Lecture 16

General Variation of a Functional
Transversality conditions
Broken extremals
Corner conditions

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Variable end conditions: motivating examples

General variation

Transversality conditions

Weierstrass-Erdman corner conditions

What we will learn:

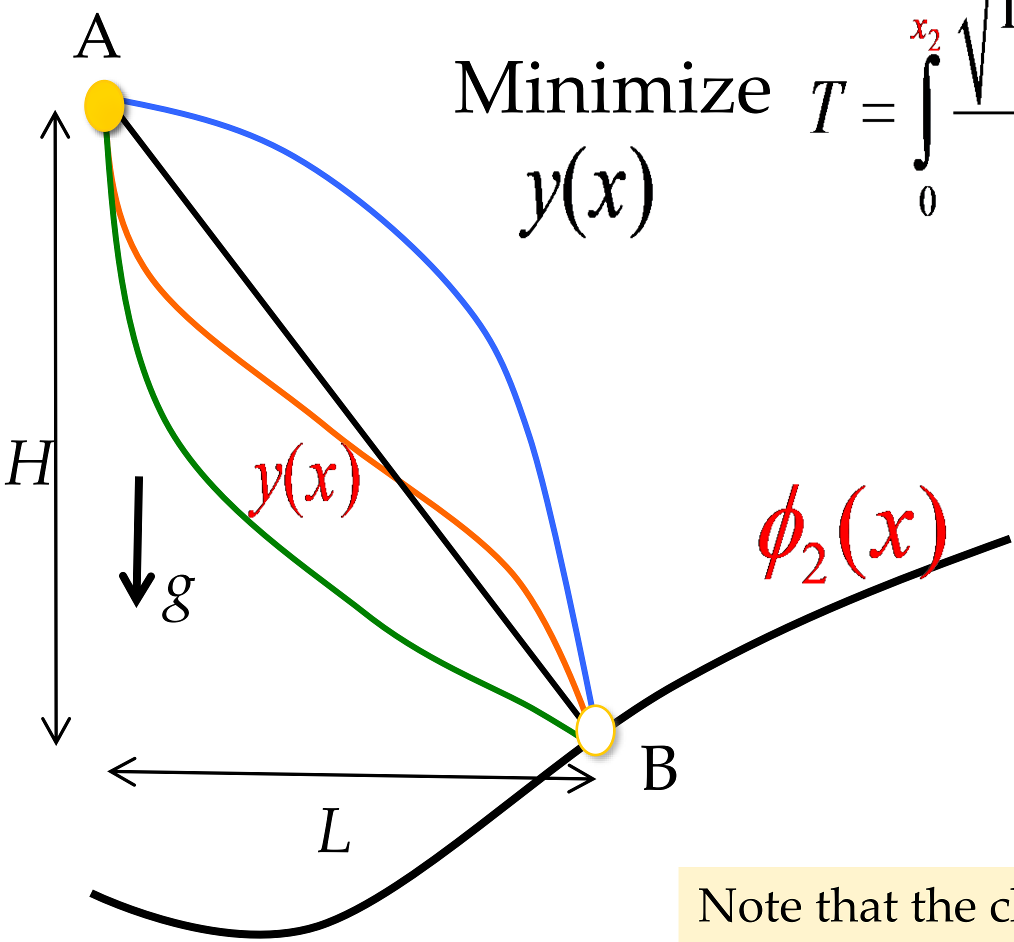
Why we need to deal with variable end conditions in calculus of variations

How to take general variation and how it affects only the boundary conditions and not the differential equation

What broken extremals are

How we can get the regular boundary conditions as special cases

Modified brachistochrone problem



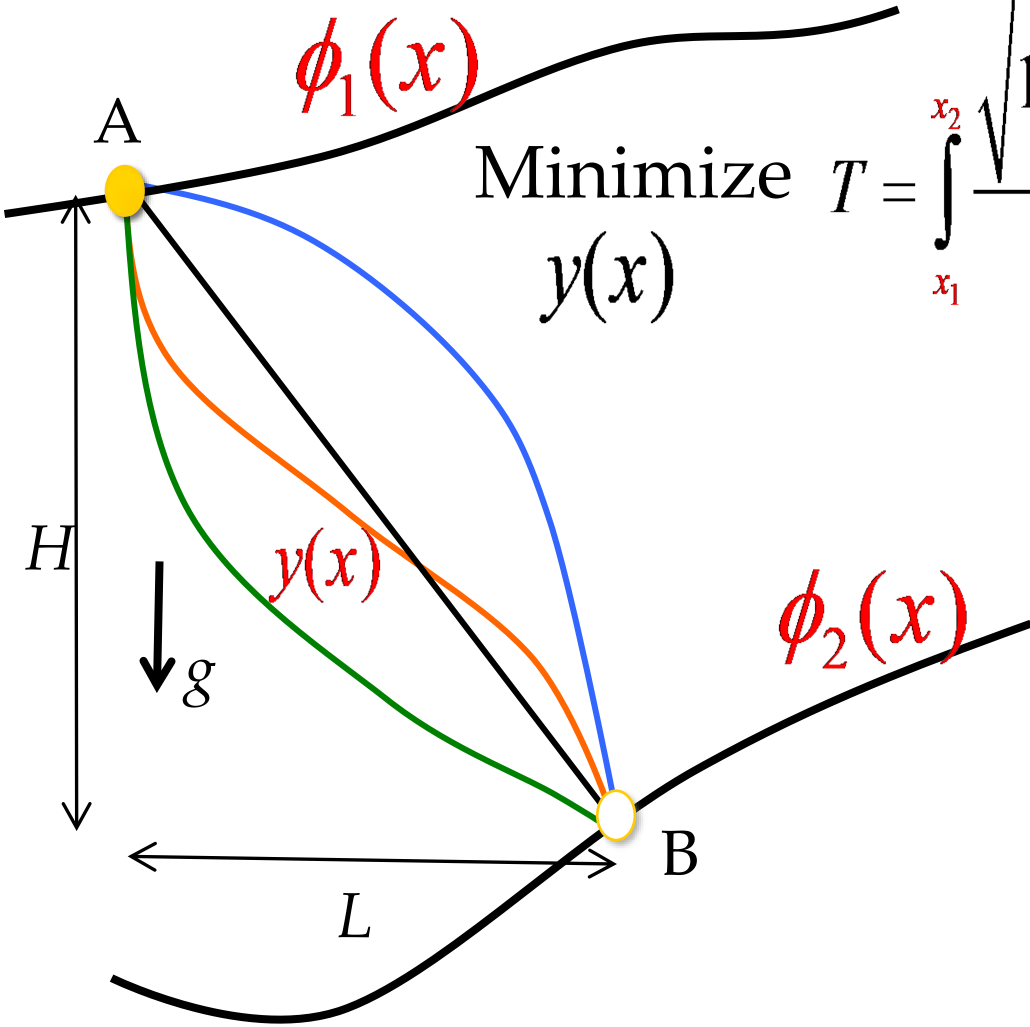
Minimize $y(x)$ $T = \int_0^{x_2} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{v(y)} dx$

Now, point B can be anywhere on a given curve represented by $\phi_2(x)$

We want to find *y(x)* such that an object will reach any point on $\phi_2(x)$ in the least time.

Note that the change in the problem statement comes only in the end condition and not in the functional.

Another modification…



Minimize $y(x)$ $\quad T = \int_{x_1}^{x_2} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{v(y)}\,dx$

Note again that the change in the problem statement comes only in the end conditions and not in the functional.

