Lecture 27

Shape Optimization of 2D elasticity for stiffness

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Posing and solving the shape optimization of 2D elastic problem in which design variables are concerned with the shape of the boundary.

Considering the objective of maximizing stiffness with area constraint.

What we will learn:

How to implement the algorithm consisting of six steps to identify the optimality criterion and use it in the numerical method to solve 2D shape optimization problems.

Steps in the solution procedure

Step 1: Write the Lagrangian

Step 2: Take variation of the Lagrangian w.r.t. the design variable and equate to zero to get the design equation.

Step 3: Re-arrange the terms in the design equation to avoid computing the sensitivity of the state variables and thereby get the adjoint equation(s).

Step 4: Collect all the equations, including the governing equation(s), complementarity condition(s), resource constraints, etc.

Step 5: Obtain the optimality criterion by substituting adjoint and equilibrium equations into the design equation, when it is possible.

Step 6: Use the optimality criteria method to solve the equations numerically.

Shape optimization

Find the optimum shape to Minimize the Strain Energy of a 2D elastic problem.

$$\underset{\partial\Omega_d}{Min}\ SE = \frac{1}{2}\int_{\Omega} \boldsymbol{\varepsilon}_u^T \mathbf{D} \boldsymbol{\varepsilon}_u \ \ d\Omega$$

Subject to

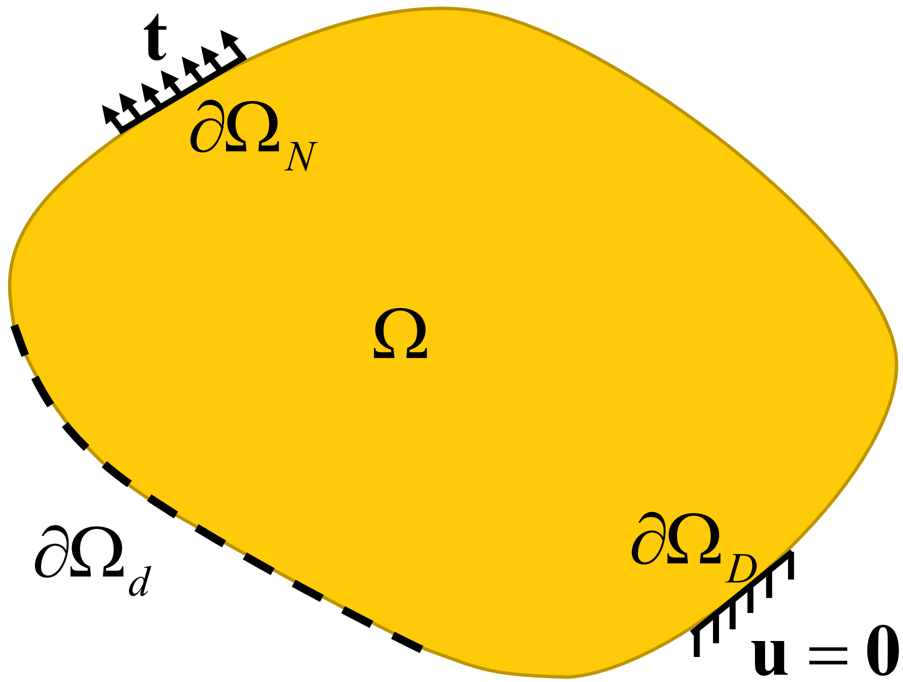
$$\Gamma : \int_{\Omega} \boldsymbol{\varepsilon}_u^T \mathbf{D} \boldsymbol{\varepsilon}_v \, d\Omega - \int_{\Omega} \mathbf{b}^T \mathbf{v} \, d\Omega - \int_{\partial\Omega_N} \mathbf{t}^T \mathbf{v} \, d\partial\Omega_N = 0$$

$$\Lambda : \quad \int_{\Omega} d\Omega - A^* \le 0$$

$$Data : \mathbf{D},\ \mathbf{b},\ \mathbf{t},\ A^*, \Omega, \partial\Omega_N$$

$\mathbf{D}$ = Elasticity Matrix
$\mathbf{b}$ = Body force
$\mathbf{t}$ = Traction
$\mathbf{u}$ = Displacement
$\mathbf{v}$ = Weak variable
A^* = Area Constraint

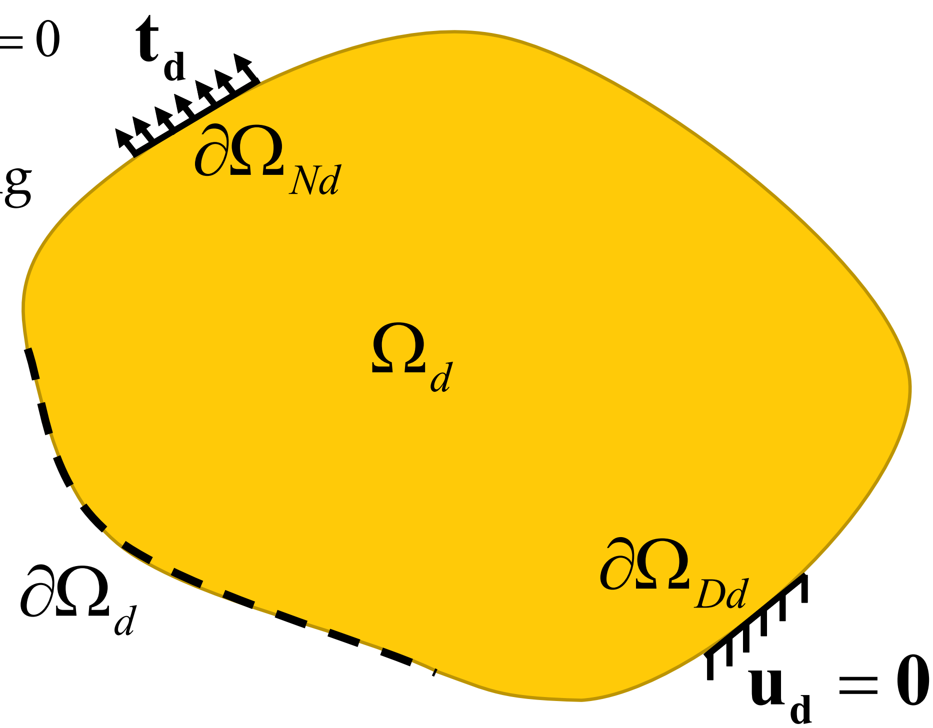
Ω = Domain
$\partial\Omega_N$ = Neumann Boundary
$\partial\Omega_D$ = Dirichlet Boundary
$\partial\Omega_d$ = Variable Boundary



