Lecture 18b

Size optimization of beams for stiffness and flexibility

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Solving two problems concerning size optimization of beams for stiffness and flexibility with volume constraint.

What we will learn:

How to apply the eight steps we had used for bars to the case of beams.

Problem 1

Minimize the mean compliance of a beam for given volume of material.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\, w\, dx$$

Subject to

$$\lambda(x): \qquad \left(EIw''\right)'' - q = 0 \Rightarrow \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L A\, dx - V^* \le 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = {t^2}/{12}, V^*$$

We assume here that only b(x) is variable.

$$I(x) = \alpha A^{\beta}$$

$$\alpha = \left(\frac{t^2}{12}\right);\ \beta = 1$$

Steps in the solution procedure

Step 1: Write the Lagrangian

Step 2: Take variation of the Lagrangian w.r.t. the design variable and equate to zero to get the design equation.

Step 3: Take variation of the Lagrangian w.r.t. state variable(s) and equate to zero to get the adjoint equation(s).

Step 4: Collect all the equations, including the governing equation(s), complementarity condition(s), resource constraints, etc.

Step 5: Obtain the optimality criterion by substituting adjoint and equilibrium equations into the design equation, when it is possible.

Step 6: Identify all boundary conditions.

Step 7: Solve the equations analytically as much as possible.

Step 8: Use the optimality criteria method to solve the equations numerically.

Solution

Minimize the mean compliance of a beam for given volume of material.

$$\underset{A(x)}{Min}\ MC = \int_0^L q\,w\,dx$$

Subject to

$$\lambda(x): \qquad \left(EIw''\right)'' - q = 0 \Rightarrow \left(E\alpha A w''\right)'' - q = 0$$

$$\Lambda: \qquad \int_0^L A\,dx - V^* \le 0$$

$$Data: L, \mathrm{q}(x), E, \alpha = {t^2}/{12}, V^*$$

Step 1

$$L = \int_0^L \left\{ q\,w + \lambda\left(\left(E\alpha A w''\right)'' - q\right) + \Lambda A \right\} dx - \Lambda V^*$$

Expand the Lagrangian

Step 1

$$L = \int_0^L \left\{ q\,w + \lambda \left(\left(E\alpha A w'' \right)'' - q \right) + \Lambda A \right\} dx - \Lambda V^*$$

$$L = \int_0^L \left\{ q\,w + \lambda \left(\left(E\alpha A w'' \right)'' - q \right) + \Lambda A \right\} dx - \Lambda V^*$$

$$= \int_0^L \left\{ qw + E\alpha\lambda \left(A'w'' + Aw''' \right)' - \lambda q + \Lambda A \right\} dx - \Lambda V^*$$

$$= \int_0^L \left\{ qw + E\alpha\lambda A''w'' + 2E\alpha\lambda A'w''' + E\alpha\lambda A w'''' - \lambda q + \Lambda A \right\} dx - \Lambda V^*$$

Design equation

Step 2

$$L = \int_0^L \{qw + E\alpha\lambda A''w'' + 2E\alpha\lambda A'w''' + E\alpha\lambda Aw'''' - \lambda q + \Lambda A\}dx - \Lambda V^*$$

$$F = qw + E\alpha\lambda A''w'' + 2E\alpha\lambda A'w''' + E\alpha\lambda Aw'''' - \lambda q + \Lambda A$$

$$\delta_A L = 0 \Rightarrow \frac{\partial F}{\partial A} - \left(\frac{\partial F}{\partial A'}\right)' + \left(\frac{\partial F}{\partial A''}\right)'' = 0$$

$$\Rightarrow \Lambda + E\alpha\lambda w'''' - \left(2E\alpha\lambda w'''\right)' + \left(E\alpha\lambda w''\right)'' = 0$$

$$\Rightarrow \Lambda + E\alpha\lambda w'''' - \left(2E\alpha\lambda w'''\right)' + \left(E\alpha\lambda' w''\right)' + \left(E\alpha\lambda w'''\right)' = 0$$

$$\Rightarrow \Lambda + E\alpha\lambda w'''' - \left(E\alpha\lambda w'''\right)' + \left(E\alpha\lambda' w''\right)' = 0$$