Now, point A can be anywhere on a given curve represented by $\phi_1(x)$

We want to find *y(x)* such that an object will reach any point on $\phi_2(x)$ starting from any point on $\phi_1(x)$ in the least time.

A general problem with variable end conditions

$$\underset{y(x)}{\text{Min}} \;\; J = \int_{x_1}^{x_2} F(y, y')\, dx$$

What do we do when ends are not given?

Recall that we had taken a variation (a perturbation) around a minimal curve $y^*(x)$ and equated the first-order term to zero to establish the necessary condition. Here, the perturbation should be taken for $y^*(x)$ and the two ends.

"Variable ends" means that both ends can also be perturbed.

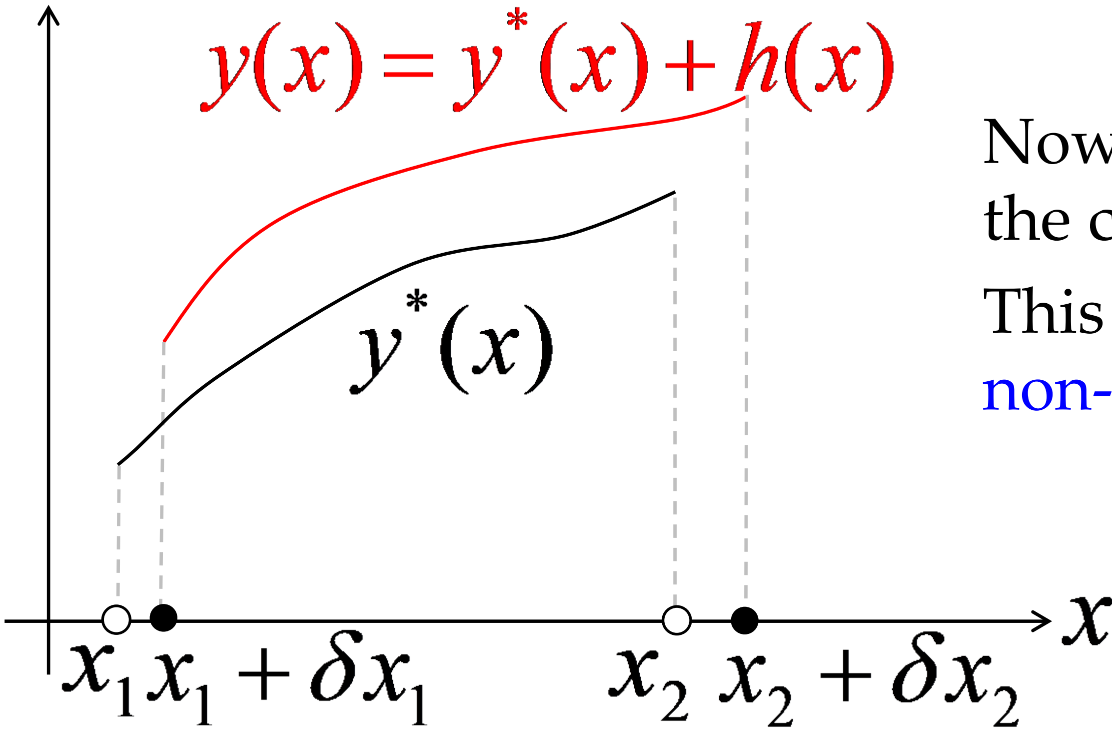
That is, the domain over which we integrate is variable.

In such a case, we take what is called a general variation in which ends are also perturbed.

See the next slide…

General non-contemporaneous variation

(related to non-contemporary)



Now we have perturbed not only the curve but also the ends!

This type of variation is called non-contemporaneous variation.

The term "non-contemporaneous" must be in the context of time-related problems. We are shifting the x-axis. So, y and y^* are not defined on the same domain.

$$\Delta J = \int_{x_1+\delta x_1}^{x_2+\delta x_2} F(y^* + h, y'^* + h')\,dx - \int_{x_1}^{x_2} F(y^*, y'^*)\,dx$$

First-order change with general variation

$$\Delta J = \int_{x_1+\delta x_1}^{x_2+\delta x_2} F(y^* + h, y'^* + h')\,dx - \int_{x_1}^{x_2} F(y^*, y'^*)\,dx$$

$$= \int_{x_1}^{x_2} F(y^* + h, y'^* + h')\,dx - \int_{x_1}^{x_2} F(y^*, y'^*)\,dx$$

We got both on the same domain.

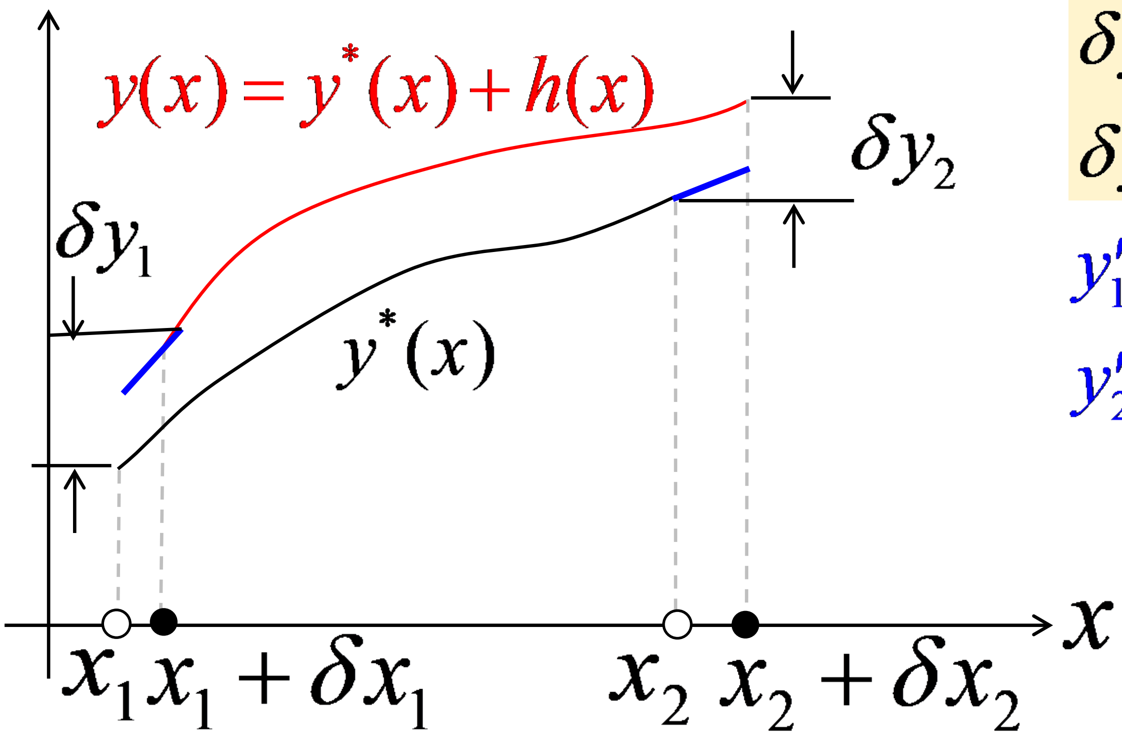
So, these two terms come out separated.

$$- \int_{x_1}^{x_1+\delta x_1} F(y^* + h, y'^* + h')\,dx + \int_{x_2}^{x_2+\delta x_2} F(y^* + h, y'^* + h')\,dx$$

$$\approx \int_{x_1}^{x_2} F(y^* + h, y'^* + h')\,dx - \int_{x_1}^{x_2} F(y^*, y'^*)\,dx \underbrace{- F\big|_{x_1} \delta x_1 + F\big|_{x_2} \delta x_2}$$

This is an approximation because the perturbed domains are very small.

