Lecture 13

Calculus of Variations with Functionals Involving Two and Three Independent Variables

ME 260, Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

G. K. Ananthasuresh

Professor, Mechanical Engineering, Indian Institute of Science, Banagalore

suresh@iisc.ac.in

Outline of the lecture

Functionals with two and three independent variables

Green and Gauss theorems for “integration by parts” in 2D and 3D

Euler-Lagrange equations

Boundary conditions

What we will learn:

How to deal with two and three independent variables.

Applying the divergence theorem to derive boundary conditions along with the differential equation.

Examples

How to deal with any unconstrained calculus of variations problems.

Functional with two independent variables, x and y

$$\underset{z(x,y)}{\text{Min}} \; J = \int_{y_1}^{y_2}\int_{x_1}^{x_2} F\left(z, z_x, z_y\right) dx\,dy = \int_S F\left(z, z_x, z_y\right) dS$$

S = closed 2D domain in the xy plane.

$$\delta_z J = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \left\{ \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial z_x}\delta z_x + \frac{\partial F}{\partial z_y}\delta z_y \right\} dxdy = 0$$

$$z_x = \frac{\partial z}{\partial x}; \quad z_y = \frac{\partial z}{\partial y}$$

Notation.

Now, we need to get rid of δz_x and δz_y and get everything in terms of δz.

A little trick to deal with δz_x and δz_y

$$\frac{\partial F}{\partial z_x}\delta z_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\delta z\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right)\delta z$$

$$\frac{\partial F}{\partial z_y}\delta z_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\delta z\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right)\delta z$$

$$\delta_z J = \int_S \left\{ \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial z_x}\delta z_x + \frac{\partial F}{\partial z_y}\delta z_y \right\} dS = 0$$

$$\Rightarrow \int_S \left\{ \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) \right\} \delta z \, dS + \int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\delta z\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\delta z\right) \right\} dS = 0$$

Suitable for the application of the fundamental lemma.

E-L equation for $F(z, z_x, z_y)$

$$\int_S \left\{ \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) \right\} \delta z \, dS = 0 \quad \text{for any} \quad \delta z$$

$$\Rightarrow \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

Thus, writing the Euler-Lagrange equation follows the same pattern as before. It is quite straightforward.

It is the boundary condition term that requires special attention.

We had done integration by parts in the case of one independent variable. Now also, we will do the same but …

Boundary condition of $F(z, z_x, z_y)$

$$\int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\delta z \right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\delta z \right) \right\} dS = 0$$

Green's theorem is the equivalent of integration by parts in the two-variables case

$$\int_S \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dS = \int_{\partial S} \left(P dx + Q dy \right)$$

∂S is the boundary and this is the boundary condition.

$$\int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\delta z \right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\delta z \right) \right\} dS = \int_{\partial S} \left(-\frac{\partial F}{\partial z_y} dx + \frac{\partial F}{\partial z_x} dy \right) \delta z = 0$$

Boundary condition of $F(z, z_x, z_y)$

$$\int_{\partial S} \left(-\frac{\partial F}{\partial z_y} dx + \frac{\partial F}{\partial z_x} dy \right) \delta z = 0$$