Weak and strong forms after the domain changes upon perturbing the boundary

Weak form becomes,

$$\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^{T} \mathbf{v}_d \, d\Omega_d - \int_{\partial\Omega_{Nd}} \mathbf{t}_d^{T} \mathbf{v}_d \, d\partial\Omega_{Nd} = 0$$



The above eq is the weak form to the following strong form

$$\begin{cases} \text{Find } \mathbf{u_d} \in \mathbf{U} \text{ such that} & \\ \nabla \cdot \mathbf{D}\boldsymbol{\varepsilon}_{ud} + \mathbf{b}_{\boldsymbol{\varepsilon}} = 0 & \text{in } \Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{ud} = \boldsymbol{\sigma}_{\mathbf{u}} & \\ \mathbf{u_d} = \mathbf{0} & \text{on } \partial\Omega_{Dd} \\ \mathbf{D}\boldsymbol{\varepsilon}_{ud}\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{ud}\mathbf{n} = \mathbf{t_d} & \text{on } \partial\Omega_{Nd} \end{cases}$$

Now, strain energy can be written as $SE = \frac{1}{2} \int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{ud} \, d\Omega_d$

Step 1: Lagrangian

Taking the adjoint variable as $\mathbf{v}$ where $\mathbf{v} = \Gamma \mathbf{v}$

$$L = \frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{ud}\, d\Omega_d + \int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^{T} \mathbf{v}_d\, d\Omega_d - \int_{\partial\Omega_{Nd}} \mathbf{t}_d^{T} \mathbf{v}_d\, d\partial\Omega_{Nd} + \Lambda\left(\int_{\Omega_d} d\Omega_d - A^{*}\right)$$

Step 2: Derivative of Lagrangian

$$\frac{dL}{d(\partial\Omega_d)} = \underbrace{\frac{d}{d(\partial\Omega_d)}\left(\frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{ud}\, d\Omega_d\right)}_{(a)} + \underbrace{\frac{d}{d(\partial\Omega_d)}\left(\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^{T} \mathbf{v}_d\, d\Omega_d - \int_{\partial\Omega_{Nd}} \mathbf{t}_d^{T} \mathbf{v}_d\, d\partial\Omega_{Nd}\right)}_{(b)} + \underbrace{\frac{d}{d(\partial\Omega_d)}\left[\Lambda\left(\int_{\Omega_d} d\Omega_d - A^{*}\right)\right]}_{(c)} = 0$$

Excluding the area term (term c); sensitivity of this will be done later in Slide 28.

$$\frac{dL}{d(\partial\Omega_d)} = \frac{dSE}{d(\partial\Omega_d)} + \Lambda\left(\frac{dA}{d(\partial\Omega_d)}\right) = 0$$

Design equation

Reynolds Transport Theorem for (a)

Reynolds Transport Theorem (RTT):

$$\frac{d}{dp}\left(\int_{\Omega_d} f\, d\Omega_d\right) = \int_{\Omega_d} f'\, d\Omega_d + \int_{\partial\Omega_d} f\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$$

Applying RTT to term (a)

$$\frac{d}{d(\partial\Omega_d)}\left(\frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d\right) = \frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}_{u'd}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d + \frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{u'd}\, d\Omega_d + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$$

$$= \int_{\Omega_d} \boldsymbol{\varepsilon}_{u'd}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$$

Converting spatial derivative to material derivative using $f' = \dot{f} - \nabla f \cdot \mathbf{V}$

$$\frac{d}{d(\partial\Omega_d)}\left(\frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d\right) = \int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d - \int_{\Omega_d} \left(\nabla\left(\boldsymbol{\varepsilon}_{ud}^{T}\right)\cdot\mathbf{V}\right)\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$$

Gauss Divergence Theorem

We know that dot product rule is $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A}$

Applying this to each second term inside square bracket

$$= \int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^{T} \mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d - \left[\int_{\Omega_d} \nabla\cdot\left(\left(\boldsymbol{\varepsilon}_{ud}^{T}\cdot\mathbf{V}\right)\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d - \int_{\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right] + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$$

Using the Gauss divergence theorem on the second term,

$$\int_{\Omega_d} \nabla\cdot\mathbf{F}\, d\Omega_d = \int_{\partial\Omega_d} \mathbf{F}\cdot\mathbf{n}\, d\partial\Omega_d$$

$$= \int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^{T} \mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d - \left[\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\cdot\mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right] + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$$

Separate the boundary terms, material derivative, and others

$$= \int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^{T} \mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d - \left[\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\cdot\mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right] + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$$

Term (a) completed

Reynolds Transport Theorem for (b)

We know that, Reynold's Transport Theorem (RTT) is

$$\frac{d}{dp}\left(\int_{\Omega_d} f\, d\Omega_d\right) = \int_{\Omega_d} f'\, d\Omega_d + \int_{\partial\Omega_d} f(\mathbf{V}\cdot\mathbf{n})\, d\partial\Omega_d$$