Extensions of the domain at either end



$$\delta y_1 = h_1 + y_1' \, \delta x_1$$
$$\delta y_2 = h_2 + y_2' \, \delta x_2$$

$y_1' =$ Slope at the first end

$y_2' =$ Slope at the second end

h_1, h_2 — Differences between the original and perturbed curves at either end

$$\Delta J \approx \int_{x_1}^{x_2} F(y^* + h, y'^* + h')\,dx - \int_{x_1}^{x_2} F(y^*, y'^*)\,dx - F\Big|_{x_1} \delta x_1 + F\Big|_{x_2} \delta x_2$$

The domains of the original curve and the perturbed curve need to be extended as shown with blue lines by maintaining tangency to the respective curves.

The first term of the first-order term…

$$\int_{x_1}^{x_2} F(y^* + h, y'^* + h')\,dx \approx \int_{x_1}^{x_2} F(y^*, y'^*)\,dx + \int_{x_1}^{x_2} \left\{ F_y h + F_{y'} h' \right\} dx$$

$$= \int_{x_1}^{x_2} F(y^*, y'^*)\,dx + \int_{x_1}^{x_2} \left\{ F_y - \frac{d}{dx}\left(F_{y'}\right) \right\} h\,dx + \left(F_{y'} h\right)\Big|_{x_1}^{x_2}$$

$$= \int_{x_1}^{x_2} F(y^*, y'^*)\,dx + \underbrace{\int_{x_1}^{x_2} \left\{ F_y - \frac{d}{dx}\left(F_{y'}\right) \right\} h\,dx + \left(F_{y'} h\right)\Big|_{x_2} - \left(F_{y'} h\right)\Big|_{x_1}}$$

A result we had derived earlier.

And now…

$$\Delta J \approx \int_{x_1}^{x_2} F(y^* + h, y'^* + h')\,dx - \int_{x_1}^{x_2} F(y^*, y'^*)\,dx - F\big|_{x_1}\delta x_1 + F\big|_{x_2}\delta x_2$$

By substituting for this from the preceding slide…

$$\Delta J \approx \int_{x_1}^{x_2} \left\{ F_y - \frac{d}{dx}\left(F_{y'}\right) \right\} h\,dx + \left(F_{y'}\,h\right)\Big|_{x_2} - \left(F_{y'}\,h\right)\Big|_{x_1} - \left(F\,\delta x\right)\Big|_{x_1} + \left(F\,\delta x\right)\Big|_{x_2}$$

Recall from slide 8:

$$\delta y_1 = h_1 + y_1'\,\delta x_1 \Rightarrow h_1 = \delta y_1 - y_1'\,\delta x_1$$

$$\delta y_2 = h_2 + y_2'\,\delta x_2 \Rightarrow h_2 = \delta y_2 - y_2'\,\delta x_2$$

$$\Rightarrow \Delta J \approx \int_{x_1}^{x_2} \left\{ F_y - \frac{d}{dx}\left(F_{y'}\right) \right\} h\,dx + \left(F_{y'}\,\delta y\right)\Big|_{x_1}^{x_2} + \left\{\left(F - F_{y'}\,y'\right)\delta x\right\}\Big|_{x1}^{x_2}$$

Necessary condition and boundary conditions…finally.

First order is equated to zero for the necessary condition, as usual.

$$\Delta J \approx \int_{x_1}^{x_2} \left\{ F_y - \frac{d}{dx}\left(F_{y'}\right) \right\} h\,dx + \left(F_{y'}\,\delta y\right)\Big|_{x_1}^{x_2} + \left\{ \left(F - F_{y'}\,y'\right)\delta x \right\}\Big|_{x_1}^{x_2} = 0$$

By invoking the fundamental lemma, we get the differential equation:

$$F_y - \frac{d}{dx}\left(F_{y'}\right) = 0$$

Boundary conditions

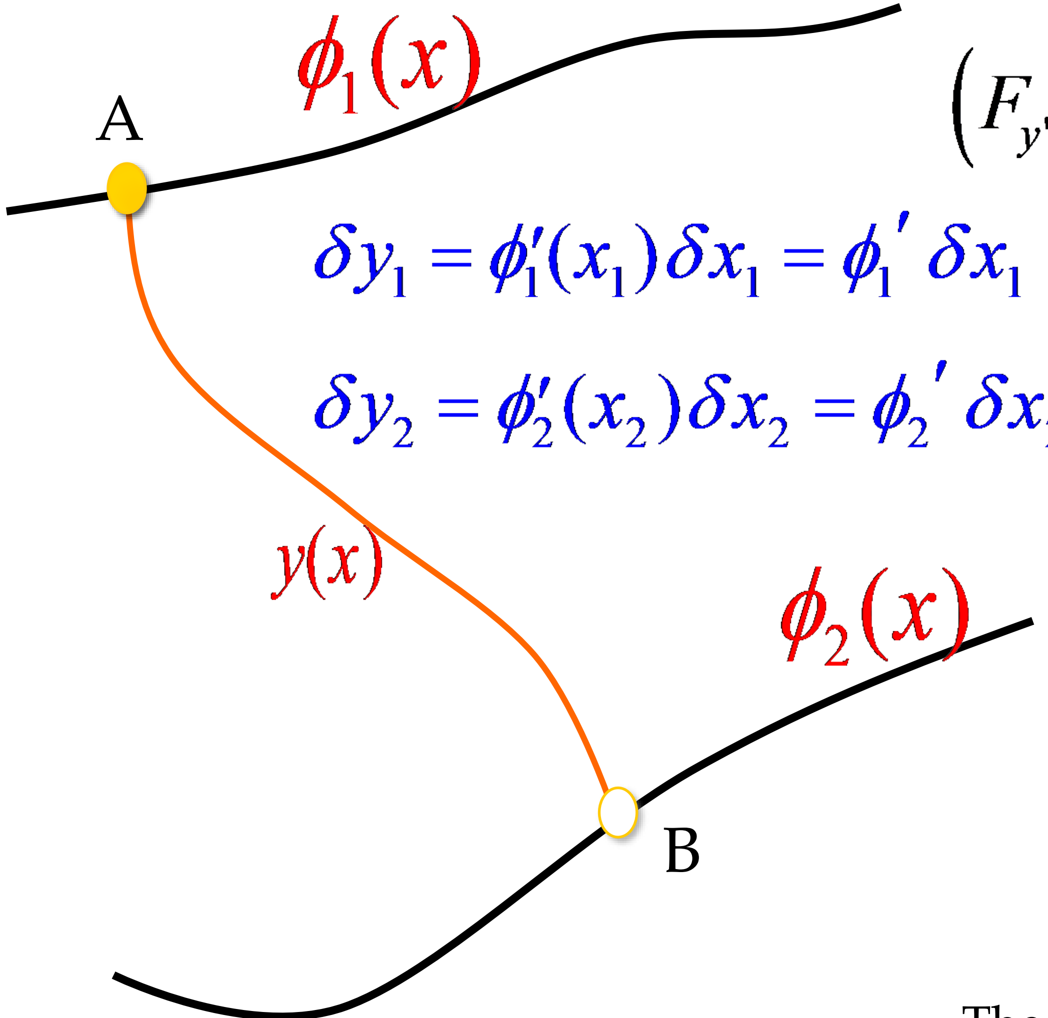
$$\left(F_{y'}\,\delta y\right)\Big|_{x_1}^{x_2} = 0 \quad \text{and}$$

$$\left\{ \left(F - F_{y'}\,y'\right)\delta x \right\}\Big|_{x_1}^{x_2} = 0$$

Note that the differential equation, the Euler-Lagrange equation, did not change!

