Lecture 18a

Many problems in optimizing a beam

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Some problems in optimizing the cross-section profile of a beam under transverse loading.

What we will learn:

How to apply the concepts and ideas learned so far to solve problems in calculus of variations with particular examples of transversely loaded beams.

How is a beam different from a bar?
(in the context of structural optimization)

A bar deforms axially whereas a beam displaces transversely. That is, a beam bends. The governing differential equation for a bar is of second order whereas that for a beam is fourth order. So, we should be prepared for tedious calculations in a beam.

Cross-section area is all that matters for volume and stiffness in a slender bar: its shape does not matter. That is not true for a beam. The volume of a beam depends on the value of cross-section area but the stiffness depends on the second moment of area of cross-section. So, shape of the cross-section matters.

In this set of problems, we consider only rectangular cross-section of beams. A rectangular has two dimensions, breadth, *b(x)*, and depth, *t(x)*. Both of these can be varied independently (provided that such a beam can be manufactured economically). However, most often, we vary only one of them.

If is *b(x)* varied, $A(x) = b(x)t; \quad I(x) = b(x)t^3/12 = b(x)t(t^2/12) = (t^2/12)A(x)$

If is *t(x)* varied, $A(x) = b\,t(x); \quad I(x) = bt(x)^3/12 = \left(bt(x)\right)^3/(12b^2) = (1/12b^2)A^3(x)$

For most cross-section shapes, we can write: $I(x) = \alpha A^{\beta}$

Problem 1

Minimize the mean compliance of a beam for given volume of material.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\, w\, dx$$

Subject to

$$\lambda(x): \qquad \left(EIw''\right)'' - q = 0 \Rightarrow \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L A\, dx - V^* \le 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = {t^2}/{12}, V^*$$

We assume here that only b(x) is variable.

$$I(x) = \alpha A^{\beta}$$

$$\alpha = \left(\frac{t^2}{12}\right);\ \beta = 1$$

Problem 2

Minimize the strain energy of a beam for given volume of material.

$$\underset{A(x)}{Min}\ SE = \int_0^L \frac{1}{2} E\alpha A w''^2\, dx$$

Subject to

$$\lambda(x): \qquad \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L A\, dx - V^* \le 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = t^2/12, V^*$$

Problem 3

Minimize the volume of a beam subject to an upper bound on the strain energy.

$$\underset{A(x)}{Min}\ V = \int_0^L A\, dx$$

Subject to

$$\lambda(x): \qquad \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L \frac{1}{2} E\alpha A w''^2\, dx - SE^* \leq 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = {t^2}/{12}, \mathrm{SE}^*$$

Problem 4

Minimize the volume of material for a given upper bound on the mean compliance.

$$\underset{A(x)}{Min}\ V = \int_0^L A\,dx$$

Subject to

$$\lambda(x): \qquad \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L q\,w\,dx - MC^* \le 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = {t^2}/{12}, \mathrm{MC}^*$$

Problem 5

Minimize the strain energy of a statically determinate beam for given volume of material. For a statically determinate beam, we can obtain the bending moment without knowing the area of cross-section of the beam.

$$\underset{A(x)}{Min}\ SE = \int_0^L \frac{M^2}{2E\alpha A} dx$$

Subject to

$$\Lambda : \quad \int_0^L A\,dx - V^* \leq 0$$

$$Data : L, M(x), E, \alpha = t^2/12, V^*$$

Problem 6

Min-max formulation for the stiffest beam for given volume of material.

$$\underset{A(x)}{Max}\ \underset{u(x)}{Min}\ PE = \int_0^L \left(\frac{1}{2} E\alpha A w''^2 - qw \right) dx$$

Subject to

$$\Lambda : \qquad \int_0^L A\,dx - V^* \le 0$$

$$Data : L, q(x), E, \alpha = t^2/12, V^*$$

Problem 7

Minimize the mean compliance of a beam for given volume of material with the governing equation in the weak form.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\, w\, dx$$

Subject to

$$\Gamma: \qquad \int_0^L \left(E\alpha A w'' v'' - qv \right) dx = 0$$

$$\Lambda: \qquad \int_0^L A\, dx - V^* \leq 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = t^2/12, V^*$$

Problem 8

Minimize the mean compliance of a beam for given volume of material and upper and lower bound constraints on the area of cross-section.

$$\underset{A(x)}{Min}\; MC = \int_0^L q\,w\,dx$$

Subject to

$$\Gamma: \qquad \int_0^L \left(E\alpha A w'' v'' - qv\right) dx = 0$$

$$\Lambda: \qquad \int_0^L A\,dx - V^* \le 0$$

$$\mu_u(x): \qquad A - A_u \le 0$$

$$\mu_l(x): \qquad A_l - A \le 0$$

$$Data: L, q(x), E, \alpha = t^2/12, V^*, A_l, A_u$$

Problem 9

Minimize the mean compliance of a beam for given volume of material where the depth of the beam is the design variable.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\,w\,dx$$

Subject to

$$\Gamma: \quad \int_0^L \left(E\alpha A^3 w'' v'' - qv \right) dx = 0$$

We assume here that only $t(x)$ is variable.

