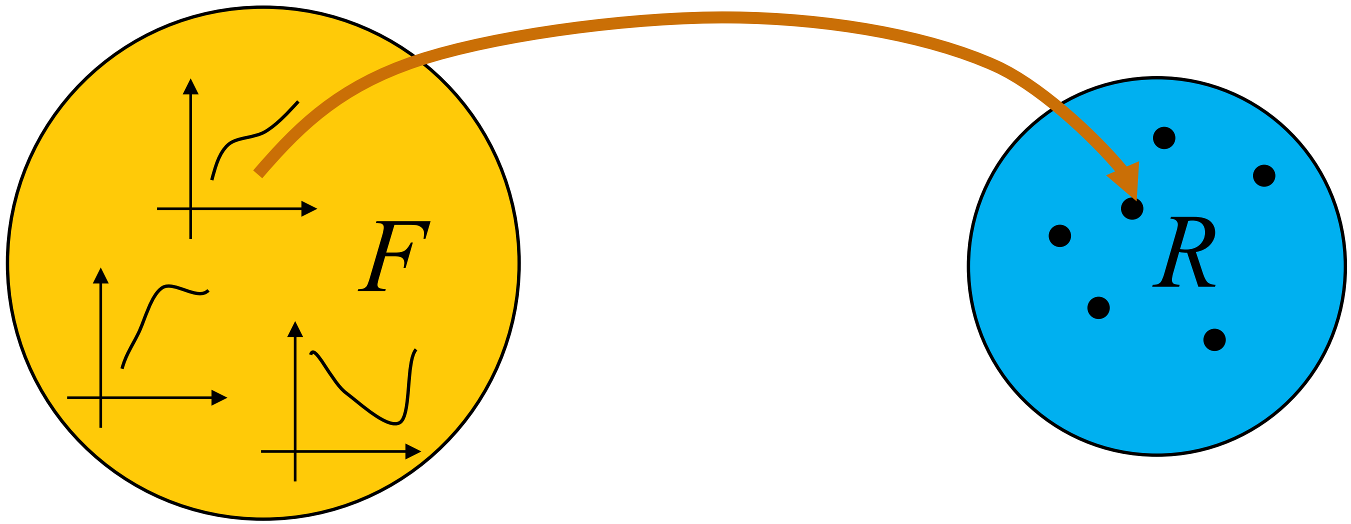
How to deal with multiple functions and multiple derivatives

Generality of Euler-Lagrange equations

Functional

A functional *J* is a mapping from a function space *F* to real number space *R*.

$$J : F \to R$$



A vector space of functions

Real number space

A vector space

Vector addition $\oplus$
Scalar multiplication $\odot$

1. Closed under vector addition

$$x \oplus y \in X \text{ for all } x, y \in X$$

2. Commutative law for addition

$$x \oplus y = y \oplus x.$$

3. Associative law for addition

$$\left(x \oplus y\right) \oplus z = x \oplus \left(y \oplus z\right)$$

4. Additive identity

$$x \oplus \theta = \theta \oplus x = x \text{ for all } x \in X$$

5. Additive inverse

$$x \oplus x' = x' \oplus x = \theta$$

6. Closed under scalar multiplication

$$\text{For all } \alpha \in K, \text{ and all } x \in X, \ \alpha \odot \mathrm{x} \in \mathrm{X}$$

7. Multiplicative identity

$$1 \odot x = x; \qquad \text{and } \left(0 \odot x \in \theta\right)$$

8. Associate law for scalar multiplication

$$\left(\alpha\beta\right) \odot x = \alpha \odot \left(\beta \odot x\right)$$

9 and 10. Distributive laws for scalar multiplication and vector addition

$$\text{For all } \alpha \in K, \text{ and all } x, y \in X, \quad \alpha \odot \left(\mathrm{x} \oplus \mathrm{y}\right) = \left(\alpha \odot x\right) \oplus \left(\alpha \odot y\right)$$

$$\left(\alpha + \beta\right) \odot x = \left(\alpha \odot x\right) + \left(\beta \odot x\right) \qquad \alpha, \beta \in K, \quad x \in X$$

A function space is a vector space.