Note that the boundary condition of the fixed end conditions comes out neatly when the variation in the end conditions are zero. That is, when $\delta x_1 = \delta x_2 = 0$

Boundary conditions when restricted to given curves



$$\left(F_{y'}\delta y\right)\Big|_{x_1}^{x_2} + \left\{\left(F - F_{y'}y'\right)\delta x\right\}\Big|_{x1}^{x_2} = 0$$

$$\delta y_1 = \phi_1'(x_1)\delta x_1 = \phi_1{}' \delta x_1$$

$$\delta y_2 = \phi_2'(x_2)\delta x_2 = \phi_2{}' \delta x_2$$

$$\left\{\left(F + F_{y'}\,(\phi_1' - y')\right)\delta x\right\}\Big|_{x_1} = 0$$

$$\left\{\left(F + F_{y'}(\phi_2' - y')\right)\delta x\right\}\Big|_{x_2} = 0$$

These are called transversality conditions.

Transversality conditions

$$\left\{\left(F + F_{y'}\left(\phi_1' - y_1'\right)\right)\delta x\right\}\Big|_{x_1} = 0$$

$$\left\{\left(F + F_{y'}\left(\phi_2' - y'\right)\right)\delta x\right\}\Big|_{x_2} = 0$$

Transversality has something to do with being orthogonal, i.e., perpendicular. It is indeed so for certain functionals.

$$J = \int_{x_1}^{x_2} f(y)\sqrt{1+y'^2}\, dx$$

$$\Rightarrow F = f(y)\sqrt{1+y'^2}$$

$$\Rightarrow F_{y'} = \frac{\partial F}{\partial y'} = \frac{f(y)y'}{\sqrt{1+y'^2}}$$

$$F + F_{y'}\left(\phi' - y'\right) = 0$$

$$\Rightarrow f\sqrt{1+y'^2} + \frac{f\,y'}{\sqrt{1+y'^2}}\left(\varphi' - y'\right) = 0$$

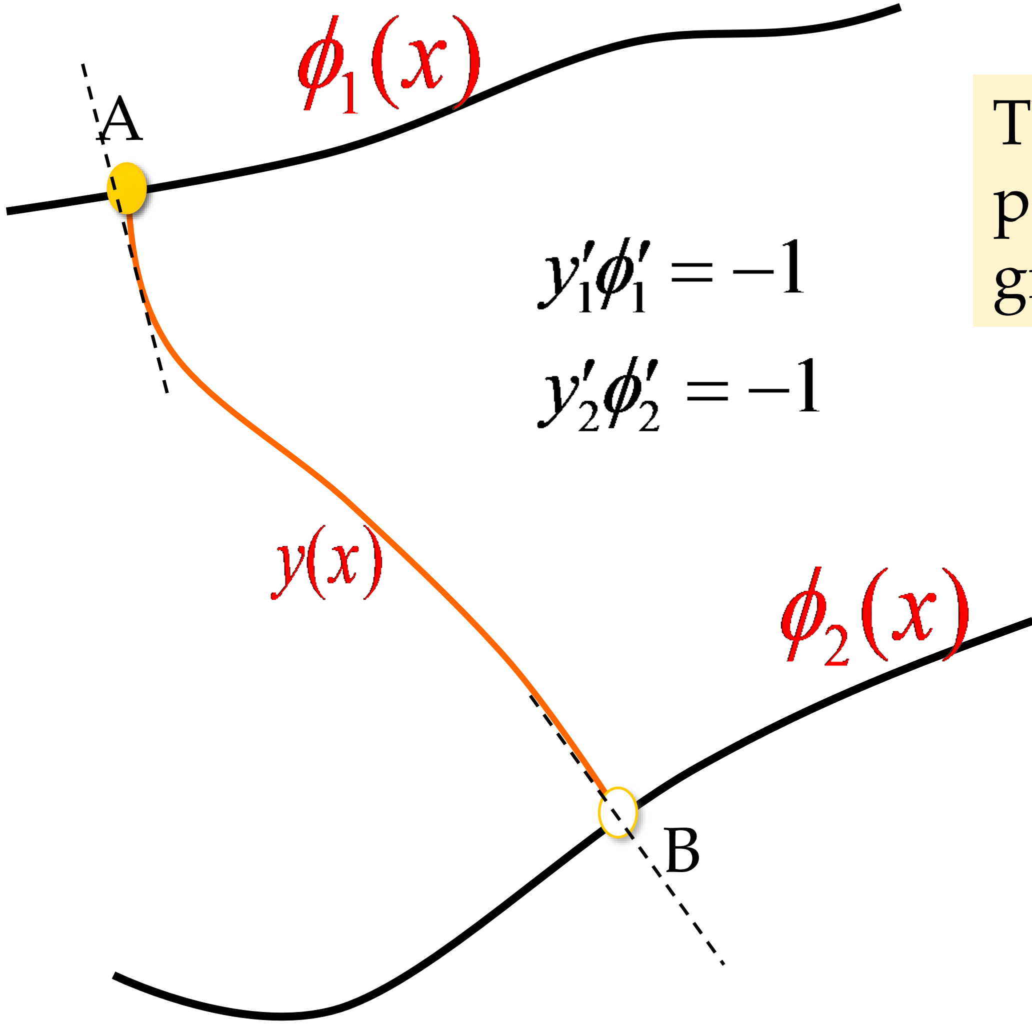
$$\Rightarrow f\left(1+y'^2\right) + fy'\phi' - fy'^2 = 0$$

$$\Rightarrow f\left(1+y'\phi'\right) = 0$$

$$\Rightarrow y'\phi' = -1$$

It means that the minimal curve is orthogonal to the boundary curve!

Transversality and brachistochrone



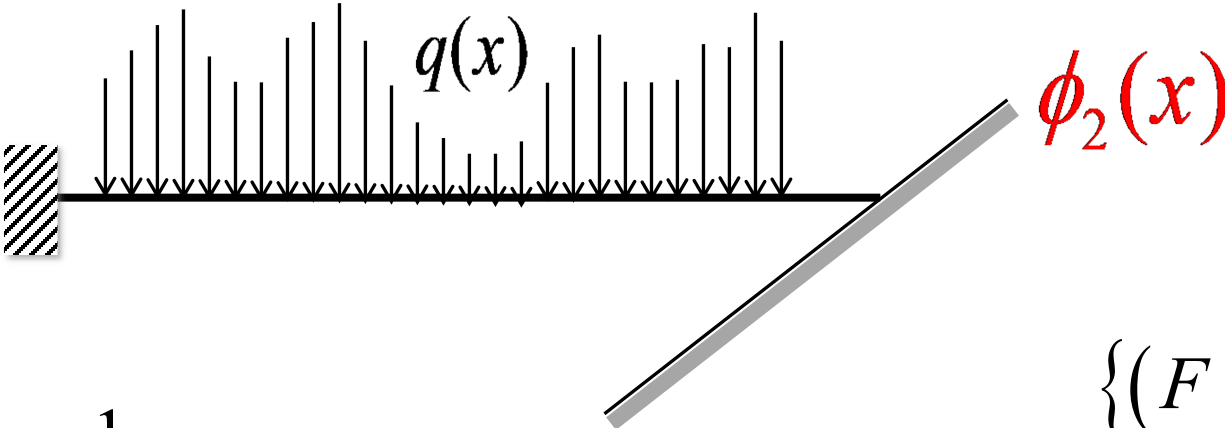
$$y_1'\phi_1' = -1$$

$$y_2'\phi_2' = -1$$

The optimal curve is perpendicular to the two given curves at either end.

