Lecture 6 extra

Duality of the Lagrangian

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

The simplest structural optimization problem of one bar

Duality of the Lagrangian using the one-bar problem

Primal and dual problems in constrained minimization

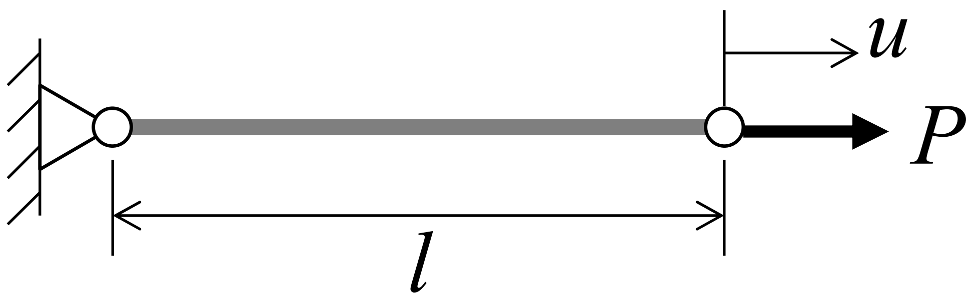
What we will learn:

How to solve a structural optimization problem in one variable.

How to use the complementarity to check if an inequality constraint is active or inactive.

How to interpret primal and dual problems, which is applicable only when the Hessian of the Lagrangian is positive definite.

Let us start with the simplest: a one-bar truss



Find the area of cross-section of a bar of given length with an upper bound on the displacement under a given load for minimizing the weight.

$$\min_{A} \; \rho l A$$

Subject to

$$\mu: \quad \frac{Pl}{AE} - u^* \le 0$$

Data: ρ, l, P, E, u^*

Mass density (ρ)

Young's modulus (E)

The solution

$$\underset{A}{\text{Min}} \quad \rho l A$$

Subject to

$$\mu: \qquad \frac{Pl}{AE} - u^* \le 0$$

Data: ρ, l, P, E, u^*

Lagrangian

$$L = \rho l A + \mu\left(\frac{Pl}{AE} - u^*\right)$$

Necessary condition

$$\frac{dL}{dA} = \rho l - \mu\left(\frac{Pl}{A^2 E}\right) = 0$$

$$\Rightarrow A = \sqrt{\frac{\mu P}{\rho E}}$$

Complementarity condition

$$\mu\left(\frac{Pl}{AE} - u^*\right) = 0 \quad \text{and} \quad \mu \ge 0$$

Non-negativity of the Lagrange multiplier

Should the constraint be active?

$$A = \sqrt{\frac{\mu P}{\rho E}} \qquad \mu\left(\frac{Pl}{AE} - u^*\right) = 0 \qquad \mu \geq 0$$

$$\frac{Pl}{AE} - u^* \leq 0$$

If the constraint is not active, area of cross-section is zero and it leads to the infeasibility of the constraint. Therefore,

$$\frac{Pl}{AE} - u^* = 0 \Rightarrow A^* = \frac{Pl}{u^* E}$$

and

since $A = \sqrt{\frac{\mu P}{\rho E}}$, $\mu^* = \frac{\rho E A^2}{P} = \frac{\rho E}{P}\left(\frac{Pl}{u^* E}\right)^2 = \frac{\rho P l^2}{u^{*2} E}$

Sufficient condition

$$L = \rho l A + \mu\left(\frac{Pl}{AE} - u^*\right) \qquad \frac{dL}{dA} = \rho l - \mu\left(\frac{Pl}{A^2 E}\right)$$

$$\frac{d^2 L}{dA^2} = \mu\left(\frac{2Pl}{A^3 E}\right) > 0$$ since $A^* = \frac{Pl}{u^* E}$ and $\mu^* = \frac{\rho P l^2}{u^{*2} E}$

So, it is indeed a minimizing solution.

Duality of the Lagrangian

$$L = \rho l A + \mu\left(\frac{Pl}{AE} - u^*\right)$$

Lagrangian is a saddle surface. It has a minimum w.r.t. the design variable and a maximum with respect to the Lagrange multiplier.

$$\underset{A}{\text{Min}}\, L = \rho l A + \mu^*\left(\frac{Pl}{AE} - u^*\right) \qquad \underset{\mu}{\text{Max}}\, L = \rho l A^* + \mu\left(\frac{Pl}{A^* E} - u^*\right)$$

$$L(A^*, \mu) \le L(A^*, \mu^*) \le L(A, \mu^*)$$

Lagrangian surface is a saddle.

