

Lecture 17b

Solutions to two problems in optimizing a bar

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Solve Problem 1 analytically.

Solve Problem 2 numerically.

What we will learn:

General procedure to write necessary conditions for a structural optimization problem.

Interpreting the optimality criterion.

Numerical solution of a structural optimization problem using the optimality criteria method.

General structure of a structural optimization problem

A simple problem

$$\underset{A(x)}{\text{Min}} \ MC = \int_0^L p u \, dx$$

Subject to

$$\lambda(x): \quad (EAu')' + p = 0$$

$$\Lambda: \quad \int_0^L A \, dx - V^* \leq 0$$

Data: $L, p(x), E, V^*$

Optimize
w.r.t. design
variables

Subject to

*They create conflict in
optimizing the objective
function.*

Data

*(This should be properly chosen although the
nature of the solution is not decided by the
data.)*

Objective function

*(It depends on design variables
and state variables.)*

(They govern state variables.)
Governing equation(s)

Resource constraint(s)
Performance constraints

Steps in the solution procedure

Step 1: Write the Lagrangian

Step 2: Take variation of the Lagrangian w.r.t. the design variable and equate to zero to get the design equation.

Step 3: Take variation of the Lagrangian w.r.t. state variable(s) and equate to zero to get the adjoint equation.

Step 4: Collect all the equations, including the governing equation(s), complementarity condition(s), resource constraints, etc.

Step 5: Obtain the optimality criterion by substituting adjoint and equilibrium equations into the design equation, when it is possible.

Step 6: Identify all boundary conditions.

Step 7: Solve the equations analytically as much as possible.

Step 8: Use the optimality criteria method to solve the equations numerically.

Consider Problem 1 to solve.

$$\underset{A(x)}{\text{Min}} \ MC = \int_0^L p u \, dx$$

Subject to

$$\lambda(x): \quad (EAu')' + p = 0$$

$$\Lambda: \quad \int_0^L A \, dx - V^* \leq 0$$

$$\text{Data: } L, p(x), E, V^*$$

Step 1 to solve Problem 1

Step 1: Write the Lagrangian

$$\underset{A(x)}{\text{Min}} \ MC = \int_0^L p u \, dx$$

Subject to

$$\lambda(x): \quad (EAu')' + p = 0$$

$$\Lambda: \quad \int_0^L A \, dx - V^* \leq 0$$

Data: $L, p(x), E, V^*$

Lagrangian

$$L = \int_0^L p u \, dx + \int_0^L \lambda \left\{ (EAu')' + p \right\} dx + \Lambda \left(\int_0^L A \, dx - V^* \right)$$

Integrand in the Lagrangian functional

$$\hat{L} = p u + \lambda \left\{ (EAu')' + p \right\} + \Lambda A$$

Step 2 to solve Problem 1

Step 2: Take variation of the Lagrangian w.r.t. the design variable and equate to zero to get the design equation.

$$\hat{L} = pu + \lambda \left\{ (EAu')' + p \right\} + \Lambda A = pu + E\lambda A'u' + E\lambda A'u'' + \Lambda A$$

$$\delta_A L = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0$$

$$\Rightarrow E\lambda u'' + \Lambda - (E\lambda u')' = 0$$

$$\Rightarrow \Lambda - E\lambda'u' = 0$$

What multiplies the Lagrange multiplier is the sensitivity of the corresponding constraint. Here, it is unity.

$$\{\Lambda\} + \{-E\lambda'u'\} = 0$$



Design equation

Sensitivity
(derivative) of the
objective function

Step 3 to solve Problem 1

Step 3: Take variation of the Lagrangian w.r.t. state variable(s) and equate to zero to get the adjoint equation.

$$\hat{L} = pu + \lambda \left\{ (EAu')' + p \right\} + \Lambda A = pu + E\lambda A'u' + E\lambda Au'' + \Lambda A$$

$$\delta_u L = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'} \right)' + \left(\frac{\partial \hat{L}}{\partial u''} \right) = 0 \quad \text{Adjoint (equilibrium) equation}$$
$$p + (EA\lambda')' = 0$$

$$\Rightarrow p - (E\lambda A')' + (E\lambda A)'' = 0$$

Adjoint load;
here it is equal to
the actual load
because the
objective function
is mean
compliance.

$$\Rightarrow p + (EA\lambda')' = 0$$

This has the same
form as the
governing
differential
equation.