$$\Lambda: \quad \int_0^L A\,dx - V^* \leq 0$$

$$\mu_u(x): \quad A - A_u \leq 0$$

$$\mu_l(x): \quad A_l - A \leq 0$$

$$Data: L, q(x), E, \alpha = \frac{1}{12b^2}, V^*, A_l, A_u$$

$$I(x) = \alpha A^\beta$$

$$\alpha = \left(\frac{1}{12b^2} \right); \ \beta = 3$$

Problem 10

Minimize the volume of a statically determinate beam with a deflection constraint in its span.

$$\underset{A(x)}{Min}\ V = \int_0^L A\,dx$$

Subject to

$$\Lambda: \qquad \int_0^L \frac{MM_d}{E\alpha A}dx - \Delta^* = 0$$

$$Data: L, M(x), M_d(x), E, \alpha = t^2/12, \Delta^*$$

Problem 11

Minimize the volume of material of a beam (statically determinate or indeterminate) for a deflection constraint in its span.

$$\underset{A(x)}{Min}\ V = \int_0^L A\,dx$$

Subject to

$$\lambda(x): \quad \left(E\alpha A w''\right)'' - q = 0$$

$$\lambda_d(x): \quad \left(E\alpha A v''\right)'' - q_d = 0$$

$$\Lambda: \quad \int_0^L E\alpha A w'' v''\,dx - \Delta^* = 0$$

$$Data: L, q(x), q_d(x), E, \alpha = t^2/12, \Delta^*$$

Problem 12

Minimize the volume of material of a beam (statically determinate or indeterminate) for a deflection constraint in its span with an upper bound on the strain energy.

$$\underset{A(x)}{Min}\ V = \int_0^L A\,dx$$

Subject to

$$\lambda(x): \quad \left(E\alpha A w''\right)'' - q = 0$$

$$\lambda_d(x): \quad \left(E\alpha A v''\right)'' - q_d = 0$$

$$\Lambda: \quad \int_0^L E\alpha A w'' v''\,dx - \Delta^* = 0$$

$$\Gamma: \quad \int_0^L \frac{1}{2} E\alpha A w''^2\,dx - SE^* = 0$$

$$Data: L, q(x), q_d(x), \alpha = t^2/12, E, \Delta^*, SE^*$$

Problem 13

Minimize the mean compliance of a beam for given volume of material and upper and lower bound constraints on the transverse displacement.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\, w\, dx$$

Subject to

$$\Gamma: \quad \int_0^L \left(E\alpha A w'' v'' - q v\right) dx = 0$$

$$\Lambda: \quad \int_0^L A\, dx - V^* \le 0$$

$$\mu_u(x): \quad w - w_u \le 0$$

$$\mu_l(x): \quad w_l - w \le 0$$

$$Data: L, q(x), E, \alpha = t^2/12, V^*, w_l, w_u$$

Problem 14

Minimize the mean compliance of a beam for given volume of material and upper and lower bound constraints on the stress.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\, w\, dx$$

Subject to

$$\Gamma: \quad \int_0^L \left(E\alpha A w'' v'' - qv \right) dx = 0$$

$$\Lambda: \quad \int_0^L A\, dx - V^* \le 0$$

$$\mu_u(x): \quad E\alpha w'' - \sigma_t \le 0$$

$$\mu_l(x): \quad \sigma_c - E\alpha w'' \le 0$$

$$Data: L, q(x), E, \alpha = t^2/12, V^*, \sigma_t, \sigma_c$$

Problem 15

Determine the worst load distribution of a beam. Note that the upper bound on the overall load is specified.

$$\underset{q(x)}{Max}\ MC = \int_0^L q\,w\,dx$$

Subject to

$$\lambda(x): \qquad \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L q\,dx - W^* \le 0$$

$$Data: L, A(x), E, \alpha = t^2/12, W^*$$

Problem 16

A general objective function for a beam problem.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\, w^2\, dx$$

Subject to

$$\lambda(x): \qquad \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L A\, dx - V^* \leq 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = t^2/12, V^*$$

Problem 17

Minimize the mean compliance of a beam for given volume of material by varying both breadth and depth of the rectangular cross-section of the beam.

$$\underset{b(x),t(x)}{Min} \; MC = \int_0^L q\, w\, dx$$

Subject to

$$\lambda(x): \qquad \left(\frac{1}{12} E b t^3 A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L A\, dx - V^* \le 0$$

$$Data: L, \text{q}(x), E, V^*$$

The end note

Practice problems in calculus of variations

- Transversely deforming beam is the second simplest structural optimization problem. Either breadth or depth can be varied; both can be varied too.
- Mean compliance and strain energy are measures of stiffness.
  Volume of material used is a cost-measure.
  Objective function and functional constraint can be interchanged without affecting the nature of the solution.
- Equilibrium equation can be posed in strong or weak form without changing the nature of the solution.
- Constraints can also be imposed on profile of the area of cross-section.
- Constraints on displacements and strains (stresses) can be imposed.

Thanks