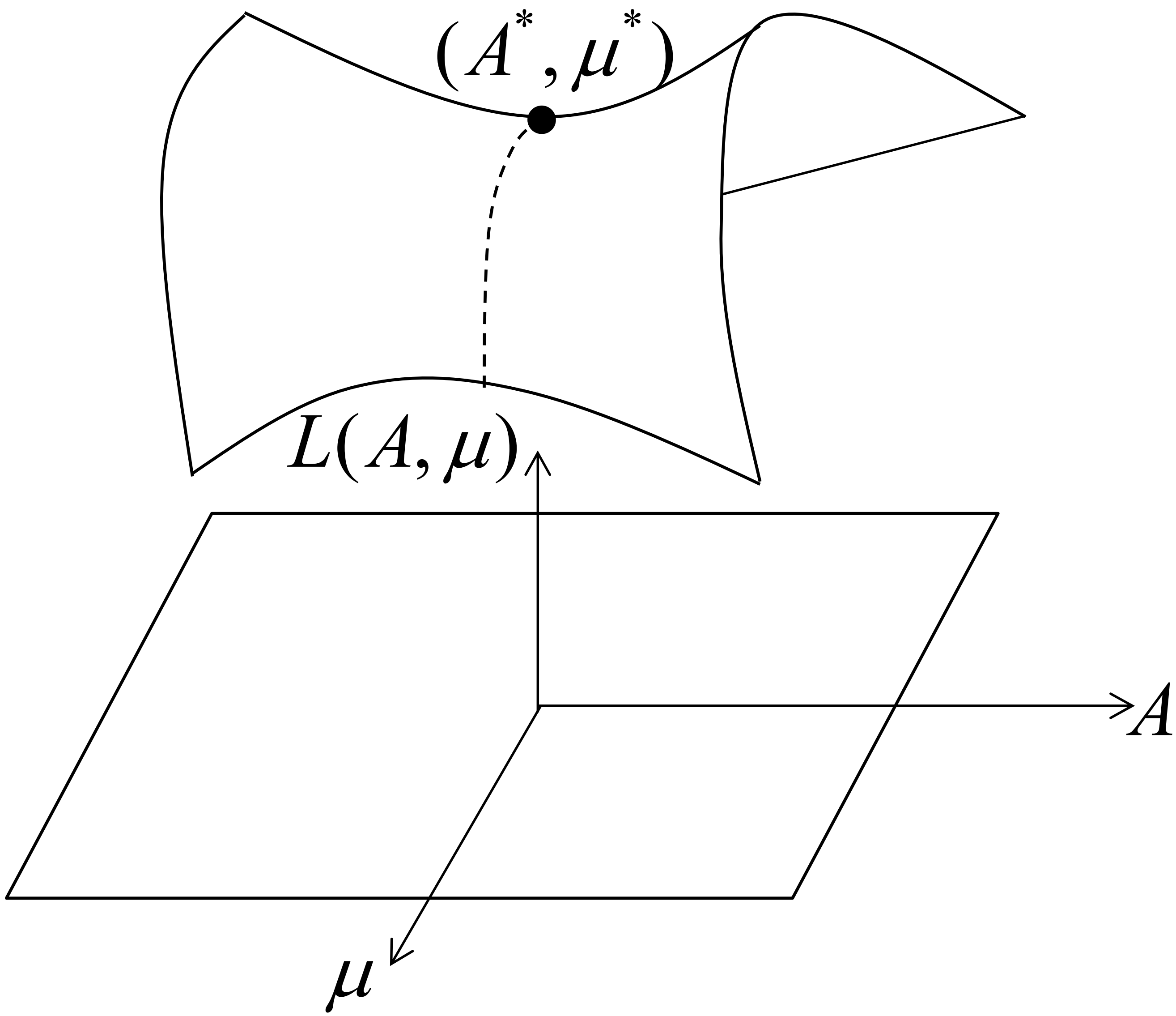
$$\underset{A}{\text{Min}}\, L = \rho l A + \mu^* \left(\frac{Pl}{AE} - u^* \right)$$

With $\left(\frac{Pl}{AE} - u^* \right) \leq 0$

$$\underset{\mu}{\text{Max}}\, L = \rho l A^* + \mu \left(\frac{Pl}{A^* E} - u^* \right)$$

With $A^* = \sqrt{\frac{\mu P}{\rho E}}$

Let us see the Lagrangian surface.