Step 4 to solve Problem 1

Step 4: Collect all the equations, including the governing equation(s), complementarity condition(s), resource constraints, etc.

$$\{\Lambda\} + \{-E\lambda'u'\} = 0 \quad \text{Design equation}$$

$$p + (EA\lambda')' = 0 \quad \text{Adjoint (equilibrium) equation}$$

$$p + (EAu')' = 0 \quad \text{Equilibrium (governing) equation}$$

$$\Lambda \left(\int_0^L A dx - V^* \right) = 0, \Lambda \geq 0 \quad \text{Complementarity conditions}$$

$$\int_0^L A dx - V^* \leq 0 \quad \text{Feasibility condition}$$

Step 5 to solve Problem 1

Step 5: Obtain the optimality criterion by substituting adjoint and equilibrium equations into the design equation, when it is possible.

$$\{\Lambda\} + \{-E\lambda'u'\} = 0$$

$$p + (EA\lambda')' = 0$$

$$p + (EAu')' = 0$$

$$\Lambda \left(\int_0^L A dx - V^* \right) = 0, \Lambda \geq 0$$

$$\int_0^L A dx - V^* \leq 0$$

$$\left. \begin{array}{l} \lambda = u \\ \Lambda - Eu'^2 = 0 \end{array} \right\}$$

Optimality criterion

$$\Lambda - Eu'^2 = 0$$

Strain energy density is uniform along the bar.

$$u' = \pm \sqrt{\frac{\Lambda}{E}}$$

Step 6 to solve Problem 1

Step 6: Identify all boundary conditions.

$$\hat{L} = p u + \lambda \left\{ (EAu')' + p \right\} + \Lambda A = p u + E\lambda A' u' + E\lambda A u'' + \Lambda A$$

$$\left. \left(\frac{\partial \hat{L}}{\partial A'} \right) \delta A \right|_0^L = 0 \Rightarrow \left. (E\lambda u') \delta A \right|_0^L = 0 \quad \text{BC1}$$

$$\left. \left\{ \frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''} \right)' \right\} \delta u \right|_0^L = 0 \Rightarrow \left. \left\{ (E\lambda A') - (E\lambda A)' \right\} \delta u \right|_0^L = 0 \Rightarrow \left. (EA\lambda') \delta u \right|_0^L = 0 \quad \text{BC2}$$

$$\left. \left(\frac{\partial \hat{L}}{\partial u''} \right) \delta u' \right|_0^L = 0 \Rightarrow \left. (EA\lambda) \delta u' \right|_0^L = 0 \quad \text{BC3}$$

Step 7 to solve Problem 1

Step 7: Solve the equations analytically as much as possible.

$$\left. \begin{aligned} \Lambda - Eu'^2 = 0 \Rightarrow u' = \pm \sqrt{\frac{\Lambda}{E}} \\ p + (EAu')' = 0 \end{aligned} \right\} p \pm EA' \sqrt{\frac{\Lambda}{E}} = 0 \Rightarrow \Lambda \text{ cannot be zero.}$$

So, $\int_0^L A dx - V^* = 0$

$$p \pm EA' \sqrt{\frac{\Lambda}{E}} = 0 \Rightarrow A(x) = \mp \frac{1}{\sqrt{E\Lambda}} \int p(x) + C$$



Solve for Λ using $\int_0^L A dx - V^* = 0$

The sign can be chosen based on the boundary condition of the bar, as in fixed free, free-fixed, fixed-fixed.

