Lecture 15

Local Constraints
in calculus of Variations

ME 260 at the Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Global and local constraints

Dealing with local constraints

Euler-Lagrange equations with constraints; Lagrange multipliers

Inequality constraints

What we will learn:

How to identify a constraint as global as local

When is Lagrange multiplier a function

How to write Euler-Lagrange equations and boundary conditions for a problem with local constraints

Interpreting the Lagrange multipliers and understanding complementarity conditions for a general problem

Global vs. local constraints

Global vs. local here pertains to whether the constraints is imposed at each point in the domain or it is imposed on a quantity that pertains to the entire domain.

- Global constraints pertain to the entire domain.
- Local constraints are imposed at every point in the domain, individually.

Mathematically, it tells whether a constraint is a functional or a function.

- Global constraint is a functional
- Local constraint is a function. It can also be a differential equation.

It also has implications when we discretize.

- Upon discretization, a global constraint gives rise to only one constraint.
- A local constraint, on the other hand, gives as many constraints as the degrees of freedom.

Examples of global and local constraints

Global constraints

Length of a curve

Area of a surface

Time of travel

Weight of a structure

Deflection at a particular point

Maximum stress

Buckling load

Natural frequency

Local constraints

Upper or lower bound on a curve

Bounds on deflection of a structure

Bounds on stress

Governing differential equation

Bounds on the mode shape

It is important to understand this difference.

Local constraint!

$$\underset{y(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F\left(y(x), y'(x)\right) dx$$

Subject to

$$g(x, y, y') = 0$$

This problem statement means that we need to find $y(x)$ that minimizes J and satisfies the equality $g = 0$

It is a local constraint because g applies to every point in the entire domain. It is a function. It has different values at different values of x.

Does this problem make sense?

Think and then see the next slide.

This problem does not make sense!

$$\underset{y(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\big(y(x), y'(x)\big)\, dx$$

Subject to

$$g(x, y, y') = 0$$

$g = 0$ is already a differential equation. So, it, in its own right, has a solution for $y(x)$. So, there is no room for minimizing another functional.

So, what do we do?

This makes sense…

$$\underset{y(x),z(x)}{\text{Min}} \ J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), z(x), z'(x)\Big) dx$$

Subject to

$$g(x, y, z) = 0$$

This problem statement means that we need to find a pair of functions *y(x)* and *z(x)* that minimize *J* and satisfy the equality $g = 0$.

So, when we have a local constraint, there should be more than one function in the functional.

How do we solve this?

$$\underset{y(x),z(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), z(x), z'(x)\Big)\,dx$$

Subject to

$$g(x, y, z) = 0$$

We follow almost the same procedure as with global constraints.
The slight difference will be in dealing with a local constraint rather than a global constraint.

Here also, we will have a Lagrange multiplier, but it will be function and not a scalar here.
The reason for this is simple: we have local constraint as a function (as opposed to a number), so its multilier is also a function.

Equivalent of first-order term of a function integrated over the domain

$$\int_{x_1}^{x_2} \left\{ g(x,y,z) - g(x^*,y^*,z^*) \right\} dx$$

g is a function. Any change in it will cause change everywhere in the domain from x_1 and x_2. So, we integrate to get the overall change.

$$= \int_{x_1}^{x_2} \left\{ g_y \delta y + g_z \delta z \right\} dx$$

Here, we have introduced variations in y and z, as we perturb from the optimum y^* and z^*.

$$= \left\{ g_y \Big|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ g_z \Big|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b$$

Here, we have introduced variations in y and z, only at specific points x_a and x_b respectively.

First-order term of the local constraint

$$\left\{ g_y \Big|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ g_z \Big|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = 0$$

Because the constraint should not change up to first order when we perturb the two functions.

$$\Rightarrow \Delta\sigma_b = -\frac{\left\{ g_y \Big|_{x=x_a} + \varepsilon_a \right\}}{\left\{ g_z \Big|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

Now, we have expressed the perturbation in function *y* at point in terms of the perturbation in *z* at another point.
We need this because we want to substitute this in the first-order term of the objective functional.

Perturbation of the objective functional at the same two points by the same amounts

$$\Delta J_a = J(y+h,z) - J(y,z) = \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a \qquad \Delta\sigma_a = \delta y_a \, \Delta x_a$$

$$\Delta J_b = J(y,z+h) - J(y,z) = \left\{ \left. \frac{\delta J}{\delta z} \right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b \qquad \Delta\sigma_b = \delta z_b \, \Delta x_b$$

$$\Delta J_a + \Delta J_b = \Delta J_{a+b}$$

$$\Rightarrow \left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \left. \frac{\delta J}{\delta z} \right|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = \Delta J_{a+b}$$