Applying RTT to the term (b)

$$\frac{d}{d(d\Omega_d)}\left(\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^T\mathbf{v}_d\, d\Omega_d - \int_{\partial\Omega_{Nd}} \mathbf{t}_d^T\mathbf{v}_d\, d\partial\Omega_d\right) = \int_{\Omega_d} \boldsymbol{\varepsilon}^T_{u'd}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d + \int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{v'd}\, d\Omega_d + \int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)(\mathbf{V}\cdot\mathbf{n})\, d\partial\Omega_d -$$

$$\left[\int_{\Omega_d} \mathbf{b}_d^T\mathbf{v}'_d\, d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T\mathbf{v}_d\right)(\mathbf{V}\cdot\mathbf{n})\, d\partial\Omega_d\right] - \left[\int_{\partial\Omega_{Nd}} \mathbf{t}_d^T\mathbf{v}'_d\, d\partial\Omega_{Nd}\right]$$

Converting spatial derivative to material derivative using $f' = \dot{f} - \nabla f\cdot\mathbf{V}$

$$= \left[\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{\dot{u}d}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \left(\int_{\Omega_d} \nabla\left(\left(\boldsymbol{\varepsilon}^T_{ud}\right)\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right) d\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right)\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right) d\Omega_d\right)\right] + \left[\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{\dot{v}d}\, d\Omega_d - \int_{\Omega_d} \left(\nabla\left(\boldsymbol{\varepsilon}^T_{vd}\right)\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d\right] +$$

$$\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)(\mathbf{V}\cdot\mathbf{n})\, d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^T\dot{\mathbf{v}}_d\, d\Omega_d - \int_{\Omega_d} \nabla\left(\mathbf{b}_d^T\mathbf{v}_d\right)\cdot\mathbf{V}\, d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T\mathbf{v}_d\right)(\mathbf{V}\cdot\mathbf{n})\, d\partial\Omega_d\right] - \left[\int_{\partial\Omega_{Nd}} \mathbf{t}_d^T\dot{\mathbf{v}}_d\, d\partial\Omega_{Nd}\right]$$

Dot Product rule

We know that dot product rule is $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A}$

Applying this to each second term inside square bracket

$$= \left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \left(\int_{\Omega_d} \nabla\left(\left(\left(\boldsymbol{\varepsilon}_{ud}^{T}\right)\cdot\mathbf{V}\right)\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)d\Omega_d - \int_{\Omega_d}\left(\left(\boldsymbol{\varepsilon}_{ud}^{T}\right)\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)d\Omega_d\right)\right] +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{\dot{v}d}\, d\Omega_d - \left(\int_{\Omega_d} \nabla\left(\left(\left(\boldsymbol{\varepsilon}_{vd}^{T}\right)\cdot\mathbf{V}\right)\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)d\Omega_d - \int_{\Omega_d}\left(\left(\boldsymbol{\varepsilon}_{vd}^{T}\right)\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)d\Omega_d\right)\right] +$$

$$\int_{\partial\Omega_d}\left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)\left(\mathbf{V}\cdot\mathbf{n}\right)d\partial\Omega_d - \left[\int_{\Omega_d}\mathbf{b}_d^{T}\dot{\mathbf{v}}_d\, d\Omega_d - \int_{\Omega_d}\left(\mathbf{b}_d^{T}\nabla\mathbf{v}_d\right)\cdot\mathbf{V}d\Omega_d + \int_{\partial\Omega_d}\left(\mathbf{b}_d^{T}\mathbf{v}_d\right)\left(\mathbf{V}\cdot\mathbf{n}\right)d\partial\Omega_d\right] - \left[\int_{\partial\Omega_{Nd}}\mathbf{t}_d^{T}\dot{\mathbf{v}}_d\, d\partial\Omega_{Nd}\right]$$

Gauss Divergence Theorem

Using the Gauss divergence theorem on each second term inside the square bracket,

$$\int_{\Omega_d} \nabla \cdot \mathbf{F} d\Omega_d = \int_{\partial\Omega_d} \mathbf{F} \cdot \mathbf{n} d\partial\Omega_d$$

$$= \left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^{T} \right) \cdot \mathbf{V} \right) \left(\mathbf{D} \boldsymbol{\varepsilon}_{vd} \cdot \mathbf{n} \right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^{T} \right) \cdot \mathbf{V} \right) \nabla \cdot \left(\mathbf{D} \boldsymbol{\varepsilon}_{vd} \right) d\Omega_d \right) \right] +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{\dot{v}d} \, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^{T} \right) \cdot \mathbf{V} \right) \left(\mathbf{D} \boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n} \right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^{T} \right) \cdot \mathbf{V} \right) \nabla \cdot \left(\mathbf{D} \boldsymbol{\varepsilon}_{ud} \right) d\Omega_d \right) \right] +$$

$$\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd} \right) \left(\mathbf{V} \cdot \mathbf{n} \right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^{T} \dot{\mathbf{v}}_d \, d\Omega_d - \int_{\Omega_d} \left(\mathbf{b}_d^{T} \nabla \mathbf{v}_d \right) \cdot \mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^{T} \mathbf{v}_d \right) \left(\mathbf{V} \cdot \mathbf{n} \right) d\partial\Omega_d \right] - \left[\int_{\partial\Omega_{Nd}} \mathbf{t}_d^{T} \dot{\mathbf{v}}_d \, d\partial\Omega_{Nd} \right]$$

Separate the terms

Separate the boundary terms, material derivative, and others

$$= \left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T \right) \cdot \mathbf{V} \right) \left(\mathbf{D} \boldsymbol{\varepsilon}_{vd} \cdot \mathbf{n} \right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T \right) \cdot \mathbf{V} \right) \nabla \cdot \left(\mathbf{D} \boldsymbol{\varepsilon}_{vd} \right) d\Omega_d \right) \right] +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{\dot{v}d} \, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^T \right) \cdot \mathbf{V} \right) \left(\mathbf{D} \boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n} \right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^T \right) \cdot \mathbf{V} \right) \nabla \cdot \left(\mathbf{D} \boldsymbol{\varepsilon}_{ud} \right) d\Omega_d \right) \right] +$$

$$\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{vd} \right) \left(\mathbf{V} \cdot \mathbf{n} \right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^T \dot{\mathbf{v}}_d \, d\Omega_d - \int_{\Omega_d} \left(\mathbf{b}_d^T \nabla \mathbf{v}_d \right) \cdot \mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{v}_d \right) \left(\mathbf{V} \cdot \mathbf{n} \right) d\partial\Omega_d \right] - \left[\int_{\partial\Omega_{Nd}} \mathbf{t}_d^T \dot{\mathbf{v}}_d \, d\partial\Omega_{Nd} \right]$$