- Two functions can be added.
- A function can be multiplied by a scalar.
- Null function exists.
- Unit function exists.
- Negative of a function exists.
- Inverse of a function exists.

- 10 rules in the previous slide ought to be satisfied by a vector space for which vector addition and scalar multiplication are defined.

What kind of functions?

- Differentiable?
- Continuous?

Norm of a function

(i) $\|f\| \geq 0$ for all $f \in F$

(ii) $\|f\| = 0$ if and only if $f = \theta$

(iii) $\|\alpha f\| = |\alpha| \|f\|$ $\alpha \in R,\ f \in F$

(iv) $\|f + g\| \leq \|f\| + \|g\|$ $f, g \in F$

Why do we need a norm?

Inner product

(i) $\langle f+g,\mathrm{h}\rangle = \langle \mathrm{f},\mathrm{h}\rangle + \langle \mathrm{g},\mathrm{h}\rangle$

(ii) $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$

(iii) $\langle f, g\rangle = \langle \overline{f, g}\rangle$

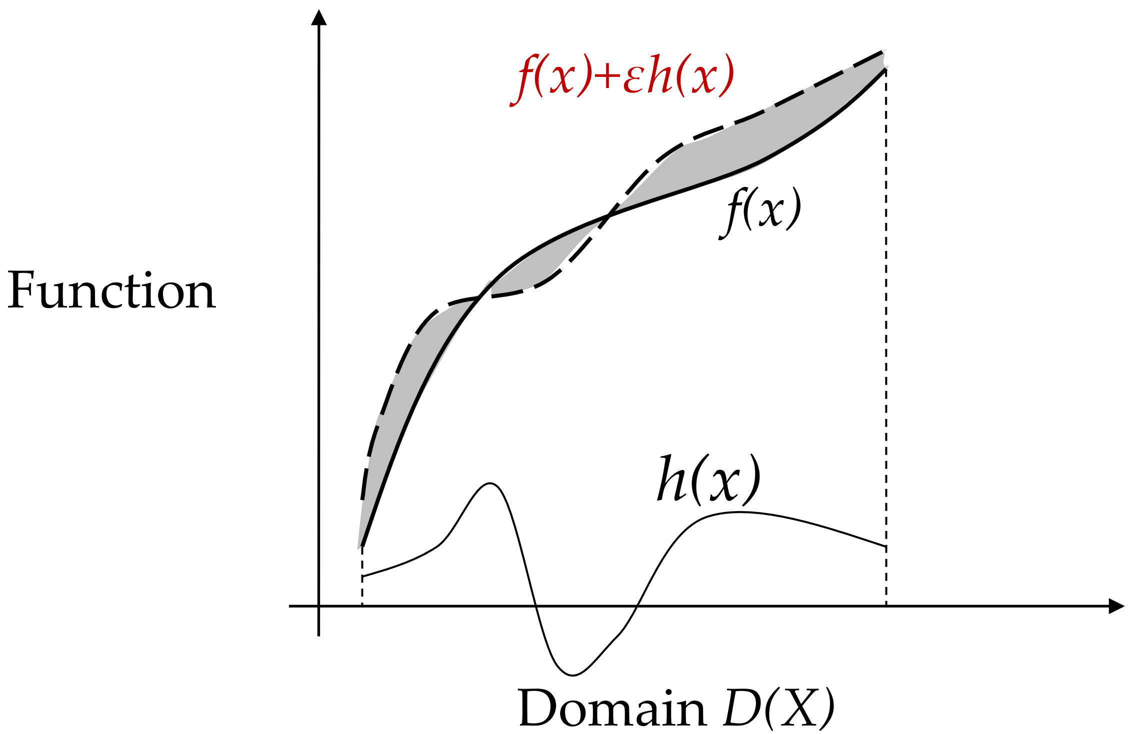
The over bar denotes conjugation and is not necessary if x, y are real.

(iv) $\langle f, f\rangle \geq 0$ and

$\langle f, f\rangle = 0$ if and only if $f = \theta$

Why do we need an inner product?