Eliminating one perturbation…

$$\Delta\sigma_b = -\frac{\left\{ g_y\Big|_{x=x_a} + \varepsilon_a \right\}}{\left\{ g_z\Big|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

$$\left\{ \frac{\delta J}{\delta y}\Big|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a + \left\{ \frac{\delta J}{\delta z}\Big|_{x=x_b} + \varepsilon_b \right\} \Delta\sigma_b = \Delta J_{a+b}$$

$$\Delta J_{a+b} = \left\{ \frac{\delta J}{\delta y}\Big|_{x=x_a} + \varepsilon_a \right\} \Delta\sigma_a - \left\{ \frac{\delta J}{\delta z}\Big|_{x=x_b} + \varepsilon_b \right\} \frac{\left\{ g_y\Big|_{x=x_a} + \varepsilon_a \right\}}{\left\{ g_z\Big|_{x=x_b} + \varepsilon_b \right\}} \Delta\sigma_a$$

First order change in objective functional

$$\Delta J_{a+b} = \left\{ \left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} + \varepsilon_a \right\} - \left\{ \left.\frac{\delta J}{\delta z}\right|_{x=x_b} + \varepsilon_b \right\} \frac{\left\{ \left. g_y \right|_{x=x_a} + \varepsilon_a \right\}}{\left\{ \left. g_z \right|_{x=x_b} + \varepsilon_b \right\}} \right\} \Delta\sigma_a = 0$$

This is zero because now it is first-order term due to one arbitrary feasible perturbation because the other one is eliminated.

$$\left\{ \frac{\left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} \right\}}{\left\{ \left. g_y \right|_{x=x_a} \right\}} - \frac{\left\{ \left.\frac{\delta J}{\delta z}\right|_{x=x_b} \right\}}{\left\{ \left. g_z \right|_{x=x_b} \right\}} \right\} = 0$$

because $\Delta\sigma_a \neq 0$

Defining a multiplier function…

$$\left\{ \frac{\left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} \right\}}{\left\{ \left. g_y \right|_{x=x_a} \right\}} - \frac{\left\{ \left. \frac{\delta J}{\delta z} \right|_{x=x_b} \right\}}{\left\{ \left. g_z \right|_{x=x_b} \right\}} \right\} = 0$$

$$\frac{\left\{ \left. \frac{\delta J}{\delta y} \right|_{x=x_a} \right\}}{\left\{ \left. g_y \right|_{x=x_a} \right\}} = \frac{\left\{ \left. \frac{\delta J}{\delta z} \right|_{x=x_b} \right\}}{\left\{ \left. g_z \right|_{x=x_b} \right\}} = -\lambda(x)$$

Since x_a and x_b are arbitrary, it means that this is true for any value of x.

Putting things together…

$$\frac{\left\{ \left.\frac{\delta J}{\delta y}\right|_{x=x_a} \right\}}{\left\{ \left. g_y \right|_{x=x_a} \right\}} = \frac{\left\{ \left.\frac{\delta J}{\delta z}\right|_{x=x_b} \right\}}{\left\{ \left. g_z \right|_{x=x_b} \right\}} = -\lambda(x)$$

Two differential equations as necessary conditions because we have two unknown functions, *y(x)* and *z(x)*

$$\frac{\delta J}{\delta y} + \lambda(x) g_y = 0$$

$$\frac{\delta J}{\delta z} + \lambda(x) g_z = 0$$

Lagrangian can now be defined.

$$\underset{y(x),z(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), z(x), z'(x)\Big)\, dx$$

Subject to

$$g(x, y, z) = 0$$

$$\underset{y(x),z(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), z(x), z'(x)\Big)\, dx + \int_{x_1}^{x_2} \lambda(x) g(x, y, z)\, dx$$

$$\frac{\delta J}{\delta y} + \lambda(x) g_y = 0$$

$$\frac{\delta J}{\delta z} + \lambda(x) g_z = 0$$

Necessary conditons

$$g(x, y, z) = 0$$

Feasibility conditon

Why do we integrate *g* multiplied by the multiplier?

$$\underset{y(x),z(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\big(y(x), y'(x), z(x), z'(x)\big)\,dx + \int_{x_1}^{x_2} \lambda(x)\, g(x,y,z)\,dx$$

Understand this point well.

Note that a local constraint is applicable at every point in the domain between x_1 and x_2.
So, when we discretize it, there will be as many constraints as there are discretization points. Then, each constraint will have a multiplier associated with it and the product of the multiplier and the constraint gets added to the Lagrangian. So, when we take a continuous (local) constraint, we need to integrate the product of the multiplier function and the constraint.