Term (b) completed

Combine terms (a) and (b)

From slides 8 and 12

$$\frac{dSE}{d(\partial\Omega_d)} = \int_{\Omega_d} \boldsymbol{\varepsilon}^T_{\dot{u}d}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d - \left[\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud}\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\cdot\mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud}\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right] + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{\dot{u}d}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right)\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\cdot\mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right)\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right) d\Omega_d \right) \right] +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{\dot{v}d}\, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{vd}\right)\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\cdot\mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{vd}\right)\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right) \right] +$$

$$\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^T \dot{\mathbf{v}}_d\, d\Omega_d - \int_{\Omega_d} \left(\mathbf{b}_d^T \nabla\mathbf{v}_d\right)\cdot\mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T\mathbf{v}_d\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d \right] - \left[\int_{\partial\Omega_{Nd}} \mathbf{t}_d^T \dot{\mathbf{v}}_d\, d\partial\Omega_{Nd} \right]$$

Strong and weak forms of adjoint/state variable

Step 3: re-arrange the terms in the design equation to avoid computing the derivative of the state variables

Collect the terms with $\dot{\mathbf{v}}_d$ and form weak form of structural problem to get $\mathbf{u}_d$

$$\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^T \mathbf{D}\boldsymbol{\varepsilon}_{\dot{v}d}\, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^T \dot{\mathbf{v}}_d\, d\Omega_d - \int_{\partial\Omega_{Nd}} \mathbf{t}_d^T \dot{\mathbf{v}}_d\, d\partial\Omega_{Nd} = 0$$

Collect the terms with $\dot{\mathbf{u}}_d$ and form adjoint structural problem

$$\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d + \int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d = 0$$

$$\Rightarrow \mathbf{v}_\mathbf{d} = -\mathbf{u}_\mathbf{d}$$

Adjoint equation

$$\begin{cases} \text{Find } \mathrm{v}_\mathbf{d} \in \mathbf{U} \text{ such that} & \\ \nabla \cdot \mathbf{D}\boldsymbol{\varepsilon}_{vd} - \mathbf{b}_d = 0 & \text{in } \Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{vd} = \boldsymbol{\sigma}_v & \\ v_\mathbf{d} = \mathbf{0} & \text{on } \partial\Omega_{Dd} \\ \mathbf{D}\boldsymbol{\varepsilon}_{vd}\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{vd}\mathbf{n} = -\mathbf{t}_\mathbf{d} & \text{on } \partial\Omega_{Nd} \end{cases}$$

Final Sensitivity Integral

Using the adjoint variable, the strong forms mentioned before and $\nabla \mathbf{u_d} = \left(\boldsymbol{\varepsilon}_{ud}\right)$, the sensitivity becomes

$$\frac{dSE}{d(\partial\Omega_d)} = -\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d + \int_{\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \cdot \mathbf{V}\right)\nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d +$$

$$+ \int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T\right) \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T\right) \cdot \mathbf{V}\right)\nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d + \int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T\right) \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d -$$

$$\int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T\right) \cdot \mathbf{V}\right)\nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d - \int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \mathbf{b}_d^T \left(\nabla \mathbf{u_d}\right) \cdot \mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{u_d}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d$$

Also since there is no load on the $\partial\Omega_d$, hence $\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right)$ becomes zero

$$\frac{dSE}{d(\partial\Omega_d)} = -\frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{u_d}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d$$

Step 4-6 will be continued from slide 29

Shape optimization-Load dependent

$$\underset{\partial\Omega_d}{Min}\ SE = \frac{1}{2}\int_{\Omega} \boldsymbol{\varepsilon}_u^T \mathbf{D} \boldsymbol{\varepsilon}_u \ d\Omega$$

Subject to

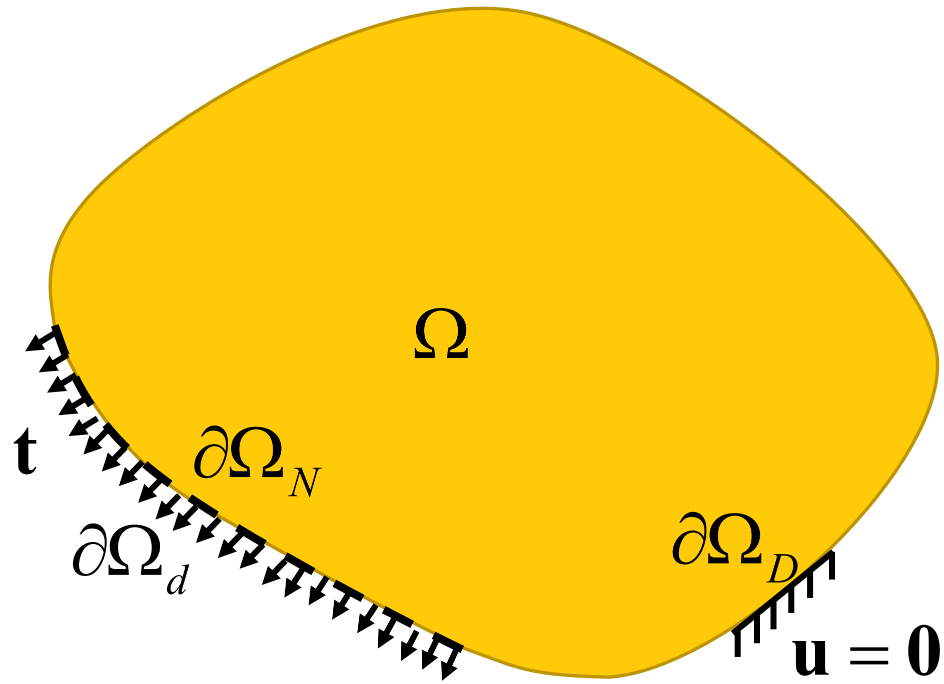
$$\Gamma : \int_{\Omega} \boldsymbol{\varepsilon}_u^T \mathbf{D} \boldsymbol{\varepsilon}_v \, d\Omega - \int_{\Omega} \mathbf{b}^T \mathbf{v} \, d\Omega - \int_{\partial\Omega_d} \mathbf{t}^T \mathbf{v} \, d\partial\Omega_d = 0$$