The concept of variation of a function



Variation of a functional

$$\delta J(x;h) = \lim_{\varepsilon \to 0} \frac{J(x+\varepsilon h) - J(x)}{\varepsilon}$$

Apply L'Hospital's rule to get:

$$\delta J(x;h) = \frac{d}{d\varepsilon} J(x+\varepsilon h)\Big|_{\varepsilon=0}$$

Convenient for evaluating the variation.

The simplest functional, *F(y,y')*

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\Big(y(x), y'(x)\Big)\, dx$$

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right\} dx = 0$$ First variation of *J* w.r.t. *y(x)*.

The condition given above should hold good for any variation of y(x), i.e., for any δy

But there is $\delta y'$, which we will get rid of it through integration by parts.

Integration by parts…

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right\} dx = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y'}\delta y' \right\} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \frac{\partial F}{\partial y'}\delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\delta y \right\} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right\} \delta y \, dx + \frac{\partial F}{\partial y'}\delta y \Big|_{x_1}^{x_2} = 0$$

We can invoke fundamental lemma of calculus of variations now.

Fundamental lemma…

$$\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right\} \delta y \, dx + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_1}^{x_2} = 0$$

$$\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right\} \delta y \, dx = 0 \quad \text{and} \quad \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_1}^{x_2} = 0$$

The two terms are equated to zero because the first term depends on the entire function whereas the second term only on the value of the function at the ends.

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0 \quad x \in (x_1, x_2)$$

The integral should be zero for any value of δy. So, by fundamental lemma (Lecture 10), the integrand should be zero at every point in the domain.

Boundary conditions

$$\left.\frac{\partial F}{\partial y'}\delta y\right|_{x_1}^{x_2} = 0$$

$$\Rightarrow \left.\frac{\partial F}{\partial y'}\delta y\right|_{x_2} - \left.\frac{\partial F}{\partial y'}\delta y\right|_{x_1} = 0$$

$$\frac{\partial F}{\partial y'}\delta y = 0$$

$$\Rightarrow \frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0$$

The algebraic sum of the two terms may be zero without the two terms being equal to zero individually. We will see those cases later. For now, we will take the general case of both terms individually being equal to zero.

Thus,

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_1$$

and

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_2$$

Euler-Lagrange (EL) equation with boundary conditions

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x)\Big)\,dx$$

Problem statement

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad x \in (x_1, x_2)$$

Differential equation

and

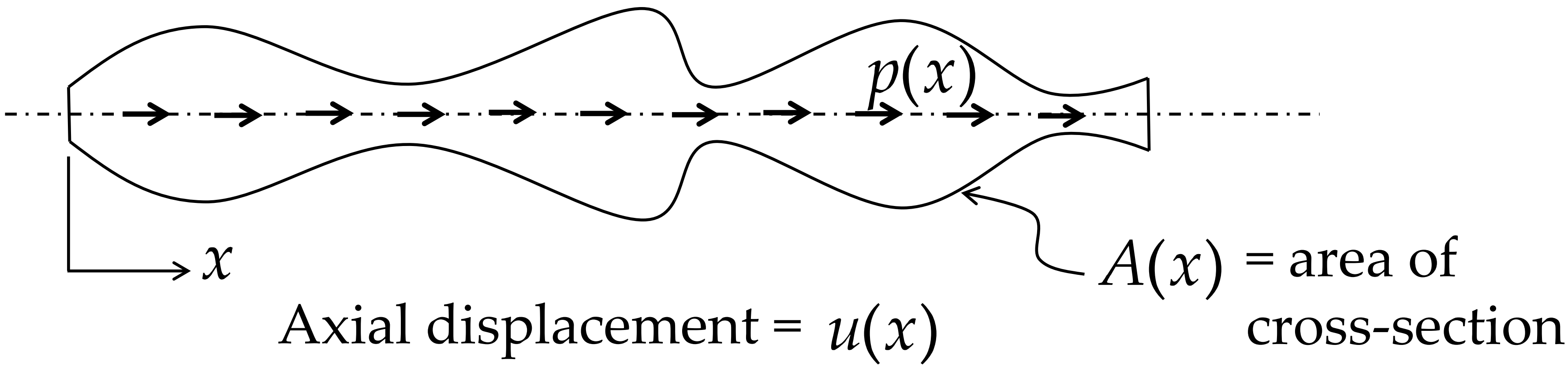
$\frac{\partial F}{\partial y'} = 0$ or $\delta y = 0$ at $x = x_1$