Necessary conditions

Equations

$$\frac{\delta J}{\delta y} + \lambda(x) g_y = 0$$

$$\frac{\delta J}{\delta z} + \lambda(x) g_z = 0$$

$$g(x, y, z) = 0$$

Two differential equations

Differential/algebraic equation

$y(x), z(x)$

Two functions

$\lambda(x)$

Scalar function

Unknowns

General form

$$\underset{y(x),z(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), z(x), z'(x)\Big)\,dx$$

Subject to

$$g(x, y, z, y', z') = 0$$

A local constraint can be differential equation.
Both objective function and the constraint can depend on any number of derivatives of any order. All the generalities discussed earlier are applicable here also.

What if we have an inequality constraint?

$$\underset{y(x),z(x)}{\text{Min}} \quad J = \int_{x_1}^{x_2} F\Big(y(x), y'(x), z(x), z'(x)\Big)\,dx$$

Subject to

$$g(x, y, z, y', z') \leq 0$$

$$\lambda(x)\, g(x, y, z, y', z') = 0$$

$$\lambda(x) \geq 0$$

We introduce complementarity condition and require non-negativity of the Lagrange multiplier…
just as we did in finite-variable optimization. The same argument applies here too.

Chatterjee's problem

$$\underset{y(x)}{\text{Min}} \; -A = -\int_0^h y \, dx$$

Subject to

$$\int_0^h \left(\sqrt{1+y'^2}\right) dx - l = 0$$

$$y(x) - r(x) \le 0$$

$$\text{Data} : l, y(0) = v_1, y(h) = v_2$$

Here, we have a global equality constraint and a local inequality constraint.

Observe how we write the Lagrangian.

$$L = -\int_0^h y \, dx + \Lambda\left(\int_0^h \left(\sqrt{1+y'^2}\right) dx - l\right) + \int_0^h \lambda(x)\left(y(x) - r(x)\right) dx$$

Necessary conditions for the Chatterjee's problem

$$L = -\int_0^h y\,dx + \Lambda\left(\int_0^h \left(\sqrt{1+y'^2}\right)dx - l\right) + \int_0^h \lambda(x)\big(y(x)-r(x)\big)dx$$

$$\delta_y L = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = (-1+\lambda) - \frac{d}{dx}\left(\frac{\Lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\lambda(y-r) = 0; \quad \lambda \geq 0$$

$$\int_0^h \left(\sqrt{1+y'^2}\right)dx - l = 0$$

$$y(x) - r(x) \leq 0$$

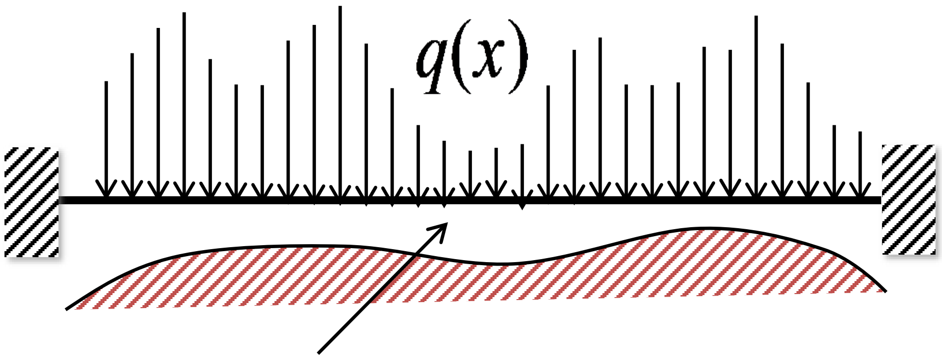
Example 2: beam contact-problem

$$\underset{w(x)}{\text{Min}} \ PE = \int_0^l \left\{ \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx$$

Subject to

$$w(x) - g(x) \leq 0$$

Data : $q(x), E, I$



$$L = \int_0^l \left\{ \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx + \int_0^l \lambda (w - g) dx$$

Necessary conditions for the beam contact problem

$$L = \int_0^l \left\{ \frac{1}{2} EI \left(\frac{d^2 w}{dx^2} \right)^2 - qw \right\} dx + \int_0^l \lambda (w - g) dx$$

$$\delta_w L = \left(EIw'' \right)'' - q + \lambda = 0$$

$$\lambda (w - g) = 0; \quad \lambda \geq 0$$

$$w - g \leq 0$$

Physical interpretation of the Lagrange multiplier

Lagrange multipliers have physical meaning in most problems. Here, it represents the contact force.

Understand the complementarity in view of this: if gap is not zero (the contact has not taken place), hence the contact force (the multiplier) is zero. But when the gap is zero, the multiplier (and hence the contact force is not zero.

And contact force is always non-negative!

The end note

Local constraints
in calculus of variations

- Distinguishing between global and local constraints
- First-order perturbation of a function
  Two perturbations to cancel the effects of each other to retain feasibility of the equality constraint.
- Concept of Lagrange multiplier and Lagrangian
  Physical meaning of the Lagrange multiplier
- Necessary conditions
  Extension to inequality constraints
- **Necessary constraints for local constraints in calculus of variations**

Thanks