Lecture 8a

Principles of minimum potential energy and virtual work and their implication in truss optimization

ME260 Indian Institute of Science

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Principle of minimum potential energy

Principle of virtual work

Unit virtual (dummy) load method

Clayperon's theorem

What we will learn:

Interpreting and using the principle of virtual work

Principle of minimum potential energy

"The minimum potential energy corresponds to stable static equilibrium."

This is an alternative view to force balance.

Potential energy = PE = Strain energy + Work potential

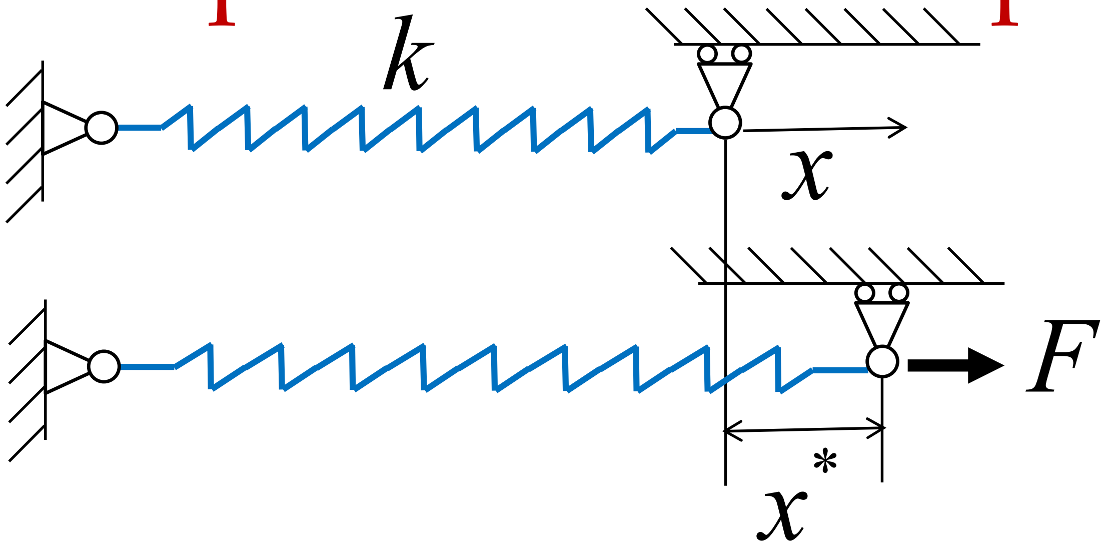
i.e., PE = SE + WP

Strain energy = deformation energy stored in a structure

Work potential = negative of the work done by the external forces

When PE is in terms of the displacements, we minimize PE with respect to the displacements.

A simple example used to understand the principle of minimum potential energy



We know from force balance:

$$F = kx^*$$

Strain energy = SE = $\frac{1}{2}kx^2$

Work potential = WP = $-Fx$

Potential energy = PE = SE + WP = $\frac{1}{2}kx^2 - Fx$

$$\underset{x}{\text{Min}} \; PE = \frac{1}{2}kx^2 - Fx$$

$$\frac{\partial PE}{\partial x} = kx - F = 0$$

So, $F = kx^*$

Let us apply the principle of minimum potential energy to trusses

Let **K** be the stiffness matrix and **u** be the displacement vector.

Strain energy = SE = 0.5$\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u}$
Work potential = WP = -$\mathbf{p}^{\mathrm{T}}\mathbf{u}$
Potential energy = SE + PE =0.5 $\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u}$ - $\mathbf{p}^{\mathrm{T}}\mathbf{u}$

$$\underset{\mathbf{u}}{\text{Min}}\ PE = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{p}^T\mathbf{u}$$

$$\frac{\partial PE}{\partial \mathbf{u}} = \mathbf{K}\mathbf{u} - \mathbf{p} = 0 \Rightarrow \underset{n\times n}{\mathbf{K}}\ \underset{n\times 1}{\mathbf{u}} = \underset{n\times 1}{\mathbf{p}}$$

Principle of virtual work

"External virtual work is equal to the internal virtual work."

This is an alternative view to force balance and the principle of minimum potential energy.

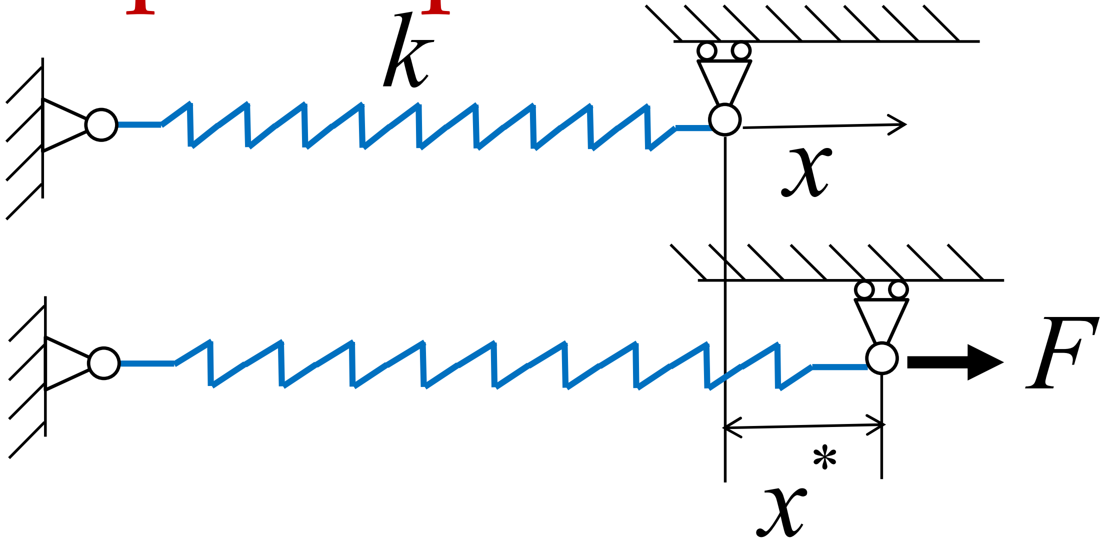
External virtual work = EVW = work done by the external real forces undergoing virtual displacements or work done by external virtual forces undergoing real displacements

