# Lecture 14

# Global Constraints
## in calculus of Variations

ME 260 at the Indian Institute of Science, Bangalore

**Structural Optimization: Size, Shape, and Topology**

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# Outline of the lecture

Global and local constraints

Dealing with global constraints

Euler-Lagrange equations with constraints; Lagrange multipliers

Inequality constraints

What we will learn:

How to identify a constraint as global as local

When is Lagrange multiplier a scalar

How to write Euler-Lagrange equations and boundary conditions for a problem with global constraints

Interpreting the Lagrange multipliers and understanding the complementarity conditions

# Global vs. local constraints

Global vs. local here pertains to whether a constraint is imposed at each point in the domain or it is imposed on a quantity that pertains to the entire domain.

- Global constraints pertain to the entire domain.
- Local constraints are imposed at every point in the domain, individually.

Mathematically, it tells whether a constraint is a functional or a function.

- Global constraint is a functional
- Local constraint is a function. It can also be a differential equation.

It also has implications when we discretize.

- Upon discretization, a global constraint gives rise to only one constraint.
- A local constraint, on the other hand, gives as many constraints as the number of discretization points.

# Examples of global and local constraints

## Global constraints

- Length of a curve
- Area of a surface
- Time of travel
- Weight of a structure
- Deflection at a particular point
- Maximum stress
- Buckling load
- Natural frequency

## Local constraints

- Upper or lower bound on a curve
- Bounds on the deflection of a structure
- Bounds on stress
- Governing differential equation
- Bounds on the mode shape

It is important to understand this difference.

# Global constraint: isoperimetric problem

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\left(y(x), y'(x)\right) dx$$

Subject to

$$K = \int_{x_1}^{x_2} G\left(y(x), y'(x)\right) dx - K^* = 0$$

This problem statement means that we need to find *y(x)* that minimizes *J* and satisfies the equality constraint, *K*.

It is a global constraint because *K* here depends on the entire domain. It is a functional. It is a single value.

A problem with a global constraint is also called isoperimetric problem. This is because the perimeter constraint is the historic global constraint.

# How do we solve this?

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\left(y(x), y'(x)\right) dx$$

Subject to

$$K = \int_{x_1}^{x_2} G\left(y(x), y'(x)\right) dx - K^* = 0$$

Recall how we handled equality constraints in finite-variable optimization.

You may recall from that…
We linearized the constraint and used the first-order term to eliminate a variable and made the problem unconstrained.
We also came up with the concept of Lagrange multiplier.
Here too, we will follow the same idea.

# Equivalent of first-order term of a functional

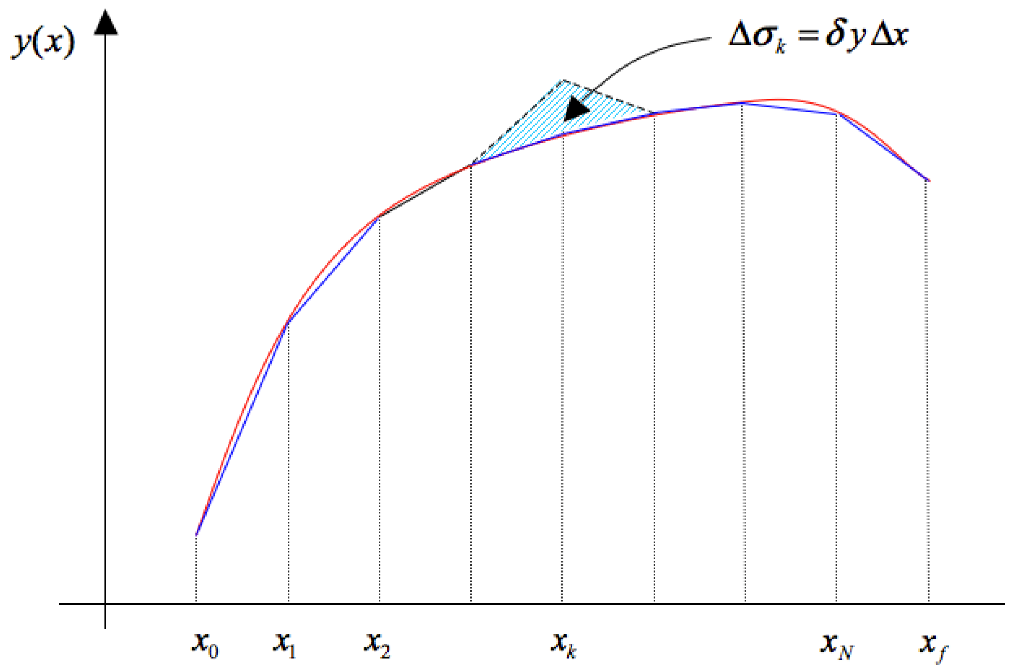
From Eq. (6) in Slide 26 of Lecture 9

$$\Delta J = J(y+h) - J(y) = \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=\hat{x}} + \varepsilon \right\} \Delta\sigma$$

$$\frac{\delta J}{\delta y} = F_y - \frac{d}{dx}(F_{y'})$$

Variational derivative, which is the expression in the Euler-Lagrange equation.