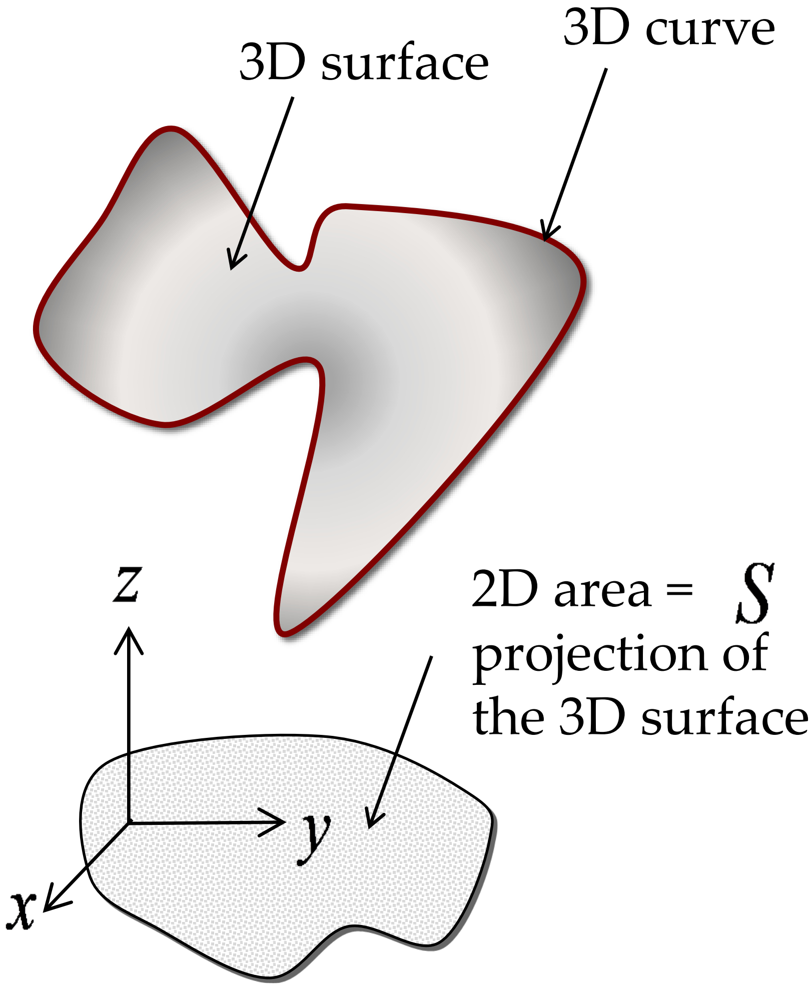
If *z(x,y)* is specified at a point on the boundary, the variation of *z* is zero there. So, the boundary condition is satisfied there.

$$\left(-\frac{\partial F}{\partial z_y} dx + \frac{\partial F}{\partial z_x} dy \right) = 0$$

$$\Rightarrow \frac{dy}{dx} = y' = \frac{\frac{\partial F}{\partial z_y}}{\frac{\partial F}{\partial z_x}}$$

at a point on the boundary where *z(x,y)* is not specified on the boundary.

Example 1: Minimal surface spanned by a given closed curve



$$\underset{z(x,y)}{\text{Min}} \quad A == \int_S \sqrt{1+z_x^2+z_y^2}\, dS$$

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0$$

(continued on the next slide)

Minimal surface (soap film) problem

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)=0$$

$$\Rightarrow z_{xx}\left(1+z_y^2\right)-2z_{xy}z_x z_y+z_{yy}\left(1+z_x^2\right)=0$$

Boundary condition is trivial here because the boundary is specified. Hence, $\delta z=0$

This equation shows that the mean curvature (if you know how it looks like) of the minimal surface is zero.

Note that calculus of variations gives only the differential equation and the boundary conditions but not the solution.

You have to use your usual bag of tricks to solve them!

What if second derivatives are present in two independent variables?

$$\underset{z(x,y)}{\text{Min}} \; J == \int_S F\left(z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}\right) dS$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2}\,;\; z_{yy} = \frac{\partial^2 z}{\partial y^2}$$

$$z_{xy} = \frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$$

S = closed 2D domain in the *xy* plane.

$$\delta_z J = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial z_x}\delta z_x + \frac{\partial F}{\partial z_y}\delta z_y + \frac{\partial F}{\partial z_{xx}}\delta z_{xx} + \frac{\partial F}{\partial z_{xy}}\delta z_{xy} + \frac{\partial F}{\partial z_{yy}}\delta z_{yy} \right\} dx\, dy = 0$$

$\left.\begin{matrix}\delta z_x \\ \delta z_y\end{matrix}\right\}$ We had used Green's theorem once to get rid of these.

$\left.\begin{matrix}\delta z_{xx} \\ \delta z_{xy} \\ \delta z_{yy}\end{matrix}\right\}$ Now, we need to apply the Green's theorem **twice.** Just like we had done for the single independent variable case in Slide 16 in Lecture 11.