Internal virtual work = IVW = work done by the real internal forces undergoing virtual displacements or work done by internal virtual forces undergoing real displacements

EVW = IVW is a powerful analytical tool.

It is a thought experiment either with virtual displacements or virtual forces.

The same simple example to illustrate the principle of virtual work



We know from force balance:

$$F = kx^*$$

$$\Rightarrow x^* = F / k$$

$$\underset{x}{\text{Min}} \;\; PE = \frac{1}{2}kx^2 - Fx$$

$$\frac{\partial PE}{\partial x} = kx - F = 0$$

So, $F = kx^*$

Imagine virtual displacement δx

$$\left.\begin{aligned} EVW &= F\delta x \\ IVW &= kx^* \delta x \end{aligned}\right\} EVW = IVW \Rightarrow F = kx^*$$

Imagine virtual force δF

$$\left.\begin{aligned} EVW &= \delta F \; x^* \\ IVW &= k\delta x \; x^* \end{aligned}\right\} \begin{aligned} &EVW = IVW \\ &\Rightarrow \delta F\left(F/k\right) = k\left(\delta F/k\right)x^* \Rightarrow F/k = x^* \end{aligned}$$

Let us apply the principle of virtual work to trusses

$$\underset{n\times n}{\mathbf{K}}\underset{n\times 1}{\mathbf{u}} = \underset{n\times 1}{\mathbf{p}}$$

Imagine virtual displacements $\delta\mathbf{u}$

$$EVW = \mathbf{p}^T\delta\mathbf{u}$$

$$IVW = (\mathbf{Ku})^T\delta\mathbf{u} = \mathbf{u}^T\mathbf{K}\delta\mathbf{u}$$ because $\mathbf{K}$ is symmetric.

$$EVW = IVW \Rightarrow \mathbf{p}^T\delta\mathbf{u} = \mathbf{u}^T\mathbf{K}\delta\mathbf{u}$$

$$\Rightarrow \mathbf{p}^T = \mathbf{u}^T\mathbf{K}$$

$$\Rightarrow \mathbf{Ku} = \mathbf{p}$$

Unit virtual (dummy) load method

$$\mathbf{K}\mathbf{u} = \mathbf{p}$$

Imagine virtual force $\delta \mathbf{p}^T = \{\overset{1}{0} \quad 0 \quad \overset{i}{1} \quad \cdots \quad \cdots \quad 0 \quad \overset{n}{0}\}$

$$EVW = \delta \mathbf{p}^T \ \mathbf{u}^*$$

$$IVW = \left(\mathbf{K}\delta\mathbf{u}\right)^T \mathbf{u}^* = \delta\mathbf{u}^T \mathbf{K}\mathbf{u}^*$$

$$EVW = IVW \Rightarrow \delta \mathbf{p}^T \ \mathbf{u}^* = \delta\mathbf{u}^T \mathbf{K}\mathbf{u}^*$$

$$\Rightarrow u_i = \left(\delta\mathbf{u}^T \mathbf{K}\right)\mathbf{u}^*$$

$\mathbf{u}^*$ = displacements due to applied real loads

$\delta\mathbf{u}$ = displacements due to the unit virtual load

Clayperon's theorem

At static equilibrium, the mean compliance is equal to twice the strain energy.

What? $MC = 2 * SE$

How? $EVW = \mathbf{p}^T \ \delta\mathbf{u}$ $\qquad IVW = \left(\mathbf{K}\mathbf{u}^*\right)^T \delta\mathbf{u} = \delta\mathbf{u}^T\mathbf{K}\mathbf{u}$

$$EVW = IVW$$

$$\Rightarrow \mathbf{p}^T \ \delta\mathbf{u} = \delta\mathbf{u}^T\mathbf{K}\mathbf{u}^*$$

Make virtual displacement equal to real equilibrium displacement.

$$\mathbf{p}^T \ \mathbf{u}^* = \mathbf{u}^{*T}\mathbf{K}\mathbf{u}^* \Rightarrow MC = 2 * SE$$

Does this make sense?

An implication from the Clayperon's theorem.

What?

$$\underset{\mathbf{u}}{\text{Min}}\ \underset{\mathbf{a}}{\text{Max}}\ PE = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{p}^T\mathbf{u}$$

Subject to

$$\mu: \qquad \rho\mathbf{a}^T\mathbf{l} - W^* \le 0$$

Data: $\rho, W^*, l_{i=1,2...,N}, N, P_{i=1,2,...,N}, E$

How?

$$PE = SE + WP = SE - MC$$

$$PE^* = SE^* - MC^* = -SE^*$$

Minimizing SE (or MC) is equivalent to maximizing PE w.r.t. to the design variable.

The end note

Truss optimization

- Principle of minimum potential energy
- Principle of virtual work
- Unit virtual load method (unit dummy load method)
- Clayperon's theorem
- **The second alternative to force balance.**
  The first alternative is the principle of minimum potential energy.

Thanks