and

$\frac{\partial F}{\partial y'} = 0$ or $\delta y = 0$ at $x = x_2$

Boundary conditions

Example 1: a bar under axial load



Strain energy — Work potential

$$\underset{u(x)}{\text{Min}} \;\; PE = \int_0^L \left(\frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x) \right) dx$$

Data: $L, E(x), A(x), p(x)$

Principle of minimum potential energy (PE)

Among all possible axial displacement functions, the one that minimizes PE is the stable static equilibrium solution.

Bar problem: E-L equation

$$\underset{u(x)}{\text{Min}}\;\; PE = \int_0^L \left(\frac{1}{2} E(x)A(x)\big(u'(x)\big)^2 - p(x)\, u(x) \right) dx$$

$$F = \frac{1}{2} E(x)A(x)\big(u'(x)\big)^2 - p(x)\, u(x)$$ Integrand of the PE

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad x \in (0, L)$$

$$\Rightarrow -p - \frac{d}{dx}\big(EAu'\big)$$

$$\Rightarrow \big(EAu'\big)' + p = 0$$ Governing differential equation

Bar problem: boundary conditions

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_1$$

and

$$\frac{\partial F}{\partial y'} = 0 \text{ or } \delta y = 0 \text{ at } x = x_2$$

$$F = \frac{1}{2} E(x) A(x) \left(u'(x)\right)^2 - p(x)\, u(x)$$

$$EAu' = 0 \text{ or } \delta u = 0 \text{ at } x = 0$$

and

$$EAu' \text{ or } \delta u = 0 \text{ at } x = L$$

$\delta u = 0$

This means that y is specified; hence, its variation is zero. This is called the **essential** or **Dirichlet** boundary condition.

$EAu' = 0$

This means that the stress is zero when the displacement is not specified. It is called the **natural** or **Neumann** boundary condition.

Weak form of the governing equation

$$\underset{u(x)}{\text{Min}}\ PE = \int_0^L \left(\frac{1}{2} E(x) A(x) \left(u'(x) \right)^2 - p(x)\, u(x) \right) dx$$

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right\} dx = 0$$

First variation is zero.

$$\delta_u PE = \int_0^L \left(E(x) A(x) u'(x) \delta u' - p(x)\, \delta u \right) dx = 0 \ \text{ for any } \delta u$$

$$\int_0^L \left(E(x) A(x) u'(x) \delta u' \right) dx = \int_0^L \left(p(x)\, \delta u \right) dx$$

Internal virtual work = external virtual work

δu
Variation of u is like virtual displacement.

Three ways for static equilibrium

$$\underset{u(x)}{\text{Min}}\; PE = \int_0^L \left(\frac{1}{2} E(x)A(x)\left(u'(x)\right)^2 - p(x)\, u(x) \right) dx$$

Minimum potential energy principle

$$\delta_y PE = \int_0^L \left(E(x)A(x)\delta u' - p(x)\, \delta u \right) dx = 0 \;\; \text{for any } \delta u$$