Even though the "transversality" is limited only to special form of the functional, the name stuck for all types of functionals. What is in a name, anyway?

Example: beam guided at one end



$$\left\{\left(F + F_{w'}\left(\phi_2' - w'\right)\right)\delta x\right\}\Big|_{x_2} = 0$$

$$F = \frac{1}{2}EI\left(w''\right)^2 - qw \quad \text{because}$$

$$\underset{w(x)}{\text{Min}}\; J = \int_0^L \left\{\frac{1}{2}EI\left(w''\right)^2 - qw\right\}dx$$

But there is no $F_{w'}$ term here. So, we need to derive the transversality condition for w'' term.

Transversality condition for y'' term

Resume from Slide 10 by including y″ term.

$$\Delta J \approx \int_{x_1}^{x_2} F(y^* + h, y'^* + h', y'' + h'')\,dx - \int_{x_1}^{x_2} F(y^*, y'^*, y''^*)\,dx - F\big|_{x_1}\delta x_1 + F\big|_{x_2}\delta x_2$$

$$= \underbrace{\int_{x_1}^{x_2} \left\{ F_y - \left(F_{y'}\right)' + \left(F_{y''}\right)'' \right\} dx + \left(F_{y''}h'\right)\Big|_{x_1}^{x_2} + \left\{ \left(F_{y'} - \left(F_{y''}\right)' \right) h \right\}\Bigg|_{x_1}^{x_2}} + \left(F\,\delta x\right)\Big|_{x_1}^{x_2}$$

From Slide 17 in Lecture 11

From Slide 10 of this lecture

$$h_1 = \delta y_1 - y_1'\,\delta x_1 \qquad h_2 = \delta y_2 - y_2'\,\delta x_2$$

$$\Rightarrow \quad h_1' = \delta y_1' - y_1''\delta x_1 \qquad h_2' = \delta y_2' - y_2''\delta x_2$$

Extended transversality conditions

$$\Delta J \approx \int_{x_1}^{x_2} \left\{ F_y - \left(F_{y'}\right)' + \left(F_{y''}\right)'' \right\} dx + \left(F_{y''} h'\right)\Big|_{x_1}^{x_2} + \left\{ \left(F_{y'} - \left(F_{y''}\right)' \right) h \right\}\Big|_{x_1}^{x_2} + \left(F\,\delta x\right)\Big|_{x_1}^{x_2} = 0$$

By invoking the fundamental lemma, we get the differential equation:

$$F_y - \left(F_{y'}\right)' + \left(F_{y''}\right)'' = 0$$

Boundary conditions

$$\left(F_{y''}\,\delta y'\right)\Big|_{x_1}^{x_2} = 0$$

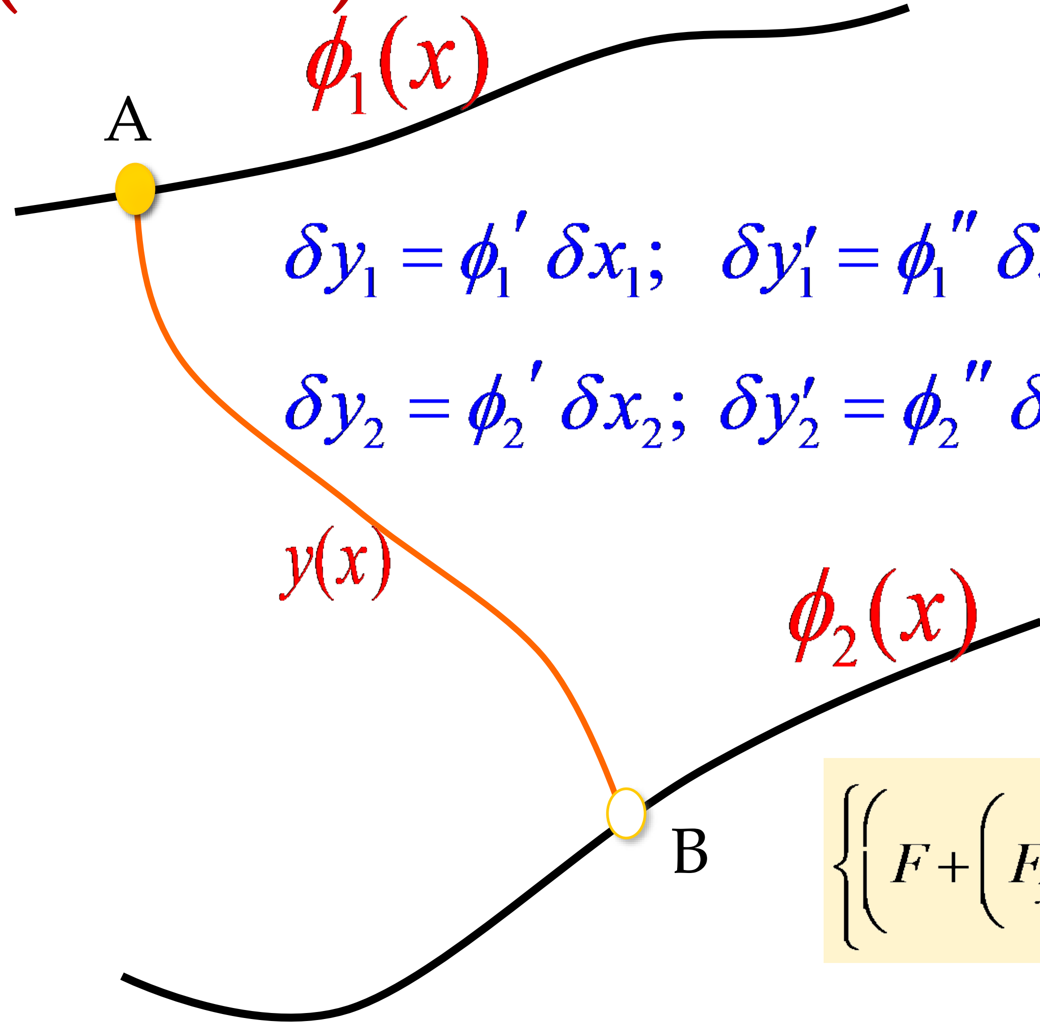
$$\left\{ \left(F_{y'} - \left(F_{y''}\right)' \right) \delta y \right\}\Big|_{x1}^{x_2} = 0 \quad \text{and}$$

$$\left\{ \left(F - F_{y'} y' + \left(F_{y''}\right)' y' - F_{y''} y'' \right) \delta x \right\}\Big|_{x1}^{x_2} = 0$$

Note that the differential equation, the Euler-Lagrange equation, did not change, once again!
It does not in all cases when the end conditions change.