Note that the design equation helped us solve for $A(x)$ even though it did not have A in it explicitly.

Step 7 to solve Problem 1 (contd.)

Step 7: Solve the equations analytically as much as possible.

As an example, take a fixed-free bar and $p(x) = p_0$ (a constant)

$$A(x) = -\frac{1}{\sqrt{E\Lambda}} \int p(x) + C = -\frac{p_0 x}{\sqrt{E\Lambda}} + C$$

$$u' = \sqrt{\frac{\Lambda}{E}}$$

Positive sign is chosen here because in a fixed-free bar, strain is positive when the load is positive.

Let us observe the boundary conditions.

For a fixed-free bar, we have: $u|_0 = 0 \Rightarrow \delta u|_0 = 0$ and $u'|_L = 0$

$$\left. \begin{array}{l} \text{BC1} (E\lambda u') \delta A |_0^L = 0 \\ \text{BC2} (EA\lambda') \delta u |_0^L = 0 \\ \text{BC3} (EA\lambda) \delta u' |_0^L = 0 \end{array} \right\} \left. \begin{array}{l} (Euu') \delta A |_0^L = 0 \\ (EAu') \delta u |_0^L = 0 \\ (EAu) \delta u' |_0^L = 0 \end{array} \right\} \left. \begin{array}{l} \text{All are satisfied by virtue of these, except} \\ (EAu) \delta u' |_0^L = 0 \Rightarrow A |_L^L = 0 \end{array} \right\}$$

Step 7 to solve Problem 1 (contd.)

Step 7: Solve the equations analytically as much as possible.

Since $A|_L = 0$

$$A(x) = -\frac{p_0 x}{\sqrt{E\Lambda}} + C \Rightarrow C = \frac{p_0 L}{\sqrt{E\Lambda}}$$

Now, let us solve for Λ using $\int_0^L A dx - V^* = 0$

$$\int_0^L A dx - V^* = 0 \Rightarrow \int_0^L \frac{p_0(L-x)}{\sqrt{E\Lambda}} dx = V^*$$

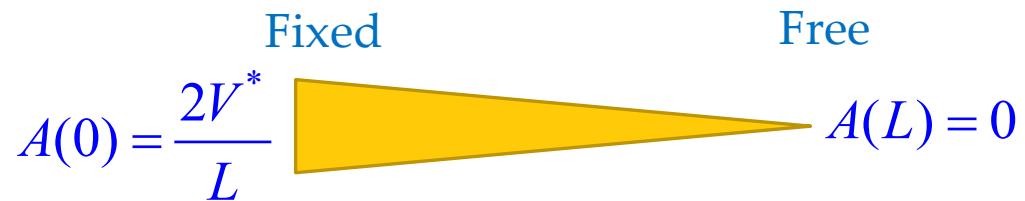
$$\Rightarrow \frac{p_0}{\sqrt{E\Lambda}} = \frac{V^*}{\int_0^L (L-x) dx} = \frac{2V^*}{L^2}$$

Thus, $A(x) = \frac{p_0(L-x)}{\sqrt{E\Lambda}}$

$$\int_0^L A dx - V^* = 0$$

Thus, $A(x) = \frac{2V^*(L-x)}{L^2}$

Note that optimal area of cross-section of a fixed-free bar under uniform loading depends neither on the value of the load nor the material property, E .
This is true in general too!



Step 8 to solve Problem 1

Step 8: Use the **optimality criteria** method to solve the equations numerically.

Set up the update scheme as:

$$A_{k+1}(x) = A_k(x) + \left(\Lambda - Eu'^2 \right)_k \quad \text{or}$$

This scheme is preferred.

$$A_{k+1}(x) = \left(\frac{Eu'^2}{\Lambda} \right)_k A_k(x)$$

This is called the fixed-point method.

k is the iteration counter.

$$A_{k+1}(x) = \left\{ \left(\frac{Eu'^2}{\Lambda} \right)^\beta \right\}_k A_k(x)$$

Choose $A_0(x)$ as the initial guess.