$$\Lambda : \quad \int_{\Omega} d\Omega - A^* \leq 0$$

$$Data : \mathbf{D},\ \mathbf{b},\ \mathbf{t},\ A^*, \Omega, \partial\Omega_N$$

Ω = Domain
$\partial\Omega_N$ = Neumann Boundary
$\partial\Omega_D$ = Dirichlet Boundary
$\partial\Omega_d$ = Variable Boundary

$\mathbf{D}$ = Elasticity Matrix
$\mathbf{b}$ = Body force
$\mathbf{t}$ = Traction
$\mathbf{u}$ = Displacement
$\mathbf{v}$ = Weak variable
A^* = Area Constraint



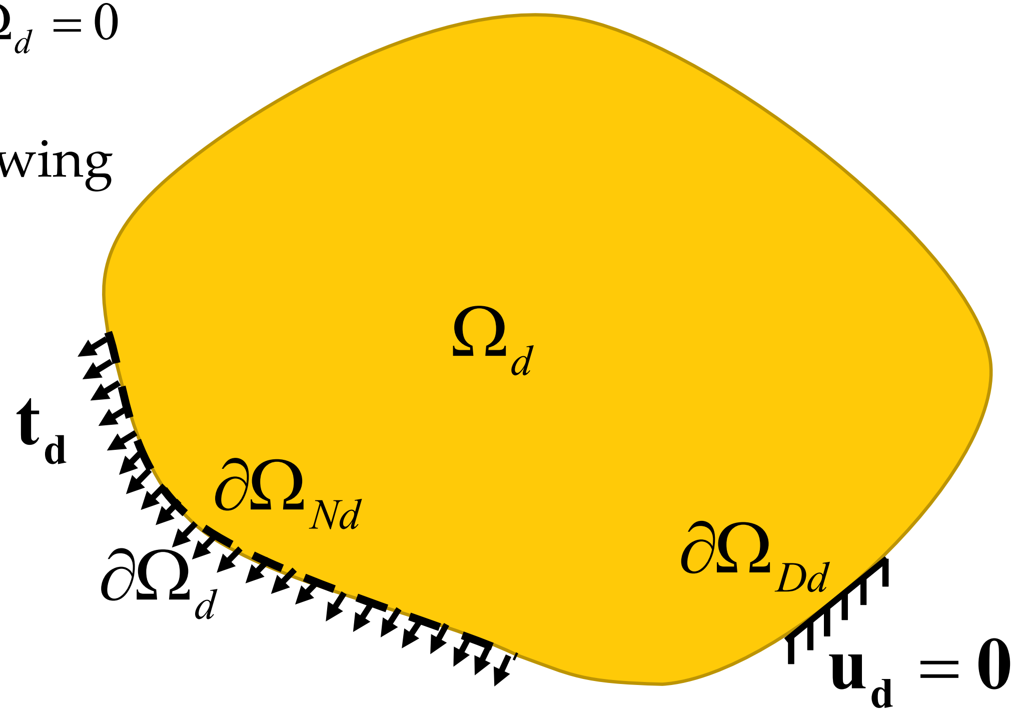
Weak and strong forms after the domain changes upon perturbing the boundary

Weak form becomes,

$$\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^{T} \mathbf{v}_d \, d\Omega_d - \int_{\partial\Omega_d} \mathbf{t}_d^{T} \mathbf{v}_d \, d\partial\Omega_d = 0$$

The above eq is the weak form to the following strong form

$$\begin{cases} \text{Find } \mathbf{u_d} \in \mathbf{U} \text{ such that} & \\ \nabla \cdot \mathbf{D}\boldsymbol{\varepsilon}_{ud} + \mathbf{b}_{\boldsymbol{\varepsilon}} = 0 & \text{in } \Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{ud} = \boldsymbol{\sigma}_{\mathbf{u}} & \\ \mathbf{u_d} = \mathbf{0} & \text{on } \partial\Omega_{Dd} \\ \mathbf{D}\boldsymbol{\varepsilon}_{ud}\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{ud}\mathbf{n} = \mathbf{t_d} & \text{on } \partial\Omega_{Nd} \end{cases}$$



Now, strain energy can be written as $SE = \frac{1}{2} \int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{ud} \, d\Omega_d$

Step 1: Lagrangian

Taking the adjoint variable as $\mathbf{v}$ where $\mathbf{v} = \Gamma \mathbf{v}$

$$L = \frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d + \int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^T \mathbf{v}_d\, d\Omega_d - \int_{\partial\Omega_d} \mathbf{t}_d^T \mathbf{v}_d\, d\partial\Omega_d + \Lambda\left(\int_{\Omega_d} d\Omega_d - A^*\right)$$

Step 2: Derivative of Lagrangian

$$\frac{dL}{d(\partial\Omega_d)} = \underbrace{\frac{d}{d(\partial\Omega_d)}\left(\frac{1}{2}\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d\right)}_{(a)} + \underbrace{\frac{d}{d(\partial\Omega_d)}\left(\int_{\Omega_d} \boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^T\mathbf{v}_d\, d\Omega_d - \int_{\partial\Omega_d} \mathbf{t}_d^T\mathbf{v}_d\, d\partial\Omega_d\right)}_{(b)} + \underbrace{\frac{d}{d(\partial\Omega_d)}\left[\Lambda\left(\int_{\Omega_d} d\Omega_d - A^*\right)\right]}_{(c)} = 0$$

Excluding the area term (term (c)); sensitivity of this will be done later on.