Note that the boundary condition of the fixed end conditions comes out neatly when the variation in the end conditions are zero. That is, when

Extended transversality conditions (contd.)



$$\delta y_1 = \phi_1' \,\delta x_1; \quad \delta y_1' = \phi_1'' \,\delta x_1$$

$$\delta y_2 = \phi_2' \,\delta x_2; \quad \delta y_2' = \phi_2'' \,\delta x_2$$

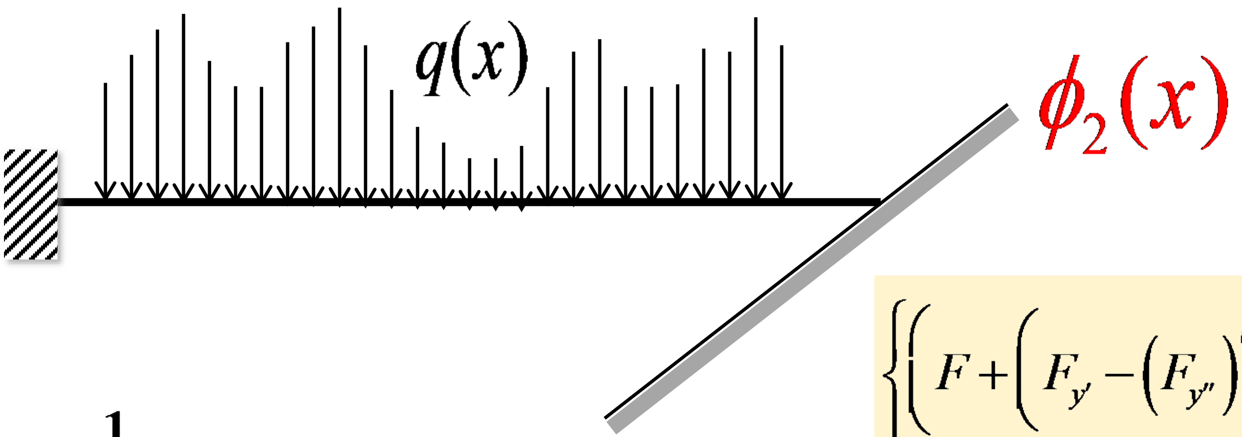
$$\left(F_{y''}\,\delta y'\right)\Big|_{x_1}^{x_2} = 0$$

$$\left\{\left(F_{y'} - \left(F_{y''}\right)'\right)\delta y\right\}\Big|_{x1}^{x_2} = 0 \quad \text{and}$$

$$\left\{\left(F - F_{y'}y' + \left(F_{y''}\right)'y' - F_{y''}y''\right)\delta x\right\}\Big|_{x1}^{x_2} = 0$$

$$\left\{\left(F + \left(F_{y'} - \left(F_{y''}\right)'\right)\left(\phi' - y'\right) + F_{y''}\left(\phi'' - y''\right)\right)\delta x\right\}\Big|_{x1}^{x_2} = 0$$

Back to the guided beam…



$$\left\{\left(F+\left(F_{y'}-\left(F_{y''}\right)'\right)\left(\phi'-y'\right)+F_{y''}\left(\phi''-y''\right)\right)\delta x\right\}\Bigg|_{x1}^{x_2}=0$$

$$F=\frac{1}{2}EI\left(y''\right)^2-qw$$ because

$$\underset{w(x)}{\text{Min}}\ J=\int_0^L\left\{\frac{1}{2}EI\left(y''\right)^2-qw\right\}dx$$

$$\left\{\left(\frac{1}{2}EI\left(w''\right)^2-qw-\left(EIw''\right)'\left(\phi_2'-y'\right)+EIw''\left(\phi_2''-y''\right)\right)\right\}\Bigg|_{x_2}=0$$

For two functions in one variable

$$\underset{y(x),z(x)}{\text{Min}} \; J = \int_{x_1}^{x_2} F(x,y,z,y',z')\, dx$$

With variable end conditions

$$x_1 = \phi_1(y,z)$$

$$x_2 = \phi_2(y,z)$$

$$F_y - \left(F_{y'}\right)' = 0$$

$$F_z - \left(F_{z'}\right)' = 0$$

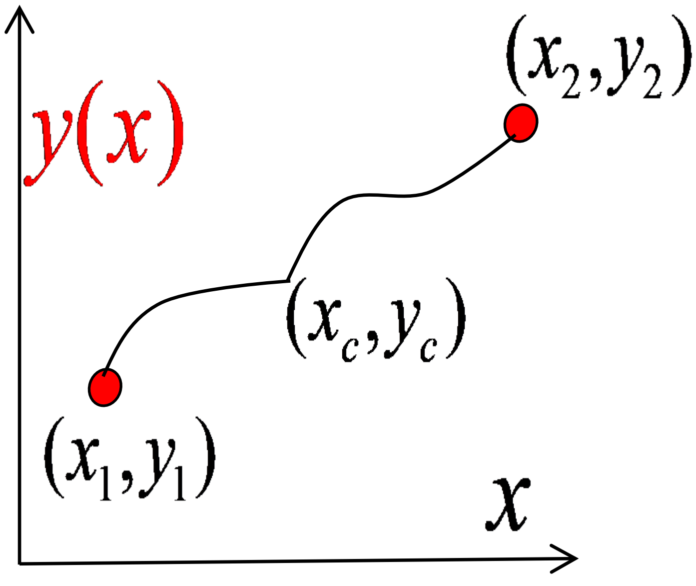
Differential equations do not change, as usual.

Transversality conditions

$$\left[F_{y'} + \frac{\partial \phi_{1\,\mathrm{or}\,2}(y,z)}{\partial y} \left(F - y'F_{y'} - z'F_{z'} \right) \right] \Bigg|_{x_1}^{x_2} = 0$$

$$\left[F_{z'} + \frac{\partial \phi_{1\,\mathrm{or}\,2}(y,z)}{\partial z} \left(F - y'F_{y'} - z'F_{z'} \right) \right] \Bigg|_{x_1}^{x_2} = 0$$

Minimal curves need not be smooth!



$$\underset{y(x)}{\text{Min}} \ J = \int_0^L \left(F(y, y')\right) dx$$

$$= \int_0^{x_c} \left(F_1(y, y')\right) dx + \int_{x_c}^{L} \left(F_2(y, y')\right) dx$$

So far, we had assumed that minimum curves are smooth, i.e., the slope of y is continuous. But what if it is not?

We get a kink or a sudden bend in the curve.

Such extremal curves are called broken extremals.

They happen in problems where something in the integrand of the function suddenly changes.

In such a case, variable conditions equations come to rescue us.

Broken extremal conditions

$$\underset{y(x)}{\text{Min}} \; J = \int_0^L \left(F(y, y') \right) dx$$

$$= \int_0^{x_c} \left(F_1(y, y') \right) dx + \int_{x_c}^{L} \left(F_2(y, y') \right) dx$$

For the two parts… for one on the right side and the other on the left side.

$$\left(F_{y'} \delta y \right) \Big|_{x_1}^{x_2} = 0 \quad \text{and}$$

$$\left\{ \left(F - F_{y'} y' \right) \delta x \right\} \Big|_{x1}^{x_2} = 0$$

$$\Rightarrow$$

$$\left((F_{y'})_1 - (F_{y'})_2 \right) \delta y \Big|_{x_c} = 0 \quad \text{and}$$

$$\left\{ \left(F - F_{y'} y' \right)_1 - \left(F - F_{y'} y' \right)_2 \right\} \delta x \Big|_{x_c} = 0$$