β is the tuning parameter in the numerical algorithm.

Evaluate Λ using $\int_0^L A dx - V^* = 0$

Update $A_{k+1}(x)$ until convergence, i.e., until $A_{k+1}(x) - A_k(x) \leq$ tolerance.

More on it when we solve Problem 8.

Note the steps once more.

Step 1: Write the Lagrangian.

Step 2: Take variation of the Lagrangian w.r.t. the design variable and equate to zero to get the design equation.

Step 3: Take variation of the Lagrangian w.r.t. state variable(s) and equate to zero to get the adjoint equation.

Step 4: Collect all the equations, including the governing equation(s), complementarity condition(s), resource constraints, etc.

Step 5: Obtain the optimality criterion by substituting adjoint and equilibrium equations into the design equation, when it is possible.

Step 6: Identify all boundary conditions.

Step 7: Solve the equations analytically as much as possible.

Step 8: Use the optimality criteria method to solve the equations numerically.

Let us consider Problem 8 now.

$$\underset{A(x)}{\text{Min}} \ MC = \int_0^L p u \, dx$$

Subject to

$$\lambda(x): \quad (EAu')' + p = 0$$

$$\Lambda: \quad \int_0^L A \, dx - V^* \leq 0$$

$$\mu_u(x): \quad A - A_u \leq 0$$

$$\mu_l(x): \quad A_l - A \leq 0$$

$$\text{Data: } L, p(x), E, V^*, A_l, A_u$$

Step 1 to solve Problem 8

Step 1: Write the Lagrangian

$$\underset{A(x)}{\text{Min}} \ MC = \int_0^L p u \, dx$$

Subject to

$$\lambda(x): \quad (EAu')' + p = 0$$

$$\Lambda: \quad \int_0^L A \, dx - V^* \leq 0$$

$$\mu_u(x): \quad A - A_u \leq 0$$

$$\mu_l(x): \quad A_l - A \leq 0$$

Data: $L, p(x), E, V^*, A_l, A_u$

Lagrangian

$$L = \int_0^L p u \, dx + \int_0^L \lambda \left\{ (EAu')' + p \right\} dx + \Lambda \left(\int_0^L A \, dx - V^* \right) \\ + \int_0^L \mu_u (A - A_u) \, dx + \int_0^L \mu_l (A_l - A) \, dx$$

Integrand in the Lagrangian functional

$$\hat{L} = p u + \lambda \left\{ (EAu')' + p \right\} + \Lambda A + \mu_u (A - A_u) + \mu_l (A_l - A)$$

Step 2 to solve Problem 8

Step 2: Take variation of the Lagrangian w.r.t. the design variable and equate to zero to get the design equation.

$$\begin{aligned}\hat{L} &= pu + \lambda \left\{ (EAu')' + p \right\} + \Lambda A + \mu_u (A - A_u) + \mu_l (A_l - A) \\ &= pu + E\lambda A'u' + E\lambda A u'' + \Lambda A + \mu_u (A - A_u) + \mu_l (A_l - A)\end{aligned}$$

$$\delta_A L = 0 \Rightarrow \frac{\partial \hat{L}}{\partial A} - \left(\frac{\partial \hat{L}}{\partial A'} \right)' = 0 \quad \text{Design equation}$$
$$\{\Lambda\} + \{\mu_u\} - \{\mu_l\} + \{-E\lambda'u'\} = 0$$

$$\begin{aligned}\Rightarrow E\lambda u'' + \Lambda + \mu_u - \mu_l - (E\lambda u')' &= 0 \\ \Rightarrow \Lambda + \mu_u - \mu_l - E\lambda'u' &= 0\end{aligned}$$

What multiplies the Lagrange multiplier is the sensitivity of the corresponding constraint. Here, it is positive or negative unity for all three.