The same little trick for δz_{xx}, δz_{xy}, and δz_{yy}

$$\frac{\partial F}{\partial z_{xx}}\delta z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\delta z_{x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\right)\delta z_{x}$$

$$\frac{\partial F}{\partial z_{yy}}\delta z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\delta z_{y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\right)\delta z_{y}$$

$$\frac{\partial F}{\partial z_{xy}}\delta z_{xy} = \frac{1}{2}\left\{\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\delta z_{x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z_{x}\right\} + \frac{1}{2}\left\{\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\delta z_{y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z_{y}\right\}$$

With the above re-arrangements,

$$\delta_z J = \int_{x_1}^{x_2}\left\{\frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial z_x}\delta z_x + \frac{\partial F}{\partial z_y}\delta z_y + \frac{\partial F}{\partial z_{xx}}\delta z_{xx} + \frac{\partial F}{\partial z_{xy}}\delta z_{xy} + \frac{\partial F}{\partial z_{yy}}\delta z_{yy}\right\} dx\, dy = 0$$

becomes…

Tedious substitutions and expansions…

$$\delta_z J = \int_S \left\{ \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right)\right\} \delta z \, dS + \int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\delta z\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\delta z\right)\right\} dS$$

$$- \int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\right)\delta z_x + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z_x + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z_y + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\right)\delta z_y \right\} dS$$

$$+ \int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\delta z_x\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\delta z_x\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\delta z_y\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\delta z_y\right)\right\} dS$$

Black part is ready for application of the fundamental lemma and thereby get the differential equation.

Red part needs another step of re-arrangement to get rid of first derivatives of variations of z.

Blue parts go to the boundary term.

Splitting of terms… once again.

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\right)\delta z_x = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\right)\delta z\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{xx}}\right)\delta z$$

$$\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z_y = \frac{1}{2}\left\{\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z\right\}$$

$$\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z_x = \frac{1}{2}\left\{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z\right\}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\right)\delta z_y = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\right)\delta z\right) - \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial z_{yy}}\right)\delta z$$

Split-terms of Slide 13 into Slide 12…

$$\delta_z J = \int_S \left\{ \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{xx}}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial F}{\partial z_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial z_{yy}}\right) \right\} \delta z \, dS$$

$$+ \int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\delta z\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\delta z\right) \right\} dS$$

$$+ \int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\delta z_x\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\delta z_x\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\delta z_y\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\delta z_y\right) \right\} dS$$

$$+ \int_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xx}}\right)\delta z\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{xy}}\right)\delta z\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_{yy}}\right)\delta z\right) \right\} dS$$

Font becomes smaller as equations become lengthier ☹

Black part is ready for application of the fundamental lemma and thereby get the differential equation.

Blue parts go to the boundary term. The last line of terms are the additional boundary terms of the second re-arrangement step. Now, these are ready for the application of the Green's theorem.

Finally… E-L equations for…

$$\underset{z(x,y)}{\text{Min}} \quad J == \int_S F\left(z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}\right) dS$$

By applying the fundamental lemma to…

$$\int_S \left\{ \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{xx}}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial F}{\partial z_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial z_{yy}}\right) \right\} \delta z \, dS = 0$$

we get the Euler-Lagrange equation:

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{xx}}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial F}{\partial z_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial z_{yy}}\right) = 0$$

Don’t you see a pattern here too?