So…

Weierstrass-Erdmann corner conditions

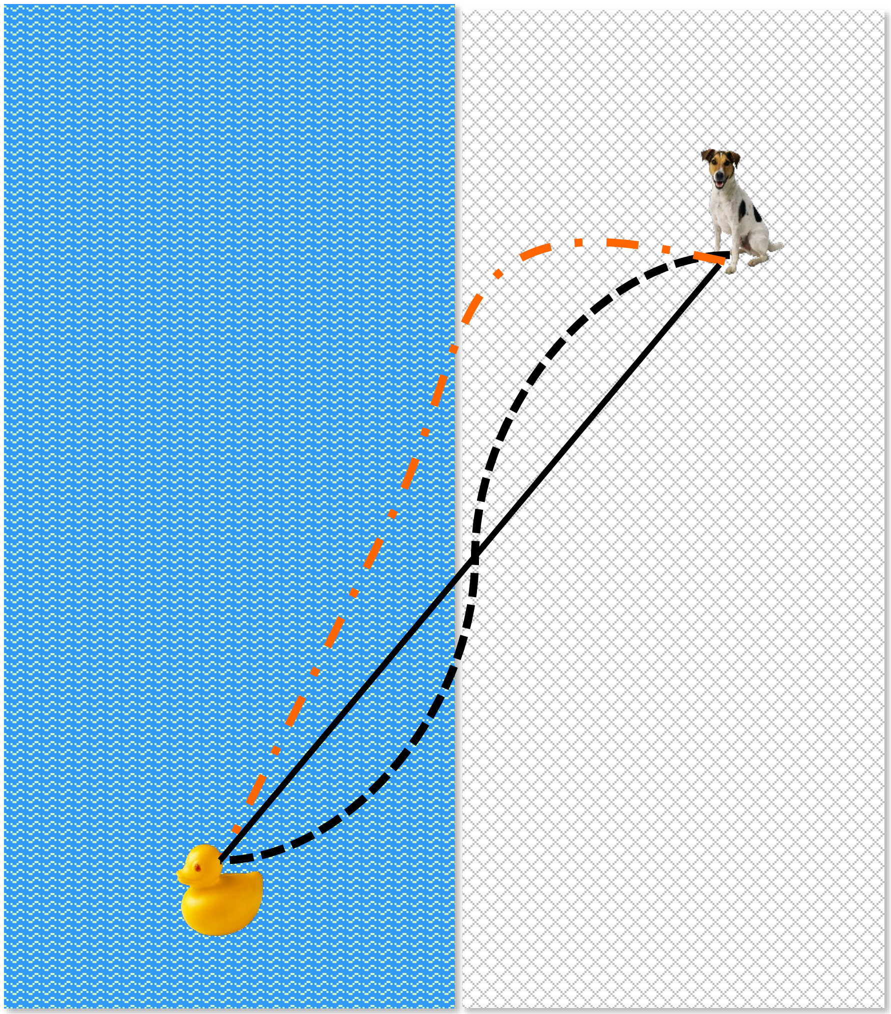
$$\left((F_{y'})_1 - (F_{y'})_2\right)\delta y\Big|_{x_c} = 0 \quad \text{and}$$

$$\left\{\left(F - F_{y'}y'\right)_1 - \left(F - F_{y'}y'\right)_2\right\}\delta x\Big|_{x_c} = 0$$

So, whenever the intermediate point is variable…

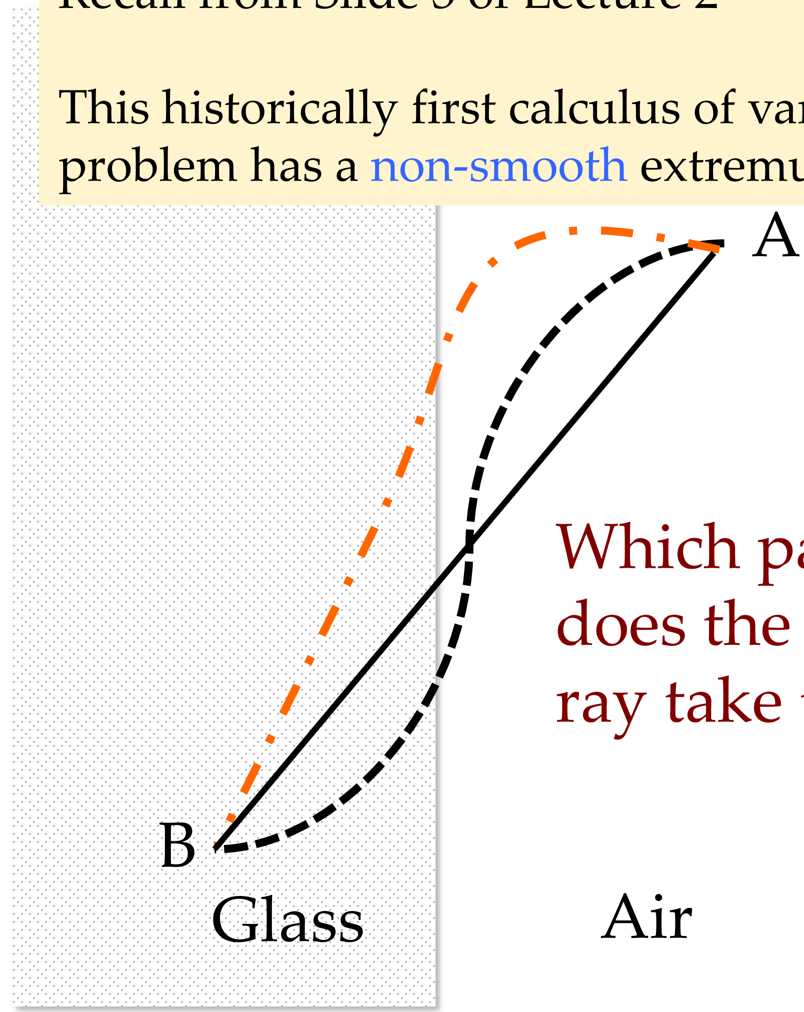
$F_{y'}$ and $\left(F - F_{y'}y'\right)$ are continuous at the intermediate corner point.

Broken (non-smooth) extremals



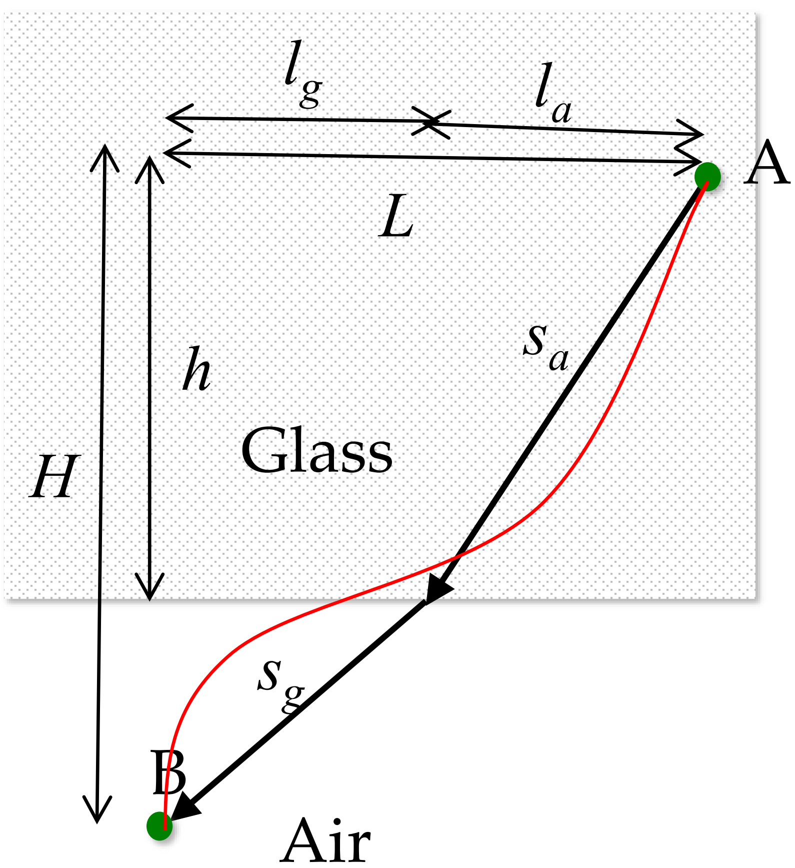
Recall from Slide 3 of Lecture 2

This historically first calculus of variations problem has a non-smooth extremum!