$$\Rightarrow \Lambda + E\alpha\lambda w'''' - \left(E\alpha\lambda' w'''\right) - \left(E\alpha\lambda w''''\right) + \left(E\alpha\lambda'' w''\right) + \left(E\alpha\lambda' w'''\right) = 0$$

$$\Rightarrow \Lambda + \left(E\alpha\lambda'' w''\right) = 0$$

Adjoint equation

Step 3

$$F = qw + E\alpha\lambda A''w'' + 2E\alpha\lambda A'w''' + E\alpha\lambda Aw'''' - \lambda q + \Lambda A$$

$$\delta_w L = 0 \Rightarrow \frac{\partial F}{\partial w} - \left(\frac{\partial F}{\partial w'}\right)' + \left(\frac{\partial F}{\partial w''}\right)'' - \left(\frac{\partial F}{\partial w'''}\right)''' + \left(\frac{\partial F}{\partial w''''}\right)'''' = 0$$

$$\Rightarrow q - (0)' + \left(E\alpha\lambda A''\right)'' - \left(2E\alpha\lambda A'\right)''' + \left(E\alpha\lambda A\right)'''' = 0$$

$$\Rightarrow q + \left(E\alpha\lambda A''\right)'' - \left(2E\alpha\lambda A'\right)''' + \left(E\alpha\lambda' A\right)''' + \left(E\alpha\lambda A'\right)''' = 0$$

$$\Rightarrow q + \left(E\alpha\lambda A''\right)'' - \left(E\alpha\lambda A'\right)''' + \left(E\alpha\lambda' A\right)''' = 0$$

$$\Rightarrow q + {\color{red}\left(E\alpha\lambda A''\right)''} - {\color{blue}\left(E\alpha\lambda' A'\right)''} - {\color{red}\left(E\alpha\lambda A''\right)''} + \left(E\alpha\lambda'' A\right)'' + {\color{blue}\left(E\alpha\lambda' A'\right)''} = 0$$

$$\Rightarrow q + \left(E\alpha\lambda'' A\right)'' = 0$$

Collect all equations

Step 4

Unknowns $A(x), w(x), \lambda(x), \Lambda$

Three functions and one scalar variable.

We have three differential equations and one scalar equation.

Design equation $$\Lambda + \left(E\alpha\lambda'' w''\right) = 0$$

Adjoint equation $$q + \left(E\alpha\lambda'' A\right)'' = 0$$

Governing equation $$\left(E\alpha A w''\right)'' - q = 0$$

(Adjoint and governing equations give) $$\lambda = -w$$

Feasibility condition $$\int_0^L A\,dx - V^* \le 0$$

Complementarity condition $$\Lambda\left(\int_0^L A\,dx - V^*\right) = 0, \Lambda \ge 0$$

Step 5

$$E\alpha w''^2 = \Lambda$$

Strain energy density is uniform throughout the beam.

And, Λ cannot be zero.

So, the volume constraint is active.

Identify all boundary conditions

Step 6 Boundary conditions for $A(x)$

$$F = qw + E\alpha\lambda A''w'' + 2E\alpha\lambda A'w''' + E\alpha\lambda Aw'''' - \lambda q + \Lambda A$$

$$\left(F_{A'} - \left(F_{A''} \right)' \right) \delta A \Big|_0^L = 0$$

$$\left\{ 2E\alpha\lambda w''' - \left(E\alpha\lambda w'' \right)' \right\} \delta A \Big|_0^L = 0$$

$$\Rightarrow \left\{ E\alpha\lambda w''' - E\alpha\lambda' w'' \right\} \delta A \Big|_0^L = 0$$

$$\Rightarrow \left\{ -E\alpha w w''' + E\alpha w' w'' \right\} \delta A \Big|_0^L = 0 \quad \text{since } \lambda = -w$$

and

$$F_{A''} \delta A' \Big|_0^L = 0$$

$$\left\{ E\alpha\lambda w'' \right\} \delta A' \Big|_0^L = 0$$

$$\Rightarrow \left\{ -E\alpha w w'' \right\} \delta A' \Big|_0^L = 0 \Rightarrow \left\{ E\alpha w w'' \right\} \delta A' \Big|_0^L = 0$$