Boundary terms

Collect terms containing these two from Slide 13 and apply the Green's theorem.

$$\frac{\partial}{\partial x}(\ ) \quad \frac{\partial}{\partial y}(\ )$$

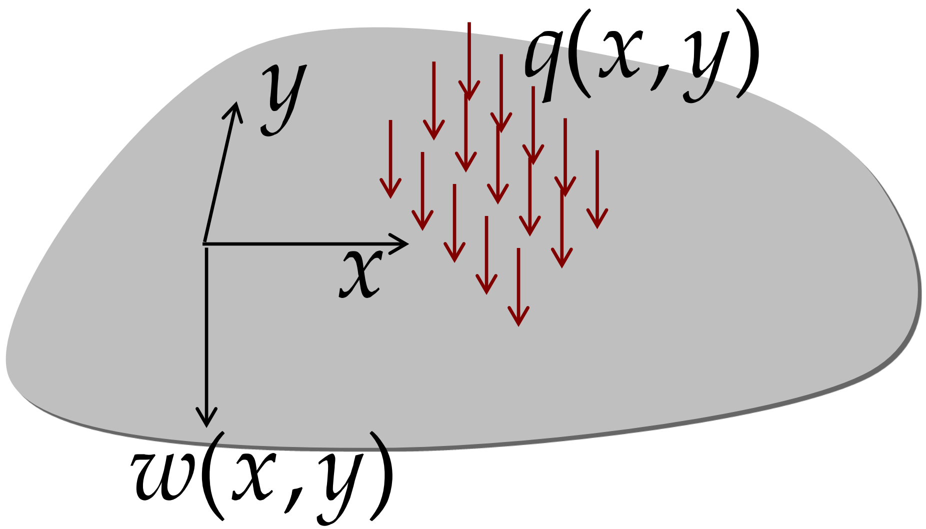
Then, we will get:

$$\int_{\partial S} (A)\delta z + \int_{\partial S} (B)\delta z_x + \int_{\partial S} (C)\delta z_y = 0$$

Write A, B, and C yourself!

Make each of the terms above go to zero.
Since we had second derivatives in the functional, we can specify the first derivatives of z here.
An example will make it clear what this means…

Example 2: deformation of a plate



A plate subjected to a transverse load *q(x,y)*. Its deformation *w(x,y)* can be determined by minimizing the potential energy.

Here, the potential energy is the functional and it depends on two independent variables, namely, *x* and *y*. It involves second derivatives of *w(x,y)*.

$$\underset{w(x,y)}{\text{Min}} \; PE = \int_S \left[\frac{D}{2} \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right\} - qw \right] dx\, dy$$

$$Data: D = \frac{2t^3 E}{3(1-\nu)^2}, t, E, \nu, q, S$$

Compare with the functional in slide 10.

Euler-Lagrange equation for a plate

$$\frac{\partial F}{\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial z_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial z_y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial z_{xx}}\right)+\frac{\partial^2}{\partial y\,\partial x}\left(\frac{\partial F}{\partial z_{xy}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial z_{yy}}\right)=0$$

$$F=\left[\frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\}\right\}-qw\right]$$

$$z=w$$

$$\frac{\partial F}{\partial w}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right)+\frac{\partial^2}{\partial y\,\partial x}\left(\frac{\partial F}{\partial w_{xy}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right)=0$$

$$\Rightarrow D\nabla^4 w = D\left\{\frac{\partial^2}{\partial x^2}\left(w_{xx}+w_{yy}\right)+\frac{\partial^2}{\partial y^2}\left(w_{xx}+w_{yy}\right)\right\}=q$$

Note that it is a fourth degree differential equation.

Boundary conditions for a plate

From slide 16

$$\int_{\partial S} (\quad)\delta w + \int_{\partial S} (\quad)\delta w_x + \int_{\partial S} (\quad)\delta w_y = 0$$

A plate may be fixed on a portion of the boundary. Then, $\delta w = 0$

It may not be allowed to bend on a portion of the boundary. Then,

$\delta w_x = 0$ or $\delta w_y = 0$ Or a linear combination of these may be zero.

The terms in the brackets will be zero when displacement or slope are not restricted… just like in a beam.

Functional with three independent variables, x_1, x_2, and x_3

$$\underset{u(x,y,z)}{\text{Min}} \quad J = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} F\left(u,u_x,u_y,u_z\right) dx\,dy\,dz = \int_V F\left(u,u_x,u_y,u_z\right) dV$$

$$\delta_u J = \int_V \left\{ \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z \right\} dV = 0$$

We need to do equivalent of integration by parts in three dimension now. The Green's theorem was integration by parts for two dimensions.