Principle of virtual work;
The weak form

$$\left(EAu'\right)' + p = 0$$

$$EAu' = 0 \;\text{ or }\; \delta u = 0 \;\text{ at }\; x = 0$$

and

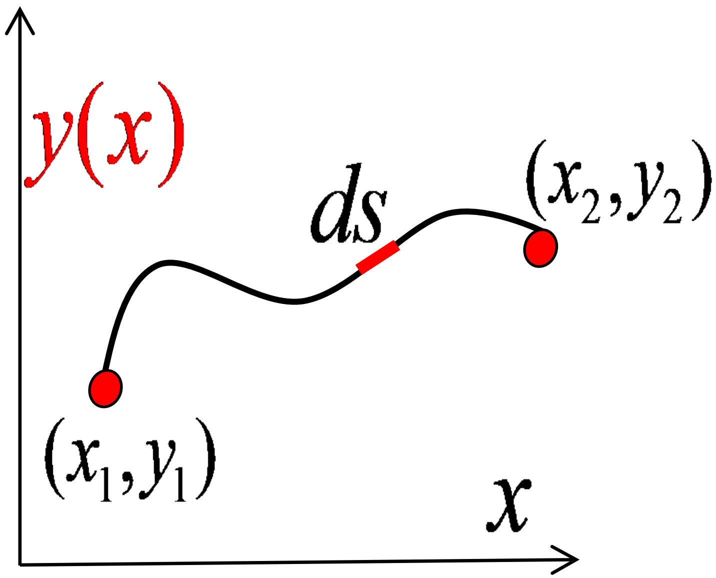
$$EAu' \;\text{ or }\; \delta u = 0 \;\text{ at }\; x = L$$

Force balance;
And boundary conditions.
The strong form.

Q: What is "weak" about the weak form?
A: It needs derivative of one less order.

Example 2: is a straight line really the least-distance curve in a plane?

From Slide 7 in Lecture 3



$$\underset{y(x)}{\text{Min}}\ L=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$$

$$\text{Data}: x_1,\ x_2,\ y(x_1)=y_1,\ y(x_2)=y_2$$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \quad x\in(x_1,x_2)$$

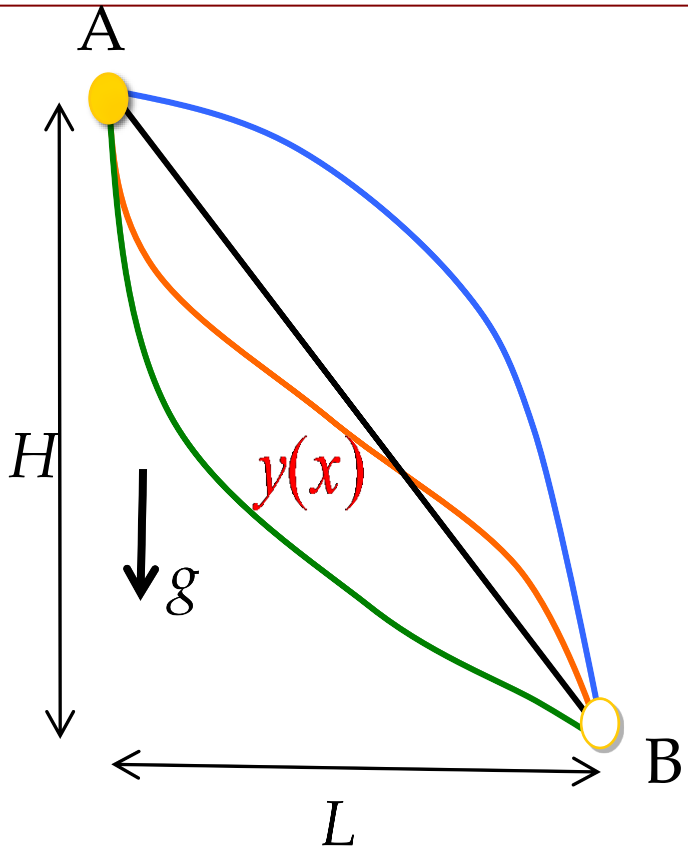
$$\Rightarrow 0-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)=0$$

$$\frac{y'}{\sqrt{1+y'^2}}=C \quad \Rightarrow \quad y'=\text{constant}$$

So, straight line is indeed the geodesic in a plane.

Example 3: Brachistochrone problem

From Slide 11 in Lecture 2



Minimize $T = \int_0^L \frac{\sqrt{1+(y')^2}}{\sqrt{2g(H-y)}}dx$

$y(x)$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\sqrt{\frac{1+y'^2}{8g}}\frac{1}{(H-y)^{3/2}} - \left(\frac{y'}{\sqrt{2g(1+y'^2)(H-y)}}\right)' = 0$$

And we have Dirichlet (essential) boundary conditions at both the ends.