Which path does the light ray take then?

Refraction of light; non-smooth solution

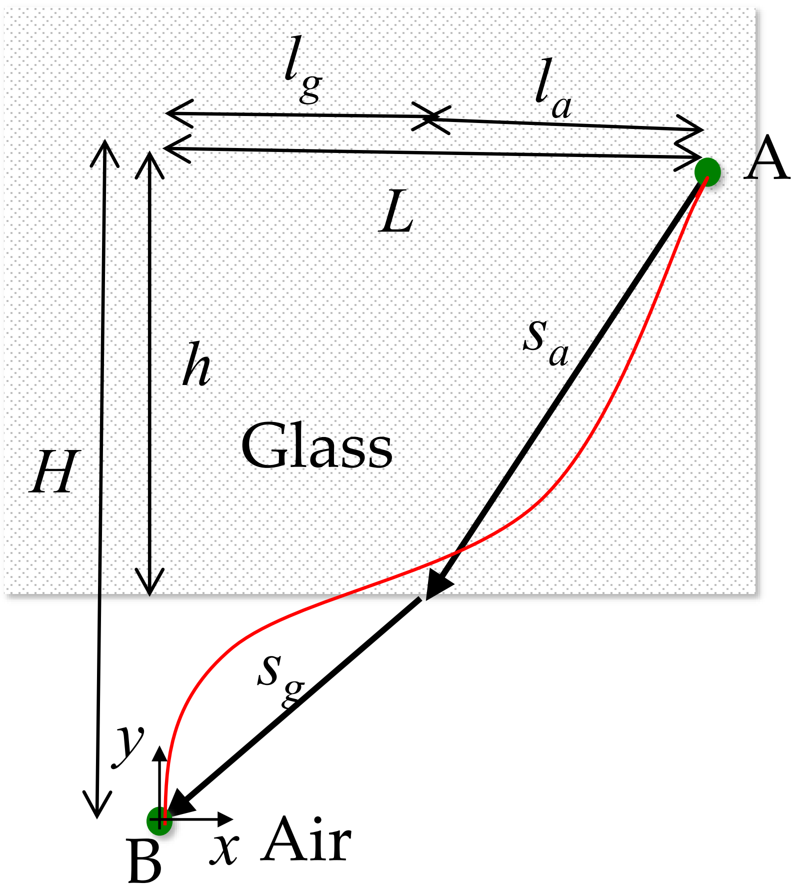


$$\underset{y(x)}{\text{Min}} \; T = \int_0^L \left(\frac{\sqrt{1+y'^2}}{v(y)} \right) dx$$

$v(x)$= speed of light ray changes at the interface between the two media.

We do not know for what x value, the bend takes place.
This is given by variable end conditions. Let us see…

Intermediate variable end condition

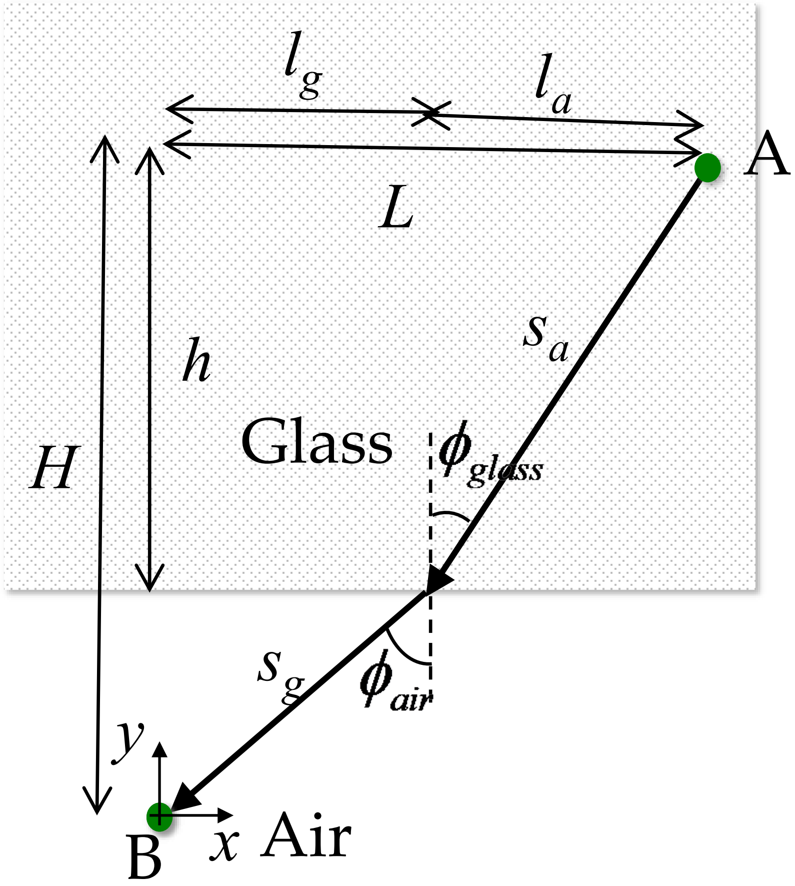


$$\underset{y(x)}{\text{Min}}\ T = \int_0^L \left(\frac{\sqrt{1+y'^2}}{v(y)} \right) dx$$

$$= \int_0^{x_c} \left(\frac{\sqrt{1+y'^2}}{v_{\text{air}}} \right) dx + \int_{x_c}^{L} \left(\frac{\sqrt{1+y'^2}}{v_{\text{glass}}} \right) dx$$

Now, for the two parts, x_c is a variable end condition!

Broken extremal conditions for a light ray



$$\left(F_{y'}\delta y\right)\Big|_{x_1}^{x_2} = 0 \quad \text{and}$$

$$\left\{\left(F - F_{y'}y'\right)\delta x\right\}\Big|_{x1}^{x_2} = 0$$

$$F = \frac{\sqrt{1+y'^2}}{v}$$

$$F_{y'} = \frac{y'}{v\sqrt{1+y'^2}}$$

$$\left(F - F_{y'}y'\right) = \frac{1}{v\sqrt{1+y'^2}}$$

Snell's law from the corner condition

$$F - y'F_{y'} = \frac{1}{v\sqrt{1+y'^2}}$$ is continuous at the corner. So, …

$$\frac{1}{v_{air}\sqrt{1+{y'_{air}}^2}} = \frac{1}{v_{glass}\sqrt{1+{y'_{glass}}^2}}$$

$$\Rightarrow \frac{1}{v_{air}\sqrt{1+\tan^2\theta_{air}}} = \frac{1}{v_{glass}\sqrt{1+\tan^2\theta_{glass}}}$$

$$\Rightarrow \frac{\cos\theta_{air}}{v_{air}} = \frac{\cos\theta_{glass}}{v_{glass}} \Rightarrow \frac{\sin\phi_{air}}{v_{air}} = \frac{\sin\phi_{glass}}{v_{glass}}$$

$$\theta = \frac{\pi}{2} - \phi$$

The first corner condition also holds good here. Because δy is zero.

Thus, we derived Snell's law using calculus of variations.

The end note

General variation; transversality conditions and broekn extremals

- Variable end conditions
- General variation
- Variable end conditions for first and second derivatives cases
  Transversality conditions
- Transversality conditions for the two-function case
- Broken extremals
  Weierstrass-Erdmann corner conditions

Thanks