Identify all boundary conditions

Step 6 Boundary conditions for $\lambda(x)$

$$F = qw + E\alpha\lambda A''w'' + 2E\alpha\lambda A'w''' + E\alpha\lambda Aw'''' - \lambda q + \Lambda A$$

$$\left(F_{w'} - \left(F_{w''}\right)' + \left(F_{w'''}\right)'' - \left(F_{w''''}\right)''' \right) \delta w \Big|_0^L = 0$$

$$\left(F_{w''} - \left(F_{w'''}\right)' + \left(F_{w''''}\right)'' \right) \delta w' \Big|_0^L = 0$$

$$\left(F_{w'''} - \left(F_{w''''}\right)' \right) \delta w'' \Big|_0^L = 0$$

$$F_{w''''} \delta w''' \Big|_0^L = 0$$

Identify all boundary conditions

Step 6 Boundary conditions for $\lambda(x)$

$$F = qw + E\alpha\lambda A''w'' + 2E\alpha\lambda A'w''' + E\alpha\lambda Aw'''' - \lambda q + \Lambda A$$

$$\left(F_{w'} - \left(F_{w''}\right)' + \left(F_{w'''}\right)'' - \left(F_{w''''}\right)'''\right)\delta w\Big|_0^L = 0 \quad \Rightarrow \quad \left\{-\left(E\alpha\lambda A''\right)' + \left(2E\alpha\lambda A'\right)'' - \left(E\alpha\lambda A\right)'''\right\}\delta w\Big|_0^L = 0$$

$$\left(F_{w''} - \left(F_{w'''}\right)' + \left(F_{w''''}\right)''\right)\delta w'\Big|_0^L = 0 \quad \Rightarrow \quad \left\{E\alpha\lambda A'' - \left(2E\alpha\lambda A'\right)' + \left(E\alpha\lambda A\right)''\right\}\delta w'\Big|_0^L = 0$$

$$\left(F_{w'''} - \left(F_{w''''}\right)'\right)\delta w''\Big|_0^L = 0 \quad \Rightarrow \quad \left\{2E\alpha\lambda A' - \left(E\alpha\lambda A\right)'\right\}\delta w''\Big|_0^L = 0$$

$$F_{w''''}\delta w'''\Big|_0^L = 0$$

$$\Downarrow$$

$$\left(E\alpha\lambda A\right)\delta w'''\Big|_0^L = 0$$

Simplify 1st adjoint boundary condition

$$\left\{-\left(E\alpha\lambda A''\right)' + \left(2E\alpha\lambda A'\right)'' - \left(E\alpha\lambda A\right)'''\right\}\delta w\Big|_0^L = 0$$

$$\Rightarrow \left\{-\left(E\alpha\lambda A''\right)' + \left(2E\alpha\lambda A'\right)'' - \left(E\alpha\lambda A'\right)'' - \left(E\alpha\lambda' A\right)''\right\}\delta w\Big|_0^L = 0$$

$$\Rightarrow \left\{-\left(E\alpha\lambda A''\right)' + \left(E\alpha\lambda A'\right)'' - \left(E\alpha\lambda' A\right)''\right\}\delta w\Big|_0^L = 0$$

$$\Rightarrow \left\{-\left(E\alpha\lambda A''\right)' + \left(E\alpha\lambda A''\right)' + \left(E\alpha\lambda' A'\right)' - \left(E\alpha\lambda' A\right)''\right\}\delta w\Big|_0^L = 0$$

$$\Rightarrow \left\{\left(E\alpha\lambda' A'\right)' - \left(E\alpha\lambda' A\right)''\right\}\delta w\Big|_0^L = 0 \Rightarrow \left\{\left(E\alpha\lambda' A'\right)' - \left(E\alpha\lambda' A'\right)' - \left(E\alpha\lambda'' A\right)'\right\}\delta w\Big|_0^L = 0$$

$$\Rightarrow \left\{-\left(E\alpha\lambda'' A\right)'\right\}\delta w\Big|_0^L = 0 \Rightarrow \left(E\alpha\lambda'' A' + E\alpha\lambda''' A\right)\delta w\Big|_0^L = 0$$