The term (a) expression remains same as slide 8

The boundary terms, material derivative, and others

$$= \int_{\Omega_d} \boldsymbol{\varepsilon}^T_{\dot{u}d}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d - \left[\int_{\partial\Omega_d}\left(\boldsymbol{\varepsilon}^T_{ud}\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\cdot\mathbf{n}\right)d\partial\Omega_d - \int_{\Omega_d}\left(\boldsymbol{\varepsilon}^T_{ud}\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)d\Omega_d\right] + \frac{1}{2}\int_{\partial\Omega_d}\left(\boldsymbol{\varepsilon}^T_{ud}\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right)d\partial\Omega_d$$

Reynolds Transport Theorem for (b)

We know that Reynolds Transport Theorem (RTT) is $\frac{d}{dp}\left(\int_{\Omega_d} f\, d\Omega_d\right) = \int_{\Omega_d} f'\, d\Omega_d + \int_{\partial\Omega_d} f\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d$

Applying RTT to the term (b)

$$\frac{d}{d(d\Omega_d)}\left(\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^{T}\mathbf{v}_d\, d\Omega_d - \int_{\partial\Omega_{Nd}} \mathbf{t}_d^{T}\mathbf{v}_d\, d\partial\Omega_d\right) = \int_{\Omega_d} \boldsymbol{\varepsilon}_{u'd}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d + \int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{v'd}\, d\Omega_d + \int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d -$$

$$\left[\int_{\Omega_d} \mathbf{b}_d^{T}\mathbf{v}'_d\, d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^{T}\mathbf{v}_d\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d\right] - \left[\int_{\partial\Omega_d} \left(\mathbf{t}_d^{T}\mathbf{v}_d\right)' d\partial\Omega_d\right]$$

Material derivative

Converting spatial derivative to material derivative using $f' = \dot{f} - \nabla f \cdot \mathbf{V}$

Material derivative of a boundary integral $\int_{\partial\Omega} f \partial\Omega = \int_{\partial\Omega} \left\{ \dot{f} + f\left(\nabla \cdot \mathbf{V} - \nabla \mathbf{V}\mathbf{n} \cdot \mathbf{n}\right) \right\} \partial\Omega$

$$= \left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d - \left(\int_{\Omega_d} \nabla\left(\left(\boldsymbol{\varepsilon}_{ud}^T\right) \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right) d\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T\right) \cdot \mathbf{V}\right) \nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right) d\Omega_d \right) \right] + \left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{\dot{v}d} \, d\Omega_d - \int_{\Omega_d} \left(\nabla\left(\boldsymbol{\varepsilon}_{vd}^T\right) \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right] +$$

$$\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{vd}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^T \dot{\mathbf{v}}_d \, d\Omega_d - \int_{\Omega_d} \nabla\left(\mathbf{b}_d^T \mathbf{v}_d\right) \cdot \mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{v}_d\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d \right] -$$

$$\left[\int_{\partial\Omega_d} \dot{\mathbf{t}}_d^T \mathbf{v}_d \, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \dot{\mathbf{v}}_d \, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \mathbf{v}_d \left(\nabla \cdot \mathbf{V} - \nabla \mathbf{V}\mathbf{n} \cdot \mathbf{n}\right) d\partial\Omega_d \right]$$

Dot Product rule

We know that dot product rule is $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A}$

Applying this to each second term inside square bracket

$$
\begin{aligned}
&= \left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^{T} \mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \left(\int_{\Omega_d} \nabla\left(\left(\left(\boldsymbol{\varepsilon}_{ud}^{T} \right)\cdot\mathbf{V} \right) \mathbf{D}\boldsymbol{\varepsilon}_{vd} \right) d\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^{T} \right)\cdot\mathbf{V} \right) \nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd} \right) d\Omega_d \right) \right] + \\
&\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D}\boldsymbol{\varepsilon}_{\dot{v}d}\, d\Omega_d - \left(\int_{\Omega_d} \nabla\left(\left(\left(\boldsymbol{\varepsilon}_{vd}^{T} \right)\cdot\mathbf{V} \right) \mathbf{D}\boldsymbol{\varepsilon}_{ud} \right) d\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^{T} \right)\cdot\mathbf{V} \right) \nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \right) d\Omega_d \right) \right] + \\
&\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D}\boldsymbol{\varepsilon}_{vd} \right)\left(\mathbf{V}\cdot\mathbf{n} \right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^{T} \dot{\mathbf{v}}_d\, d\Omega_d - \int_{\Omega_d} \left(\mathbf{b}_d^{T} \nabla\mathbf{v}_d \right)\cdot\mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^{T} \mathbf{v}_d \right)\left(\mathbf{V}\cdot\mathbf{n} \right) d\partial\Omega_d \right] - \\
&\left[\int_{\partial\Omega_d} \dot{\mathbf{t}}_d^{T} \mathbf{v}_d\, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^{T} \dot{\mathbf{v}}_d\, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^{T} \mathbf{v}_d \left(\nabla\cdot\mathbf{V} - \nabla\mathbf{V}\mathbf{n}\cdot\mathbf{n} \right) d\partial\Omega_d \right]
\end{aligned}
$$