Let us check if L is maximized.

$$\underset{\mu}{\text{Max}}\, L = \rho l A^* + \mu\left(\frac{Pl}{A^* E} - u^*\right) \quad \text{with} \quad A^* = \sqrt{\frac{\mu P}{\rho E}}$$

$$\underset{\mu}{\text{Max}}\, L = \rho l \sqrt{\frac{\mu P}{\rho E}} + \mu\left(\frac{Pl}{\sqrt{\frac{\mu P}{\rho E}}E} - u^*\right)$$

$$= l\sqrt{\frac{\rho \mu P}{E}} + l\sqrt{\frac{\rho \mu P}{E}} - \mu u^* = 2l\sqrt{\frac{\rho \mu P}{E}} - \mu u^*$$

Lagrangian is maximized w.r.t. μ

$$\operatorname*{Max}_{\mu} L = 2l\sqrt{\frac{\rho\mu P}{E}} - \mu u^*$$

$$\frac{\partial L}{\partial \mu} = l\sqrt{\frac{\rho P}{\mu E}} - u^* = 0 \;\Rightarrow\; \mu^* = \frac{l^2 \rho P}{u^{*2} E}$$

$$\frac{\partial^2 L}{\partial \mu^2} = -\frac{l}{2}\sqrt{\frac{\rho P}{E}}\;\mu^{-3/2} < 0$$

The Lagrangian is indeed a maximum.

Value of the optimum Lagrangian in primal and dual problems is the same.

$$\underset{A}{\text{Min}}\, L = \rho l A + \mu^* \left(\frac{Pl}{AE} - u^* \right)$$

$$= \rho l A^* + \mu^* \left(\frac{Pl}{AE} - u^* \right)$$

$$L_{\min} = f_{\min} = \rho l \frac{Pl}{u^* E} = \frac{\rho P l^2}{u^* E}$$

$$\underset{\mu}{\text{Max}}\, L = 2l \sqrt{\frac{\rho \mu P}{E}} - \mu u^*$$

$$\mu^* = \frac{\rho P l^2}{u^{*2} E}$$

$$L_{\max} = 2l \sqrt{\frac{\rho \frac{\rho P l^2}{u^{*2} E} P}{E}} - \frac{\rho P l^2}{u^{*2} E} u^*$$

$$= 2l \sqrt{\frac{\rho^2 l^2 P^2}{u^{*2} E^2}} - \frac{\rho P l^2}{u^{*2} E} u^*$$

$$= 2 \frac{\rho P l^2}{u^* E} - \frac{\rho P l^2}{u^* E} = \frac{\rho P l^2}{u^* E}$$

Let us plot and see…

$$\underset{A}{\text{Min}}\, L = \rho l A + \mu^*\left(\frac{Pl}{AE} - u^*\right)$$

$$\left(\frac{Pl}{AE} - u^*\right) \le 0$$

$$\underset{\mu}{\text{Max}}\, L = \rho l A^* + \mu\left(\frac{Pl}{A^* E} - u^*\right)$$

$$A^* = \sqrt{\frac{\mu P}{\rho E}}$$

Use $l = 1; P = 100; \rho = 7800; E = 210\times10^9; u^* = 10^{-3}$

All in SI units

$$\underset{A}{\text{Min}}\, L = 7800 A + \mu^*\left(\frac{1}{150\times10^7 A} - 10^{-3}\right)$$