Sensitivity (derivative) of the objective function

Step 3 to solve Problem 8

Step 3: Take variation of the Lagrangian w.r.t. state variable(s) and equate to zero to get the adjoint equation.

$$\hat{L} = p u + E \lambda A' u' + E \lambda A u'' + \Lambda A + \mu_u (A - A_u) + \mu_l (A_l - A)$$

$$\delta_u L = 0 \Rightarrow \frac{\partial \hat{L}}{\partial u} - \left(\frac{\partial \hat{L}}{\partial u'} \right)' + \left(\frac{\partial \hat{L}}{\partial u''} \right) = 0 \quad \text{Adjoint (equilibrium) equation}$$
$$p + (EA\lambda')' = 0$$

$$\Rightarrow p - (EA\lambda')' + (EA\lambda)'' = 0$$

Adjoint load;
here it is equal to
the actual load
because the
objective function
is mean
compliance.

$$\Rightarrow p + (EA\lambda')' = 0$$

No change here
as compared to
Problem 1.

Step 4 to solve Problem 8

Step 4: Collect all the equations, including the governing equation(s), complementarity condition(s), resource constraints, etc.

$$\Lambda + \mu_u - \mu_l - E\lambda' u' = 0 \quad \text{Design equation}$$

$$p + (EA\lambda')' = 0 \quad \text{Adjoint (equilibrium) equation}$$

$$p + (EAu')' = 0 \quad \text{Equilibrium (governing) equation}$$

$$\Lambda \left(\int_0^L A dx - V^* \right) = 0, \Lambda \geq 0; \quad \mu_u (A - A_u) = 0, \mu_u \geq 0; \quad \mu_l (A_l - A) = 0, \mu_l \geq 0; \quad \text{Complementarity conditions}$$

$$\int_0^L A dx - V^* \leq 0; \quad A - A_u \leq 0; \quad (A_l - A) \leq 0 \quad \text{Feasibility conditions}$$

Step 5 to solve Problem 8

Step 5: Obtain the optimality criterion by substituting adjoint and equilibrium equations into the design equation, when it is possible.

$$\left. \begin{array}{l} \Lambda + \mu_u - \mu_l - E\lambda'u' = 0 \\ p + (EA\lambda')' = 0 \\ p + (EAu')' = 0 \end{array} \right\} \lambda = u \quad \left. \begin{array}{l} \text{Optimality criterion} \\ \Lambda + \mu_u - \mu_l - Eu'^2 = 0 \end{array} \right\}$$

These are functions of x ; so, the strain energy density is not necessarily constant throughout.

Step 6 to solve Problem 8

Step 6: Identify all boundary conditions.

$$\hat{L} = pu + E\lambda A'u' + E\lambda Au'' + \Lambda A + \mu_u(A - A_u) + \mu_l(A_l - A)$$

$$\left. \left(\frac{\partial \hat{L}}{\partial A'} \right) \delta A \right|_0^L = 0 \Rightarrow (E\lambda u' + \mu_u - \mu_l) \delta A \Big|_0^L = 0 \quad \text{BC1}$$

$$\left. \left\{ \frac{\partial \hat{L}}{\partial u'} - \left(\frac{\partial \hat{L}}{\partial u''} \right)' \right\} \delta u \right|_0^L = 0 \Rightarrow \left. \left\{ (E\lambda A') - (E\lambda A)' \right\} \delta u \right|_0^L = 0 \Rightarrow (EA\lambda') \delta u \Big|_0^L = 0 \quad \text{BC2}$$

$$\left. \left(\frac{\partial \hat{L}}{\partial u''} \right) \delta u' \right|_0^L = 0 \Rightarrow (EA\lambda) \delta u' \Big|_0^L = 0 \quad \text{BC3}$$

Step 7 to solve Problem 8

Step 7: Solve the equations analytically as much as possible.