The Gauss divergence theorem is "integration by parts" for three dimensions!

Splitting of terms… as before.

$$\frac{\partial F}{\partial u_x}\delta u_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)\delta u$$

$$\frac{\partial F}{\partial u_y}\delta u_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\delta u$$

$$\frac{\partial F}{\partial u_z}\delta u_z = \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right)\delta u$$

Red goes to boundary term And blue to the differential equation.

Substitution leads to…

$$\delta_u J = \int_V \left\{ \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial u_z}\delta u_z \right\} dV = 0$$

$$\Rightarrow \int_V \left\{ \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) \right\} \delta u \, dV$$

$$+ \int_V \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\delta u\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\delta u\right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\delta u\right) \right\} dV = 0$$

Ready for application of the fundamental lemma

Needs the application of the divergence theorem.

Application of divergence theorem

$$\int_V (\nabla \cdot \mathbf{U})\, dV = \int_S (\mathbf{U} \cdot \mathbf{n})\, dS \longleftarrow$$

Divergence theorem. **n** is the unit outer normal to the surface S that encloses volume V.

$$\int_V \left\{ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x} \delta u \right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y} \delta u \right) + \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z} \delta u \right) \right\} dV$$

$$= \int_V \left[\nabla \cdot \left\{ \left(\frac{\partial F}{\partial u_x} \hat{i} + \frac{\partial F}{\partial u_y} \hat{j} + \frac{\partial F}{\partial u_z} \hat{k} \right) \delta u \right\} \right] dV$$

$$= \int_S \left\{ \left(\frac{\partial F}{\partial u_x} \hat{i} + \frac{\partial F}{\partial u_y} \hat{j} + \frac{\partial F}{\partial u_z} \hat{k} \right) \cdot \mathbf{n} \right\} \delta u \, dS$$

Now, the application of the fundamental lemma gives the condition for the boundary.

EL equation and BC for the 3D case

$$\underset{u(x,y,z)}{\text{Min}} \quad J = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} F\left(u, u_x, u_y, u_z\right) dx\, dy\, dz = \int_V F\left(u, u_x, u_y, u_z\right) dV$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_z}\right) = 0$$

Differential equation

$$\left\{\left(\frac{\partial F}{\partial u_x}\hat{i} + \frac{\partial F}{\partial u_y}\hat{j} + \frac{\partial F}{\partial u_z}\hat{k}\right)\cdot \mathbf{n}\right\}\delta u = 0$$

Boundary conditions

Either one is zero on the boundary.

Example 3: Elastic deformation of a 3D body

Here we have three functions in three independent variables.

$$\underset{\mathbf{u}}{\text{Min}} \; PE = \int_{\Omega} \left(\frac{1}{2} \boldsymbol{\varepsilon} : \mathbf{D} : \boldsymbol{\varepsilon} - \mathbf{b} \cdot \mathbf{u} \right) d\Omega$$

$$\text{Data} : \mathbf{D}, \mathbf{b}, \Omega$$

$$\mathbf{u} = \begin{Bmatrix} u_1(x, y, z) \\ u_2(x, y, z) \\ u_3(x, y, z \end{Bmatrix} = \begin{Bmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{Bmatrix}$$

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad i, j = 1, 2, 3$$

Note the potential energy functional is of the same form as the functional on Slide 20.

$$\nabla \cdot (\mathbf{D} : \varepsilon) + \mathbf{b} = 0$$ Euler-Lagrange equation

$$\{ (\mathbf{D} : \varepsilon) \mathbf{n} \} \delta \mathbf{u} = 0$$ Boundary condition; the traction condition

The end note

Functionals involving two and three independent variables

- Functionals with two independent variables and first derivatives
- Splitting of terms
  Application of the Green's theorem as equivalent of integration of parts in two dimensions
  Soap-film problem as an example
- Functionals with two independent variables and second derivatives
  Plate problem as an example
- Functionals involving three independent variables
- Splitting of terms; application of the divergence theorm
  Example of a 3D elastic body

Thanks