(first-order approximation of a perturbed functional)



# First-order term of the global constraint

$$K = \int_{x_1}^{x_2} G\left(y(x), y'(x)\right) dx - K^* = 0$$

$$\Delta K = K(y+h) - K(y) = \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=\hat{x}} + \varepsilon \right\} \Delta\sigma$$

$$\frac{\delta K}{\delta y} = G_y - \frac{d}{dx}(G_{y'})$$

The first-order term shows that the constraint has non-zero value whenever we perturb the function at a point. So, it won't satisfy the equality constraint anymore.

So, we will perturb *y(x)* at two points…

# Two perturbations of the global constraint

$$\Delta K_a = K(y+h) - K(y) = \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a \qquad \Delta\sigma_a = \delta y_a \, \Delta x_a$$

$$\Delta K_b = K(y+h) - K(y) = \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b \qquad \Delta\sigma_b = \delta y_b \, \Delta x_b$$

We choose $x_a$ and $x_b$ such that the first-order changes due to the two perturbations cancel each other and we retain the feasibility of the constraint.

$$\Delta K_a + \Delta K_b = 0$$

$$\Rightarrow \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = 0$$

# One perturbation of the function in terms of the other

$$\left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = 0$$

$$\Rightarrow \Delta\sigma_b = -\frac{\left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\}}{\left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

In order to divide like this, we require that there should be at least one point $x$ where the variational derivative is not zero. This is the equivalent of constraint qualification of finite-variable optimization. See Slide 13 of Lecture 5.

# Perturbation of the objective functional at the same two points by the same amounts

$$\Delta J_a = J(y+h) - J(y) = \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a \qquad \Delta\sigma_a = \delta y_a \, \Delta x_a$$

$$\Delta J_b = J(y+h) - J(y) = \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b \qquad \Delta\sigma_b = \delta y_b \, \Delta x_b$$

$$\Delta J_a + \Delta J_b = \Delta J_{a+b}$$

$$\Rightarrow \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = \Delta J_{a+b}$$

# Eliminating one perturbation…

$$\Delta\sigma_b = -\frac{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

$$\Delta J_{a+b} = \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b$$

$$\Delta J_{a+b} = \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a - \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\} \frac{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

# Defining a multiplier…

$$\Delta J_{a+b} = \left[ \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} - \frac{\left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}} \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \right] \Delta\sigma_a$$

$$\Lambda = - \frac{\left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}}{\left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_b} + \varepsilon_b \right\}}$$

$$\Delta J_{a+b} = \left[ \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} + \Lambda \left\{ \left.\frac{\delta K}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} \right] \Delta\sigma_a$$

# First order change in the objective functional

$$\Delta J_{a+b} = \left[ \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} + \Lambda \left\{ \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \right] \Delta \sigma_a$$

$$\Rightarrow \Delta J_{a+b} = \left[ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \Lambda \left. \frac{\delta K}{\delta y} \right|_{x=x_a} + \varepsilon \right] \Delta \sigma_a = 0$$

This is zero because now it is the first-order term due to one arbitrary feasible perturbation because the other one is eliminated.

$$\left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \Lambda \left. \frac{\delta K}{\delta y} \right|_{x=x_a} = 0$$

because $\Delta \sigma_a \neq 0$

and $\varepsilon \Delta \sigma_a = 0$ (the second order term)

# Putting things together…

From Slide 13…

$$-\frac{\left\{\left.\frac{\delta J}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}}{\left\{\left.\frac{\delta K}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}}=\Lambda \Rightarrow \left\{\left.\frac{\delta J}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}+\Lambda\left\{\left.\frac{\delta K}{\delta y}\right|_{x=x_b}+\varepsilon_b\right\}=0$$

$$\Rightarrow \left.\frac{\delta J}{\delta y}\right|_{x=x_b}+\Lambda\left.\frac{\delta K}{\delta y}\right|_{x=x_b}=0$$

From Slide 14…

$$\left.\frac{\delta J}{\delta y}\right|_{x=x_a}+\Lambda\left.\frac{\delta K}{\delta y}\right|_{x=x_b}=0$$

Since $x_a$ and $x_b$ are arbitrary, the following should be true for any $x$. And $\Lambda$ must be a constant.

$$\boxed{\frac{\delta J}{\delta y}+\Lambda\frac{\delta K}{\delta y}=0}$$

# Lagrangian can now be defined.

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\, dx$$

Subject to

$$K = \int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\, dx - K^* = 0$$

$$\underset{y(x)}{\text{Min}} \quad L = \left\{ \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\, dx \right\} + \Lambda \left\{ \int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\, dx \right\}$$

$$\frac{\delta J}{\delta y} + \Lambda \frac{\delta K}{\delta y} = 0$$

Necessary coniditon

$$\int_{x_1}^{x_2} G\big(y(x), y'(x)\big)\, dx - K^* = 0$$

Feasibility coniditon

# Necessary conditions

Equations

$$\frac{\delta J}{\delta y} + \Lambda \frac{\delta K}{\delta y} = 0$$

$$\int_{x_1}^{x_2} G\Big(y(x), y'(x)\Big) dx - K^* = 0$$

Differential equation

Scalar equation

$y(x)$

Function

$\Lambda$

Scalar variable

Unknowns

# What if we have an inequality constraint?

$$\underset{y(x)}{\text{Min}} \;\; J = \int_{x_1}^{x_2} F\left(y(x), y'(x)\right) dx$$