Simplify 2nd-4th adjoint boundary conditions

$$\left\{E\alpha\lambda A'' - \left(2E\alpha\lambda A'\right)' + \left(E\alpha\lambda A\right)''\right\}\delta w'\Big|_0^L = 0$$

$$\Rightarrow \left\{E\alpha\lambda A'' - \left(2E\alpha\lambda A'\right)' + \left(E\alpha\lambda A'\right)' + \left(E\alpha\lambda' A\right)'\right\}\delta w'\Big|_0^L = 0$$

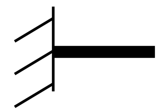
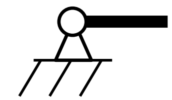
$$\Rightarrow \left\{E\alpha\lambda A'' - \left(E\alpha\lambda A'\right)' + \left(E\alpha\lambda' A\right)'\right\}\delta w'\Big|_0^L = 0$$

$$\Rightarrow \left\{-\left(E\alpha\lambda' A'\right) + \left(E\alpha\lambda' A\right)'\right\}\delta w'\Big|_0^L = 0 \Rightarrow \left(E\alpha\lambda'' A\right)\delta w'\Big|_0^L = 0$$

$$\left\{2E\alpha\lambda A' - \left(E\alpha\lambda A\right)'\right\}\delta w''\Big|_0^L = 0 \Rightarrow \left(E\alpha\lambda A' - E\alpha\lambda' A\right)\delta w''\Big|_0^L = 0$$

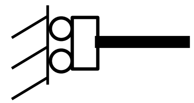
$$\left(E\alpha\lambda A\right)\delta w'''\Big|_0^L = 0$$

All four adjoint boundary conditions

	$\delta w = \delta w' = 0$ Fixed 	$\delta w = w'' = 0$ Pinned 
$\left(E\alpha\lambda'' A' + E\alpha\lambda''' A\right)\delta w\Big\vert_0^L = 0$	No BC for $\lambda(x)$	No BC for $\lambda(x)$
$\left(E\alpha\lambda'' A\right)\delta w'\Big\vert_0^L = 0$	No BC for $\lambda(x)$	$\lambda'' A = 0$
$\left(E\alpha\lambda A' - E\alpha\lambda' A\right)\delta w''\Big\vert_0^L = 0$	$\lambda A' - \lambda' A = 0$	No BC for $\lambda(x)$
$\left(E\alpha\lambda A\right)\delta w'''\Big\vert_0^L = 0$	$\lambda A = 0$	$\lambda A = 0$

Notice that BCs of the state variable transfer to adjoint variable (most often).
But be sure to keep the BCs on the design variable in mind.

All four adjoint boundary conditions

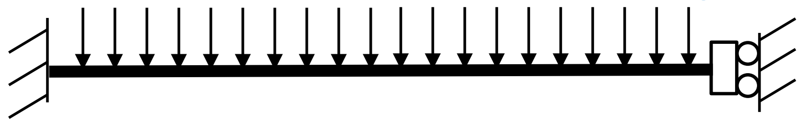
	Transversely guided	Free
	$\delta w' = w''' = 0$ 	$w'' = w''' = 0$
$\left(E\alpha\lambda'' A' + E\alpha\lambda''' A\right)\delta w\big\|_0^L = 0$	$\lambda'' A' + \lambda''' A = 0$	$\lambda'' A' + \lambda''' A = 0$
$\left(E\alpha\lambda'' A\right)\delta w'\big\|_0^L = 0$	No BC for $\lambda(x)$	$\lambda'' A = 0$
$\left(E\alpha\lambda A' - E\alpha\lambda' A\right)\delta w''\big\|_0^L = 0$	$\lambda A' - \lambda' A = 0$	No BC for $\lambda(x)$
$\left(E\alpha\lambda A\right)\delta w'''\big\|_0^L = 0$	No BC for $\lambda(x)$	No BC for $\lambda(x)$

Notice that BCs of the state variable transfer to adjoint variable (most often).
But be sure to keep the BCs on the design variable in mind.