A functional with two derivatives: *F(y,y',y'')*

$$\underset{y(x)}{\text{Min}} \;\; J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), y''(x)\Big)dx$$

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' \right\} dx = 0$$

First variation of J w.r.t. $y(x)$.

We now need to integrate by parts twice to get rid of the second derivative of y.

Integration by parts… twice!

$$\delta_y J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' \right\} dx = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y'}\delta y' \right\} dx + \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y''}\delta y'' \right\} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y}\delta y \right\} dx + \frac{\partial F}{\partial y'}\delta y \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y \right\} dx + \frac{\partial F}{\partial y''}\delta y' \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y' \right\} dx = 0$$

$$\Rightarrow \underbrace{\int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \right\} \delta y \, dx}_{} + \underbrace{\left(\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \right) \delta y \Big|_{x_1}^{x_2} + \frac{\partial F}{\partial y''}\delta y' \Big|_{x_1}^{x_2}}_{} = 0$$

= 0 gives differential equation by using the fundamental lemma.

Two sets of boundary conditions

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \quad \text{for } x \in (x_1, x_2)$$

E-L equation and BCs for *F(y,y',y'')*

$$\underset{y(x)}{\text{Min}} \;\; J=\int_{x_1}^{x_2} F\Big(y(x),y'(x),y''(x)\Big)\,dx$$

Things are getting lengthy;
Let us use short-hand notation.

$$\frac{\partial F}{\partial y}=F_y;\;\frac{\partial F}{\partial y'}=F_{y'};\;\frac{\partial F}{\partial y''}=F_{y''}$$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0 \;\text{ for } x\in(x_1,x_2)$$

$$F_y-(F_{y'})'+(F_{y''})''=0$$

$$\left(\frac{\partial F}{\partial y'}-\frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right)\delta y\Bigg|_{x_1}^{x_2}=0$$

$$\left(F_{y'}-\left(F_{y''}\right)'\right)\delta y\Bigg|_{x_1}^{x_2}=0$$

and

$$\frac{\partial F}{\partial y''}\delta y'\Bigg|_{x_1}^{x_2}=0$$

and

$$F_{y''}\delta y'\Big|_{x_1}^{x_2}=0$$

Example 4: beam deformation

From Slide 27 in Lecture 3

$$\underset{w(x)}{\text{Min}} \ PE = \int_0^L \left\{ \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx$$

$$\text{Data}: q(x), E, I$$

$$F = \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw$$

$$w(x)$$

$$F_y - (F_{y'})' + (F_{y''})'' = 0$$

$$-q - 0 + \left(EIw'' \right)'' = 0$$

$$\Rightarrow \left(EIw'' \right)'' = q$$

When E and I are uniform, we get the familiar: $EIw^{iv} = q$

Boundary conditions for the beam

$$F = \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw$$

$w(x)$

$$\left(F_{y'} - \left(F_{y''} \right)' \right) \delta y \Big|_{x_1}^{x_2} = 0$$

$$\left(EIw'' \right)' \delta w \Big|_0^L = 0$$

and

$$F_{y''} \delta y' \Big|_{x_1}^{x_2} = 0$$

$$\left(EIw'' \right) \delta w' \Big|_0^L = 0$$

Physical interpretation

Either shear stress is zero or the transverse displacement is specified.

Either bending moment is zero or the slope is specified.

Do we see a trend for multiple derivatives in the functional?

$$\underset{y(x)}{\text{Min}} \;\; J = \int_{x_1}^{x_2} F\Big(y(x), y'(x)\Big)dx$$

$$F_y - \Big(F_{y'}\Big)' = 0$$

$$\Big(F_{y'}\Big)\delta y\Big|_{x_1}^{x_2} = 0$$

$$\underset{y(x)}{\text{Min}} \;\; J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), y''(x)\Big)dx$$