Optimality criterion

$$\Lambda + \mu_u - \mu_l - Eu'^2 = 0$$

We have three cases now.

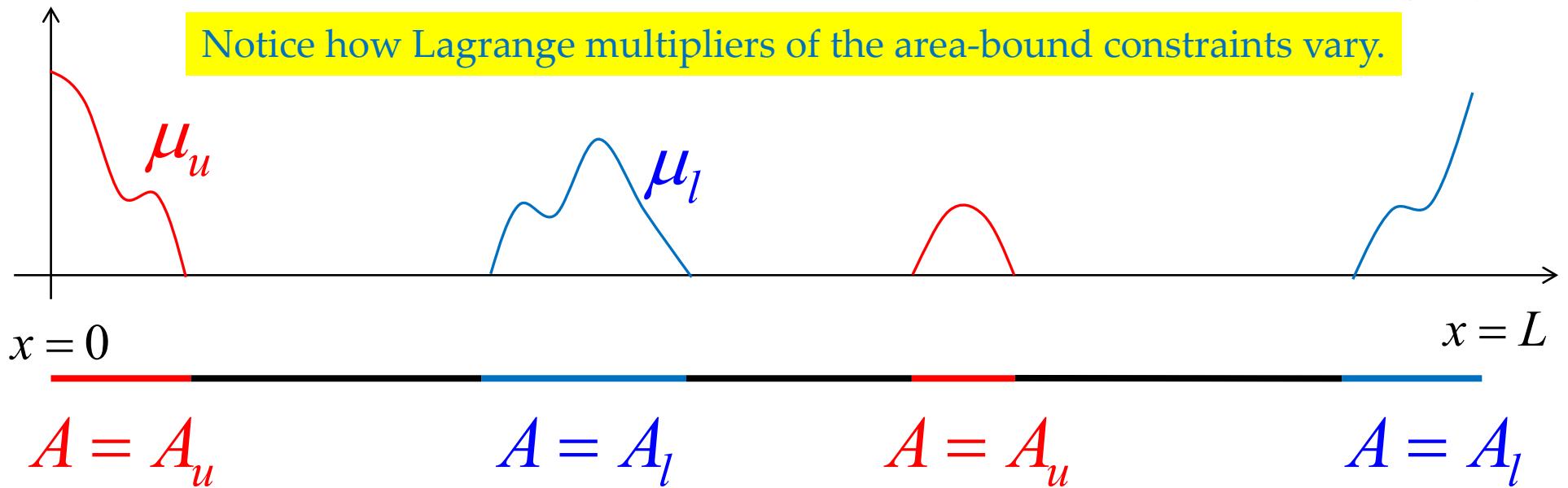
Case 1: $\mu_u > 0, \mu_l = 0 \Rightarrow A = A_u$ Area of cross-section is thus known.

Case 2: $\mu_l > 0, \mu_u = 0 \Rightarrow A = A_l$ Area of cross-section is thus known.

Case 3: $\mu_l = 0, \mu_u = 0 \Rightarrow$ We have dealt with this in Problem 1.

We need to partition the domain $(0, L)$ into these three cases. We do this numerically using the optimality criteria method.

Partitioning the domain into three types



$$A_{k+1}(x) = \left\{ \left(\frac{E u'^2}{\Lambda} \right)^\beta \right\}_k A_k(x)$$

As per the optimality criterion

This is how we update when $A(x)$ is in between the bounds.

While we partition the domain, we should ensure that the volume constraint is satisfied.