Solving for a particular beam BCs

Step 7

Fixed $\quad q(x) = q_0 \quad$ Transversely guided



$w = w' = 0$ $\qquad$ $w' = w''' = 0$

$\lambda A = 0 \quad \& \quad \lambda A' - \lambda' A = 0$ $\qquad$ $\lambda'' A' + \lambda''' A = 0 \quad \& \quad \lambda A' - \lambda' A = 0$

Take $\lambda = \lambda' = 0$ $\qquad$ Take $\lambda''' = \lambda' = A' = 0$

$$\left\{ E\alpha w w'' \right\} \delta A' \Big|_0^L = 0 \quad \& \quad \left\{ -E\alpha w w''' + E\alpha w' w'' \right\} \delta A \Big|_0^L = 0$$

Satisfied at $x = 0$ and $x =$ L $\qquad$ Satisfied at $x = 0$ and $x =$ L

$$E\alpha w''^2 = \Lambda$$

$$\Rightarrow w'' = \pm \sqrt{\frac{\Lambda}{E\alpha}}$$

$$\left(E\alpha A w'' \right)'' - q = 0$$

$$\left(\pm A \sqrt{E\alpha\Lambda} \right)'' = q_0 \Rightarrow A = \pm \frac{q_0 x^2}{2\sqrt{E\alpha\Lambda}} + C_1 x + C_0$$

$$C_1 = -\frac{q_0 L}{\sqrt{E\alpha\Lambda}}$$

Solve for Λ using $\int_0^L A\, dx - V^* \le 0$

Reconciliation of BCs for $A(x)$

Step 7 From the previous slide, we have $A = \pm \frac{q_0 x^2}{2\sqrt{E\alpha\Lambda}} - \frac{q_0 L}{\sqrt{E\alpha\Lambda}} x + C_0$

We can use the active volume constraint to solve for Λ

$$\Rightarrow \pm \frac{q_0 L^3}{6\sqrt{E\alpha\Lambda}} - \frac{q_0 L^3}{2\sqrt{E\alpha\Lambda}} + C_0 L = V^*$$

$$\Rightarrow \sqrt{E\alpha\Lambda} = -\frac{q_0 L^3}{3\left(V^* - C_0 L\right)}$$ for + sign & $$\sqrt{E\alpha\Lambda} = -\frac{2 q_0 L^3}{3\left(V^* - C_0 L\right)}$$ for - sign

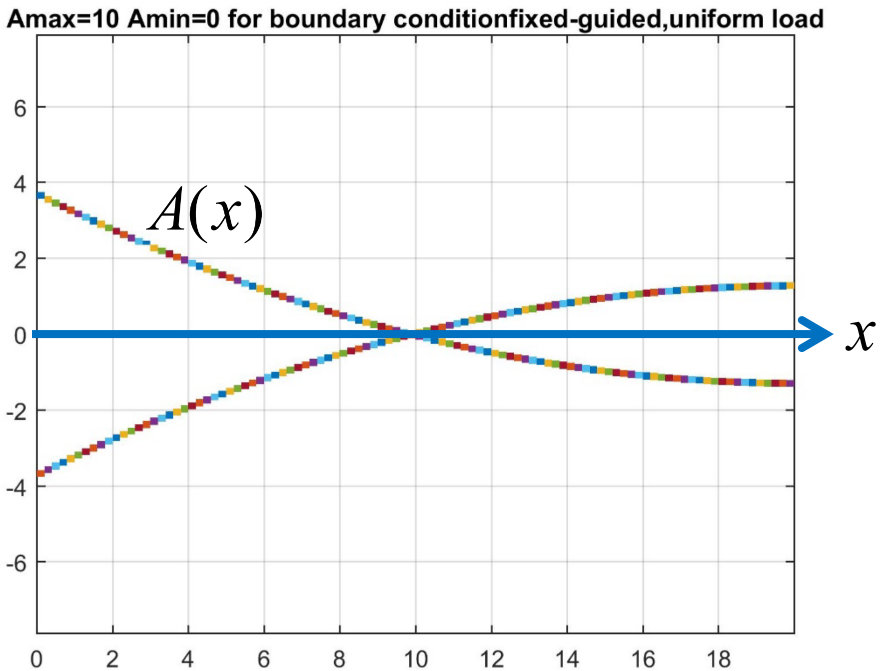
$$A(x) = \begin{cases} -\dfrac{3x^2\left(V^* - C_0 L\right)}{2L^3} - \dfrac{3\left(V^* - C_0 L\right)}{L^2} x + C_0 \\ \dfrac{3x^2\left(V^* - C_0 L\right)}{4L^3} - \dfrac{3\left(V^* - C_0 L\right)}{2L^2} x + C_0 \end{cases}$$

How do we choose the two possibilities (+ or -) and determine C_0?