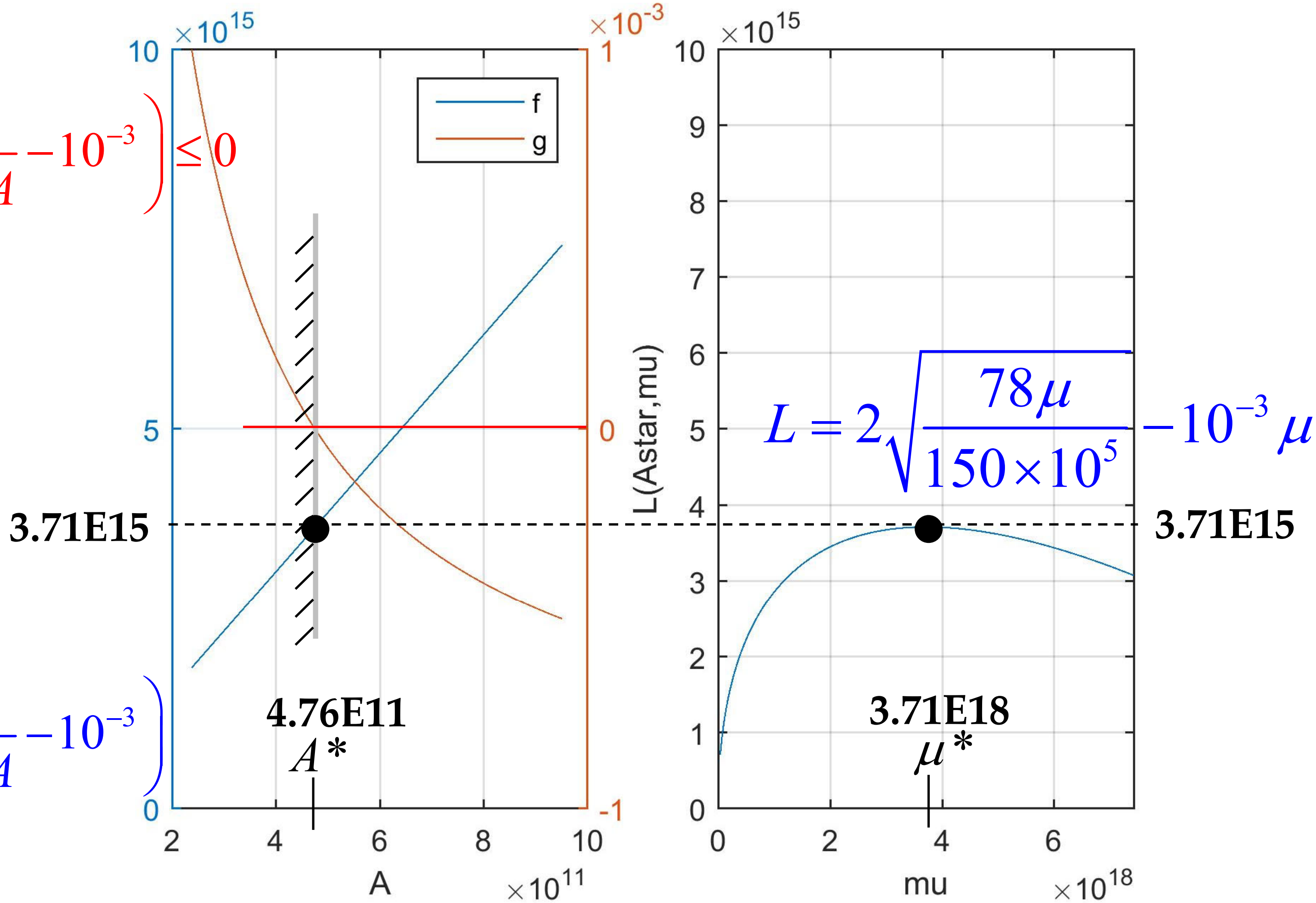
$$\left(\frac{1}{150\times10^7 A} - 10^{-3}\right) \le 0$$

$$\underset{\mu}{\text{Max}}\, L = 2\sqrt{\frac{78\mu}{150\times10^5}} - 10^{-3}\mu$$

Plot and see

$$g = \left(\frac{1}{150 \times 10^7 A} - 10^{-3} \right) \le 0$$

$$L = f = 7800A + \mu^* \left(\frac{1}{150 \times 10^7 A} - 10^{-3} \right)$$



$$L = 2\sqrt{\frac{78\mu}{150 \times 10^5}} - 10^{-3}\mu$$

In general…

Primal problem

$$\underset{x}{\text{Min}}\, L(x,\mu) = f + \mu g$$

$$g \le 0$$

Dual problem

$$\underset{\mu}{\text{Max}}\, L(x^*(\mu),\mu) = f + \mu g$$

$$\frac{\partial L}{\partial x} = 0, \ \mu \ge 0$$

This is true only if the Hessian of the Lagrangian is positive definite.

The end note

Duality of Lagrangian explained using one-bar optimization

- One-bar optimization
- How to use the complementarity condition to argue if an inequality constraint is active or inactive
- Pose the dual problem for one-bar optimization
- **Duality demonstration for the simplest structural optimization problem**
- Dual problem is applicable only when the Hessian of the Lagrangian is positive definite.

Thanks