Subject to

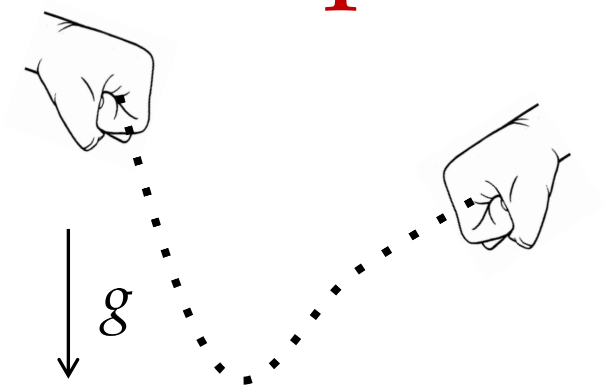
$$K = \int_{x_1}^{x_2} G\left(y(x), y'(x)\right) dx - K^* \leq 0$$

$$\Lambda \left( \int_{x_1}^{x_2} G\left(y(x), y'(x)\right) dx - K^* \right) = 0$$

$$\Lambda \geq 0$$

We introduce complementarity condition and require non-negativity of the Lagrange multiplier… just as we did in finite-variable optimization. The same argument applies here too.

# Example 1: hanging chain problem

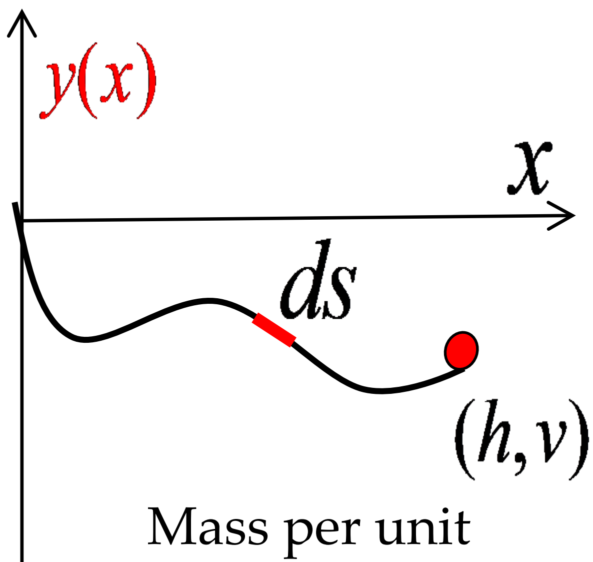



$$\underset{y(x)}{\text{Min}} \quad PE = \int_0^h (\rho g y)\, ds = \int_0^h \rho g y \sqrt{1+y'^2}\, dx$$

Subject to

$$\int_0^h \left(\sqrt{1+y'^2}\right) dx - L = 0$$

$$\text{Data}: L, y(0)=0, h, y(h)=v, \rho, g$$




$\rho =$ Mass per unit length of the chain

$$\underset{y(x)}{\text{Min}} \quad L = \int_0^h \rho g y \sqrt{1+y'^2}\, dx + \Lambda\left(\int_0^h \left(\sqrt{1+y'^2}\right) dx - L\right)$$

$$\text{Data}: L, y(0)=0, h, y(h)=v, \rho, g$$

# Necessary conditions for the hanging chain problem

$$\underset{y(x)}{\text{Min}} \quad L = \int_0^h \rho g y \sqrt{1+y'^2}\, dx + \Lambda\left( \int_0^h \left(\sqrt{1+y'^2}\right) dx - L \right)$$

$$\text{Data}: L, y(0)=0, h, y(h)=v, \rho, g$$

$$\delta_y L = 0$$

$$\int_0^h \left(\sqrt{1+y'^2}\right) dx - L = 0$$

$$\delta_y L = \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

Differential equation

# Example 2: Stiffest beam of given volume

$$\underset{b(x)}{\text{Min}} \;\; SE = \int_0^L \left\{ \frac{1}{2} \frac{Ebd^3}{12} \left( \frac{d^2 w}{dx^2} \right)^2 \right\} dx$$

Subject to

$$\frac{d^2}{dx^2}\left( Ebd^3 \frac{d^2 w}{dx^2} \right) + q = 0$$

This is a local constraint; it is valid at every point in the domain.

$$\int_0^L bd \, dx - V^* \leq 0$$

We now know how to deal with this global constraint

$$\text{Data}: L, q(x), d, V^*, E$$

# The end note

Global constraints in calculus of variations

- Distinguishing between global and local constraints
- First-order perturbation of a functional using the concept of Variational derivative
  Two perturbations to cancel the effects of each other to retain feasibility of The equality constraint.
- Concept of Lagrange multiplier and Lagrangian
- Necessary conditions
  Extension to inequality constraints
- **Necessary constraints for global constraints in calculus of variations**

Thanks