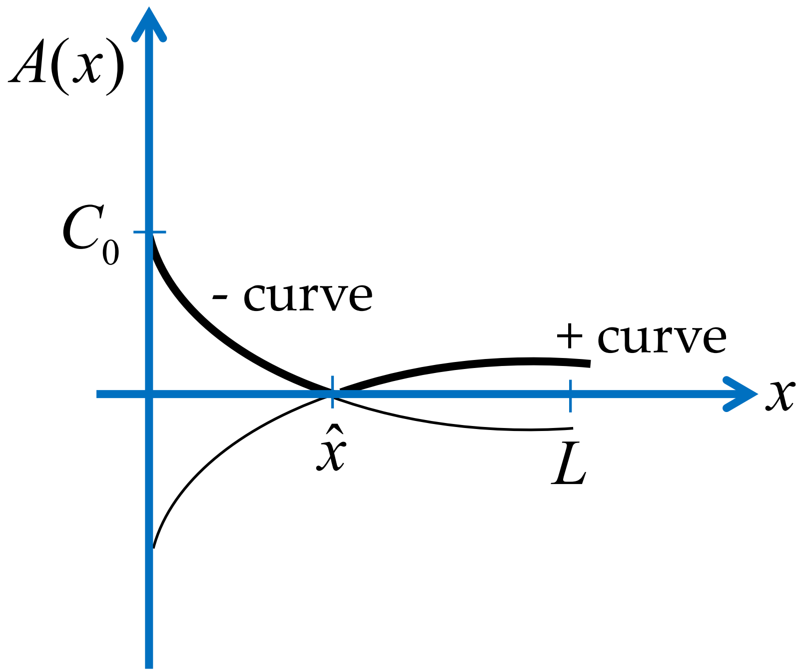
Validating with the numerical solution…

$$A(x)=\begin{cases} A^{+}=-\dfrac{3x^{2}\left(V^{*}-C_0L\right)}{2L^{3}}-\dfrac{3\left(V^{*}-C_0L\right)}{L^{2}}x+C_0 & \text{+ curve} \\ A^{-}=\dfrac{3x^{2}\left(V^{*}-C_0L\right)}{4L^{3}}-\dfrac{3\left(V^{*}-C_0L\right)}{2L^{2}}x+C_0 & \text{- curve} \end{cases}$$

$L=20; d=1; V^{*}=50; E=210; A_{\min}=1E-3; A_{\max}=10; q_0=1$



We need to see which part of the domain should have the "+ curve" and which part should have "- curve".



We need to find $\hat{x}$ and C_0 using $A^{+}(\hat{x})=A^{-}(\hat{x})=0$

Problem 12

Minimize the volume of material of a beam (statically determinate or indeterminate) for a deflection constraint in its span with an upper bound on the strain energy.

$$\underset{A(x)}{Min}\ V = \int_0^L A\,dx$$

Subject to

$$\lambda(x):\quad \left(E\alpha A w''\right)'' - q = 0$$

$$\lambda_d(x):\quad \left(E\alpha A v''\right)'' - q_d = 0$$

$$\Lambda:\quad \int_0^L E\alpha A w'' v''\,dx - \Delta^* = 0$$

$$\Gamma:\quad \int_0^L \frac{1}{2} E\alpha A w''^2\,dx - SE^* = 0$$

$$Data: L, q(x), q_d(x), \alpha = t^2/12, E, \Delta^*, SE^*$$

The end note

Size optimization of beams

- We follow essentially the same eight steps
- Identifying the optimality criterion is the highlight.
- Boundary conditions for the adjoint variable need to be carefully done. See the correlation between BCs and optimal profiles
- Analytical solution may be segmented with multiple possibilities because of + and – of constant strain.
- Iterative numerical solution, when it is needed, remains the same.

Thanks