

Lecture 7a

Truss optimization

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Size optimization of multi-bar trusses

Adjoint method of sensitivity analysis

Optimality criterion

What we will learn:

Solution to the size optimization of trusses

Algorithm for the optimality criteria method

Size optimization of a truss

$$\text{Min}_{\mathbf{A}} MC = \sum_{i=1}^N P_i u_i = \mathbf{p}^T \mathbf{u}$$

Subject to

$$\lambda: \quad \mathbf{K}\mathbf{u} - \mathbf{p} = \mathbf{0}$$

$$\mu: \quad \sum_{i=1}^N \rho A_i l_i - W^* \leq 0 \quad \text{or} \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

Data: $\rho, W^*, l_{i=1,2,\dots,N}, N, P_{i=1,2,\dots,N}, E$

MC = mean compliance = work done by the external forces

MC is a measure of stiffness:
the lower the value of MC, the stiffer the truss.

\mathbf{K} depends on areas of cross section of all truss elements and Young's modulus.

Size optimization of a truss

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Data: $\rho, W^*, l_{i=1,2,\dots,N}, N, P_{i=1,2,\dots,N}, E$

Lagrangian

$$L = \mathbf{p}^T \mathbf{u} + \lambda^T (\mathbf{K}\mathbf{u} - \mathbf{p}) + \mu (\rho \mathbf{a}^T \mathbf{l} - W^*)$$

Necessary conditions

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \lambda^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0$$
$$i = 1, 2, \dots, N$$

$$\mathbf{K}\mathbf{u} = \mathbf{p}, \quad \rho \mathbf{a}^T \mathbf{l} - W^* \leq 0$$

$$\mu (\rho \mathbf{a}^T \mathbf{l} - W^*) = 0$$

Tally the number of unknowns and the number of equations

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \quad i = 1, 2, \dots, N \quad N \text{ equations}$$

$$\mathbf{K} \mathbf{u} = \mathbf{p} \quad n \text{ equations}$$

\mathbf{A} N unknowns

$$\rho \mathbf{a}^T \mathbf{1} - W^* \leq 0$$

\mathbf{u} n unknowns

$$\mu (\rho \mathbf{a}^T \mathbf{1} - W^*) = 0 \quad 1 \text{ equation}$$

$\boldsymbol{\lambda}$ n unknowns

μ 1 unknown

n = twice the number of vertices in the truss = total number of DoF

$(N+n+1)$ equations in $(N+2n+1)$ unknowns.

Ouch... what do we do? Where are n more equations?

Finding n more equations

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \quad i = 1, 2, \dots, N \quad N \text{ equations}$$


Let us see hidden n more equations by recasting the above equation.

$$\left(\mathbf{p}^T + \boldsymbol{\lambda}^T \mathbf{K} \right) \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i = 0$$

Do you see n more equations yet?

No? Let us see what we can compute and what we cannot if we know the areas of cross section.

Here are n more equations

$$\left(\mathbf{p}^T + \boldsymbol{\lambda}^T \mathbf{K}\right) \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i = 0 \quad \Rightarrow \quad \boldsymbol{\lambda}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i = 0$$


$$\left(\mathbf{p}^T + \boldsymbol{\lambda}^T \mathbf{K}\right) = \mathbf{0}^T \quad \text{or} \quad \mathbf{K} \boldsymbol{\lambda} = -\mathbf{p}$$

This clever way of finding n more equations is called the **adjoint method of sensitivity analysis**.

$\mathbf{K} \boldsymbol{\lambda} = -\mathbf{p}$ This is called the adjoint (equilibrium) equation.

$\boldsymbol{\lambda}$ is called the adjoint (state) variable.

Now, we are set to solve the problem.

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \boldsymbol{\lambda}^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \quad i = 1, 2, \dots, N \quad N \text{ equations}$$

$$\mathbf{K}\mathbf{u} = \mathbf{p} \quad n \text{ equations}$$

$$\rho \mathbf{a}^T \mathbf{1} - W^* \leq 0$$

$$\mu (\rho \mathbf{a}^T \mathbf{1} - W^*) = 0 \quad 1 \text{ equation}$$

$$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{p} \quad n \text{ equations}$$

\mathbf{A} N unknowns

\mathbf{u} n unknowns

$\boldsymbol{\lambda}$ n unknowns

μ 1 unknown

$(N+2n+1)$ equations in $(N+2n+1)$ unknowns.