Calculate the Lagrange multiplier

Substitute $A_{k+1}(x) = \left\{ \left(\frac{Eu'^2}{\Lambda} \right)^\beta \right\}_k A_k(x)$ into $\int_0^L A dx - V^* = 0$



$x = 0$ $x = L$

$A = A_u$ $A = A_l$ $A = A_u$ $A = A_l$

$$\int_{\Omega_u} A_u dx + \int_{\Omega} A dx + \int_{\Omega_l} A_l dx = V^* \Rightarrow \int_{\Omega_u} A_u dx + \int_{\Omega} \left\{ \left(\frac{Eu'^2}{\Lambda} \right)^\beta \right\}_k A_k(x) dx + \int_{\Omega_l} A_l dx = V^*$$

$$\Rightarrow \int_{\Omega} \left\{ \left(\frac{Eu'^2}{\Lambda} \right)^\beta \right\}_k A_k(x) dx = V^* - \int_{\Omega_u} A_u dx - \int_{\Omega_l} A_l dx \Rightarrow \Lambda = \left[\frac{\int_{\Omega} \left\{ \left(Eu'^2 \right)^\beta \right\}_k A_k(x) dx}{V^* - \int_{\Omega_u} A_u dx - \int_{\Omega_l} A_l dx} \right]^{1/\beta}$$

Step 8 to solve Problem 8

Step 8: Use the **optimality criteria** method to solve the equations numerically.

This is the step-wise procedure of the optimality criteria method.

(i) Choose $A_0(x)$ as the initial guess.

$$(ii) \quad A_{k+1}(x) = \left\{ \left(\frac{Eu'^2}{\Lambda} \right)^\beta \right\}_k A_k(x)$$

(iii) Evaluate Λ using $\int_0^L A dx - V^* = 0$

(iv) Ensure that everywhere $A \leq A_u ; A_l \leq A$

This step is iterative; we call it the inner iterative loop. This is where we partition the domain into three types.

(v) Update $A_{k+1}(x)$ until convergence, i.e., until $A_{k+1}(x) - A_k(x) \leq \text{tolerance}$.

Outer and inner loop iterations

Begin the outer loop with $A_0(x)$

$$\Lambda = \left[\frac{\int_{\Omega} \left\{ (Eu'^2)^{\beta} \right\}_k A_k(x) dx}{V^*} \right]^{1/\beta}$$

$$A_k(x)$$

Compute

$$\text{Update } A_{k+1}(x) = \left\{ \left(\frac{Eu'^2}{\Lambda} \right)^{\beta} \right\}_k A_k(x)$$

Inner loop begins here.

Check against bounds: *If* $A_{k+1}(x) > A_u$, $A_{k+1}(x) = A_u$ Add that x to Ω_u

If $A_{k+1}(x) < A_l$, $A_{k+1}(x) = A_l$ Add that x to Ω_l

Now, recalculate Λ As show in the next slide.

Inner loop (contd.)

$$\Lambda = \left[\frac{\int_{\Omega} \left\{ \left(Eu'^2 \right)^{\beta} \right\}_k A_k(x) dx}{V^* - \int_{\Omega_u} A_u dx - \int_{\Omega_l} A_l dx} \right]^{1/\beta}$$

Update again $A_{k+1}(x) = \left\{ \left(\frac{Eu'^2}{\Lambda} \right)^{\beta} \right\}_k A_k(x)$

Check against bounds If $A_{k+1}(x) > A_u$, $A_{k+1}(x) = A_u$ update Ω_u
again

 If $A_{k+1}(x) < A_l$, $A_{k+1}(x) = A_l$ update Ω_l

Now, recalculate Λ

Repeat this until partitioning does not change.