Gauss Divergence Theorem

Using the Gauss divergence theorem on each second term inside the square bracket,

$$\int_{\Omega_d} \nabla \cdot \mathbf{F} d\Omega_d = \int_{\partial\Omega_d} \mathbf{F} \cdot \mathbf{n} d\partial\Omega_d$$

$$= \left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T \right) \cdot \mathbf{V} \right) \left(\mathbf{D} \boldsymbol{\varepsilon}_{vd} \cdot \mathbf{n} \right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T \right) \cdot \mathbf{V} \right) \nabla \cdot \left(\mathbf{D} \boldsymbol{\varepsilon}_{vd} \right) d\Omega_d \right) \right] +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{\dot{v}d} \, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^T \right) \cdot \mathbf{V} \right) \left(\mathbf{D} \boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n} \right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^T \right) \cdot \mathbf{V} \right) \nabla \cdot \left(\mathbf{D} \boldsymbol{\varepsilon}_{ud} \right) d\Omega_d \right) \right] +$$

$$\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{vd} \right) \left(\mathbf{V} \cdot \mathbf{n} \right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^T \dot{\mathbf{v}}_d \, d\Omega_d - \int_{\Omega_d} \left(\mathbf{b}_d^T \nabla \mathbf{v}_d \right) \cdot \mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{v}_d \right) \left(\mathbf{V} \cdot \mathbf{n} \right) d\partial\Omega_d \right] -$$

$$\left[\int_{\partial\Omega_d} \dot{\mathbf{t}}_d^T \mathbf{v}_d \, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \dot{\mathbf{v}}_d \, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \mathbf{v}_d \left(\nabla \cdot \mathbf{V} - \nabla \mathbf{V} \mathbf{n} \cdot \mathbf{n} \right) d\partial\Omega_d \right]$$

Separate the terms

Separate the boundary terms, material derivative, and others

$$
=\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^{T}\right)\cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\cdot \mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^{T}\right)\cdot \mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right) d\Omega_d\right)\right]+
$$

$$
\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{\dot{v}d}\, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^{T}\right)\cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\cdot \mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^{T}\right)\cdot \mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d\right)\right]+
$$

$$
\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^{T}\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^{T}\dot{\mathbf{v}}_d\, d\Omega_d - \int_{\Omega_d} \left(\mathbf{b}_d^{T}\nabla\mathbf{v}_d\right)\cdot \mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^{T}\mathbf{v}_d\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d\right]-
$$

$$
\left[\int_{\partial\Omega_d} \dot{\mathbf{t}}_d^{T}\mathbf{v}_d\, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^{T}\dot{\mathbf{v}}_d\, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^{T}\mathbf{v}_d\left(\nabla\cdot\mathbf{V} - \nabla\mathbf{V}\mathbf{n}\cdot\mathbf{n}\right) d\partial\Omega_d\right]
$$

Term (b) completed

Combine terms (a) and (b)

From slides 8 and 24

$$\frac{dSE}{d(\partial\Omega_d)} = \int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D}\boldsymbol{\varepsilon}_{ud}\, d\Omega_d - \left[\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \cdot \mathbf{V}\right) \nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right] + \frac{1}{2} \int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D}\boldsymbol{\varepsilon}_{vd}\, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T\right)\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\cdot\mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{ud}^T\right)\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{vd}\right) d\Omega_d \right) \right] +$$

$$\left[\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^T \mathbf{D}\boldsymbol{\varepsilon}_{\dot{v}d}\, d\Omega_d - \left(\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^T\right)\cdot\mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\cdot\mathbf{n}\right) d\partial\Omega_d - \int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}_{vd}^T\right)\cdot\mathbf{V}\right)\nabla\cdot\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d \right) \right] +$$

$$\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}_{ud}^T \mathbf{D}\boldsymbol{\varepsilon}_{vd}\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d - \left[\int_{\Omega_d} \mathbf{b}_d^T \dot{\mathbf{v}}_d\, d\Omega_d - \int_{\Omega_d} \left(\mathbf{b}_d^T \nabla \mathbf{v}_d\right)\cdot \mathbf{V} d\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{v}_d\right)\left(\mathbf{V}\cdot\mathbf{n}\right) d\partial\Omega_d \right] -$$

$$\left[\int_{\partial\Omega_d} \dot{\mathbf{t}}_d^T \mathbf{v}_d\, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \dot{\mathbf{v}}_d\, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \mathbf{v}_d \left(\nabla\cdot\mathbf{V} - \nabla\mathbf{V}\mathbf{n}\cdot\mathbf{n}\right) d\partial\Omega_d \right]$$

Strong and weak forms of adjoint/state variable

Step 3: re-arrange the terms in the design equation to avoid computing the derivative of the state variables

Collect the terms with $\dot{\mathbf{v}}_d$ and form weak form of structural problem to get $\mathbf{u}_d$

$$\int_{\Omega_d} \boldsymbol{\varepsilon}_{ud}^T \mathbf{D} \boldsymbol{\varepsilon}_{\dot{v}d} \, d\Omega_d - \int_{\Omega_d} \mathbf{b}_d^T \dot{\mathbf{v}}_d \, d\Omega_d - \int_{\partial\Omega_d} \mathbf{t}_d^T \dot{\mathbf{v}}_d \, d\partial\Omega_d = 0$$

Collect the terms with $\dot{\mathbf{u}}_d$ and form adjoint structural problem

$$\int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D} \boldsymbol{\varepsilon}_{ud} \, d\Omega_d + \int_{\Omega_d} \boldsymbol{\varepsilon}_{\dot{u}d}^T \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d = 0$$

$$\Rightarrow \mathbf{v}_\mathbf{d} = -\mathbf{u}_\mathbf{d}$$

Adjoint equation

$$\begin{cases} \text{Find } \mathrm{v}_\mathbf{d} \in \mathbf{U} \text{ such that} & \\ \nabla \cdot \mathbf{D}\boldsymbol{\varepsilon}_{vd} - \mathbf{b}_d = 0 & \text{in } \Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{vd} = \boldsymbol{\sigma}_v & \\ v_\mathbf{d} = \mathbf{0} & \text{on } \partial\Omega_{Dd} \\ \mathbf{D}\boldsymbol{\varepsilon}_{vd}\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_d \\ \mathbf{D}\boldsymbol{\varepsilon}_{vd}\mathbf{n} = -\mathbf{t}_\mathbf{d} & \text{on } \partial\Omega_{Nd} \end{cases}$$