We also have two inequalities too: $\rho \mathbf{a}^T \mathbf{1} - W^* \leq 0$ and $\mu \geq 0$

Solving...

$$\frac{\partial L}{\partial A_i} = 0 \Rightarrow \mathbf{p}^T \frac{\partial \mathbf{u}}{\partial A_i} + \lambda^T \left(\mathbf{K} \frac{\partial \mathbf{u}}{\partial A_i} + \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) + \mu \rho l_i = 0 \quad i = 1, 2, \dots, N \quad N \text{ equations}$$

$$\mathbf{K}\mathbf{u} = \mathbf{p} \quad n \text{ equations}$$

$$\rho \mathbf{a}^T \mathbf{1} - W^* \leq 0$$

$$\mu(\rho \mathbf{a}^T \mathbf{1} - W^*) = 0 \quad 1 \text{ equation}$$

$$\mathbf{K}\boldsymbol{\lambda} = -\mathbf{p} \quad n \text{ equations}$$

$$\boldsymbol{\lambda} = -\mathbf{u}$$

State variable and adjoint variable are equal.

Solving (contd.)

$$-\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu \rho l_i = 0 \Rightarrow \mu = \frac{1}{\rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) \quad i = 1, 2, \dots, N \quad N \text{ equations}$$

$$\mathbf{K} \mathbf{u} = \mathbf{p} \quad n \text{ equations}$$

$$\rho \mathbf{a}^T \mathbf{1} - W^* \leq 0$$

$$\mu (\rho \mathbf{a}^T \mathbf{1} - W^*) = 0 \quad 1 \text{ equation and two inequalities}$$

$$\mu \geq 0$$

$$\frac{1}{\rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) = ? \quad \text{What is this, physically?}$$

Interpretation: $\frac{1}{\rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) = se_i$

$$SE = \frac{1}{2} (\mathbf{u}^T \mathbf{K} \mathbf{u}) = \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N k_{jk} u_j u_k$$

$$SE_i = \frac{1}{2} \sum_{j \in DoF} \sum_{k \in DoF} k_{jk} u_j u_k$$

$$se_i = \frac{SE_i}{\rho l_i A_i} = \frac{1}{2 \rho l_i A_i} \sum_{j \in DoF} \sum_{k \in DoF} k_{jk} u_j u_k$$

$$\frac{\partial SE}{\partial A_i} = \frac{1}{2} \sum_{j \in DoF} \sum_{k \in DoF} \frac{k_{jk}}{A_i} u_j u_k$$

This is only partial sensitivity; not the total sensitivity to which we will return to later.

Strain energy density of the i^{th} element

Since the strain energy density of all elements cannot be zero, we say that μ cannot be zero. Hence, the weight constraint is active.

Solving (contd.): Optimality criterion

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$
$$\left(\rho\mathbf{a}^T\mathbf{1} - W^*\right) = 0$$
$$\mu = \frac{1}{\rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right)$$

This implies that the strain energy density is constant for all elements.

How do we find A_i s?

Since $1 = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right)$

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

k is the iteration number in optimization.

Fixed point method of solution

$$A_i^{(k+1)} = \frac{1}{\mu\rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Multiplicative version of the optimality criteria method.

Additive version

$$A_i^{(k+1)} = A_i^{(k)} + \left(-\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} + \mu\rho l_i \right)^{(k)}$$

$$f(x) = 0$$

$$\tan x - x^2 - 1 = 0$$

$$g(x) = f(x) + x$$

Example

$$x^{(k+1)} = g(x^{(k)}) = f(x^{(k)}) + x^{(k)}$$

Algorithm

Step 0 Assume initial guess for all areas of cross section so that the weight constraint is satisfied.

$$A_i^{(0)}, i = 1, 2, \dots, N$$

Step 1 Solve for the displacements using the loads.

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$

Step 2 Solve $(\rho\mathbf{a}^T\mathbf{1} - W^*) = 0$ to find μ

$$A_i^{(k+1)} = \frac{1}{\mu\rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Step 2 and farther...

$$\begin{aligned}(\rho \mathbf{a}^T \mathbf{1} - W^*) &= 0 \Rightarrow \sum_{i=1}^N \rho A_i l_i = W^* \\ \Rightarrow \sum_{i=1}^N \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)} \rho l_i &= W^* \\ \Rightarrow \mu &= \frac{\sum_{i=1}^N \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}}{W^*}\end{aligned}$$

Step 3 Update areas of cross section.

$$A_i^{(k+1)} = \frac{1}{\mu \rho l_i} \left(\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial A_i} \mathbf{u} \right) A_i^{(k)}$$

Go to Step 1 until convergence.

Do you see any other issues here?

The end note

- Truss optimization
- Size optimization: problem statement and necessary conditions
 - Counting unknowns and equations
 - Rational behind the adjoint method of sensitivity analysis
 - Algorithm for size optimization of trusses**
 - We are we missing anything else in the algorithm? Think.

Thanks