Optimality criteria method

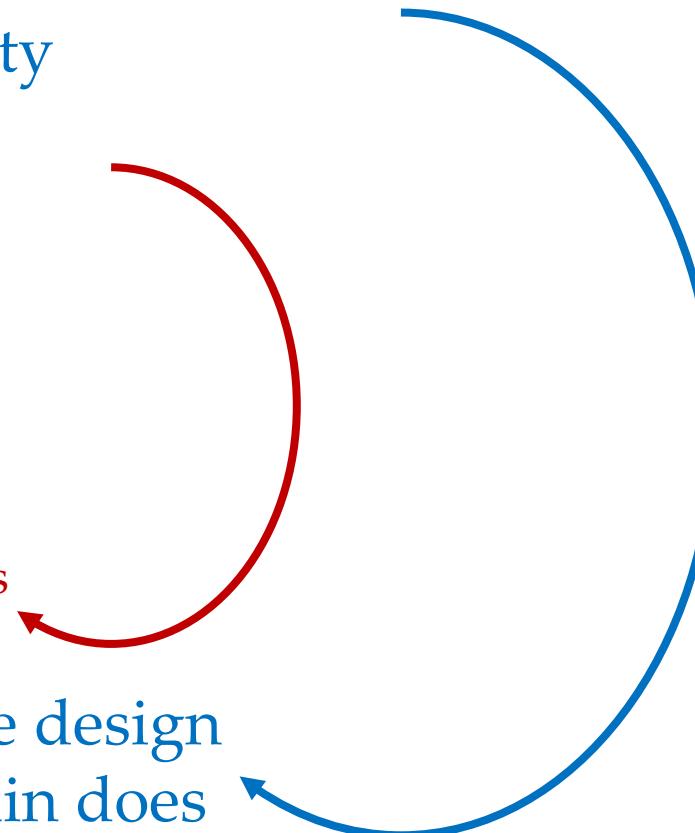
Updating area of cross-section is the outer loop using the optimality criterion.

Updating Lambda and partitioning of the domain is the inner loop.

Note that area of cross-section gets update in the inner loop also.

This ends when partitioning does not change anymore.

The outer loop ends when the design variable over the entire domain does not change anymore.



Check out the BarOpt Matlab code.

BarOpt has four files:

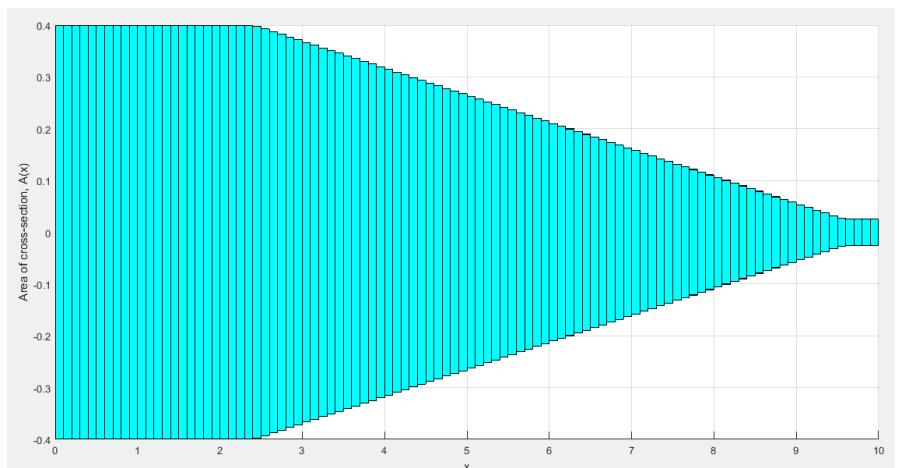
`baropt.m` >> implements optimality criteria method.

`fembar.m` >> finite element code for bar elements.

`matcut.m` and `veccut.m` >> These are used by `fembar.m`

`baropt.m` may be modified by you to change the data such as the length of the bar, loading, displacement boundary conditions, the number of elements in the bar, total number of iterations, the tolerance to stop the iterative process, etc.

A representative result of optimized area of cross-section.



The end note

Two simple and illustrative structural optimization problems pertaining to a bar.

Axially deforming bar is the simplest structural optimization problem.

We considered two cases of this.

The first is general but it illustrates how optimality criterion comes about. The second considers restrictions on the design variable, which results in the optimality criterion not being satisfied everywhere.

Problem 1 was solved completely analytically. Problem 8 needed numerical method.

Optimality criteria method and its algorithm were discussed.

Problem 8 is solved numerically; Matlab script provided separately.

Thanks