$$F_y - (F_{y'})' + (F_{y''})'' = 0$$

$$\left(F_{y'} - \Big(F_{y''}\Big)'\right)\delta y\Big|_{x_1}^{x_2} = 0$$

and

$$F_{y''}\delta y'\Big|_{x_1}^{x_2} = 0$$

Three derivatives… *F(y,y',y'',y''')*

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), y''(x), y'''(x)\Big)dx$$

$$F_y - (F_{y'})' + (F_{y''})'' - (F_{y'''})''' = 0$$

$$\left(F_{y'} - \left(F_{y''}\right)' + \left(F_{y'''}\right)''\right)\delta y\Big|_{x_1}^{x_2} = 0 \;, \qquad \left(F_{y''} - \left(F_{y'''}\right)'\right)\delta y'\Big|_{x_1}^{x_2} = 0 \;\; \text{and}$$

$$F_{y'''}\delta y''\Big|_{x_1}^{x_2} = 0$$

Many derivatives… $F(y,y',y'',\dots y^{(n)})$

$$\underset{y(x)}{\text{Min}}\ J=\int_{x_1}^{x_2} F\left(y(x),y'(x),y''(x),\cdots,y^{(n)}(x)\right)dx$$

$$F_y-(F_{y'})'+(F_{y''})''-(F_{y'''})'''+\ldots=\sum_{i=0}^{n}\left(-1\right)^{i}\left(F_{y^{(i)}}\right)^{(i)}=0$$

$$\left(\sum_{i=j}^{n}\left(-1\right)^{i-j}\left(F_{y^{(i)}}\right)^{(i-j)}\right)\delta y^{(j-1)}=0 \quad \text{for } j=1,2,\cdots n$$

Most general form with one function and many derivatives

What if we have two functions?

$$\underset{y_1(x),y_2(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y_1(x), y_1'(x), y_2(x), y_2'(x)\Big)dx$$

$$\delta_{y_1} J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_1'}\delta y_1' \right\} dx = 0$$

$$\delta_{y_2} J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y_2}\delta y_2 + \frac{\partial F}{\partial y_2'}\delta y_2' \right\} dx = 0$$

Now, we need to take the first variation with respect to both the functions, **separately**.

What if we have two functions? (contd.)

$$\underset{y_1(x),y_2(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y_1(x), y_1'(x), y_2(x), y_2'(x)\Big)dx$$

$$F_{y_1} - \left(F_{y_1'}\right)' = 0 \quad \text{and} \quad \left(F_{y_1'}\right)\delta y_1\Big|_{x_1}^{x_2} = 0$$

$$F_{y_2} - \left(F_{y_2'}\right)' = 0 \quad \text{and} \quad \left(F_{y_2'}\right)\delta y_2\Big|_{x_1}^{x_2} = 0$$

And, we will have two differential equations and two sets of boundary conditions.

Two unknown functions need two differential equations and two sets of BCs. That is all!

Most general form: *m* functions with *n* derivatives.

The most general form when we have one independent variable x.

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\left(y_1, y_1', \cdots, y_1^{(n)}, y_2, y_2', \cdots, y_2^{(n)}, \cdots, y_m, y_m', \cdots, y_m^{(n)}\right) dx$$

$$\left.\begin{aligned} &F_{y_k} - (F_{y_k'})' + (F_{y_k''})'' - (F_{y_k'''})''' + \ldots = \sum_{i=0}^{n} (-1)^i \left(F_{y_k^{(i)}}\right)^{(i)} = 0 \\ &\left(\sum_{i=j}^{n} (-1)^{i-j} \left(F_{y_k^{(i)}}\right)^{(i-j)}\right) \delta y_k^{(j-1)} = 0 \quad \text{for } j = 1, 2, \cdots n \end{aligned}\right\} k = 1, 2, \cdots, m$$

The end note

Euler-Lagrange equations and their extension to multiple functions and multiple derivatives

- Euler-Lagrange equations = first variation + integration by parts + fundamental lemma
- Boundary conditions
  Essential (Dirichlet)
  Natural (Neumann)
- Dealing with multiple derivatives along with boundary conditions (need to do integration by parts as many times as the order of the highest derivative)
- Dealing with multiple functions (rather easy)
- **General form of Euler-Lagrange equations in one independent variable**

Thanks