Final Sensitivity Integral

Using the adjoint variable, the strong forms mentioned before and $\nabla \mathbf{u_d} = \left(\boldsymbol{\varepsilon}_{ud}\right)$, the sensitivity becomes

$$\frac{dSE}{d(\partial\Omega_d)} = \cancel{-\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud} \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d} + \cancel{\int_{\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud} \cdot \mathbf{V}\right) \nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d} + \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud} \mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d +$$

$$+ \cancel{\int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right) \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d} - \cancel{\int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right) \cdot \mathbf{V}\right) \nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d} + \int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right) \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d -$$

$$\cancel{\int_{\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right) \cdot \mathbf{V}\right) \nabla \cdot \left(\mathbf{D}\boldsymbol{\varepsilon}_{ud}\right) d\Omega_d} - \int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud} \mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d - \cancel{\int_{\Omega_d} \mathbf{b}_d^T \left(\nabla \mathbf{u_d}\right) \cdot \mathbf{V} d\Omega_d} + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{u_d}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d +$$

$$\left[\int_{\partial\Omega_d} \dot{\mathbf{t}}_d^T \mathbf{u}_d \, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \mathbf{u}_d \left(\nabla \cdot \mathbf{V} - \nabla \mathbf{V}\mathbf{n} \cdot \mathbf{n}\right) d\partial\Omega_d\right]$$

$$\frac{dSE}{d(\partial\Omega_d)} = \int_{\partial\Omega_d} \left(\left(\boldsymbol{\varepsilon}^T_{ud}\right) \cdot \mathbf{V}\right)\left(\mathbf{D}\boldsymbol{\varepsilon}_{ud} \cdot \mathbf{n}\right) d\partial\Omega_d - \frac{1}{2}\int_{\partial\Omega_d} \left(\boldsymbol{\varepsilon}^T_{ud} \mathbf{D}\boldsymbol{\varepsilon}_{ud}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d + \int_{\partial\Omega_d} \left(\mathbf{b}_d^T \mathbf{u_d}\right)\left(\mathbf{V} \cdot \mathbf{n}\right) d\partial\Omega_d +$$

$$\int_{\partial\Omega_d} \dot{\mathbf{t}}_d^T \mathbf{u}_d \, d\partial\Omega_d + \int_{\partial\Omega_d} \mathbf{t}_d^T \mathbf{u}_d \left(\nabla \cdot \mathbf{V} - \nabla \mathbf{V}\mathbf{n} \cdot \mathbf{n}\right) d\partial\Omega_d$$

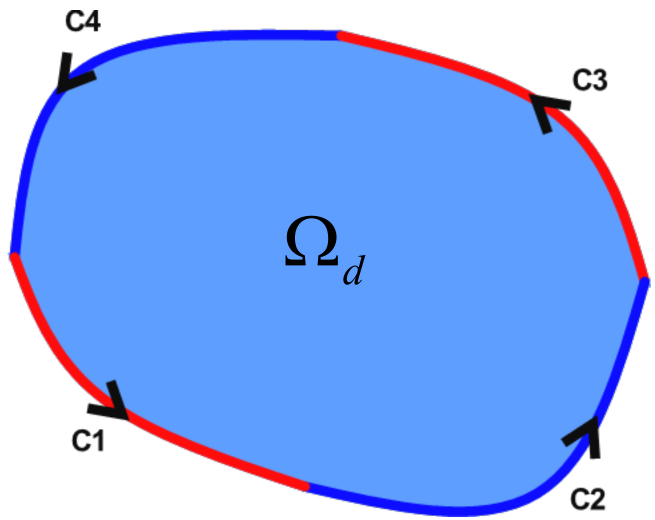
Area

The area can be defined as, $A = \int_{\Omega_d} d\Omega$

Green's theorem can be used to convert the domain integral to the line integral.

The green's theorem states that

$$\int_{\Omega} \left[\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right] d\Omega = \oint_{C} (L dx + M dy)$$



Where C is the boundary of the domain.
For the Area, the RHS terms are 1.

$$\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 1$$

Which means $M = \frac{x}{2}, L = \frac{-y}{2}$

So the area can be defined as $A = \int_{\Omega_d} d\Omega = \frac{1}{2} \int_{C} (x\,dy - y\,dx)$

Area derivative $\frac{dA}{d(\partial\Omega_d)} = \frac{d}{d(\partial\Omega_d)} \int_{\Omega_d} d\Omega_d = \frac{d}{d(\partial\Omega_d)} \left(\frac{1}{2} \int_{C} (x\,dy - y\,dx) \right)$

All equations

Step 4: Collect all Equations

Design equation

$$\frac{dL}{d\left(\partial\Omega_d\right)} = \frac{dSE}{d\left(\partial\Omega_d\right)} + \Lambda\left(\frac{dA}{d\left(\partial\Omega_d\right)}\right) = 0$$

Adjoint equation

$$\int_{\Omega_d} \boldsymbol{\varepsilon}_{\tilde{u}d}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{ud} \, d\Omega_d + \int_{\Omega_d} \boldsymbol{\varepsilon}_{\tilde{u}d}^{T} \mathbf{D} \boldsymbol{\varepsilon}_{vd} \, d\Omega_d = 0$$

$$\Rightarrow \mathbf{v_d} = \mathbf{-u_d}$$

Feasibility equation

$$A - A^* \leq 0$$

Complementarity condition

$$\Lambda\left(A - A^*\right) = 0; \quad \Lambda \geq 0$$

Optimality criterion

Step 5: Optimality criterion by rearranging

$$\frac{dL}{d\left(\partial\Omega_d\right)} = \frac{dSE}{d\left(\partial\Omega_d\right)} + \Lambda\left(\frac{dA}{d\left(\partial\Omega_d\right)}\right) = 0$$

$$\text{Ratio} = \frac{\dfrac{dSE}{d\left(\partial\Omega_d\right)}}{-\Lambda\left(\dfrac{dA}{d\left(\partial\Omega_d\right)}\right)} = 1$$

Numerical solution

Step 6: Use Optimality criterion to find variable

Initial guess for $\Lambda, \partial\Omega_d$

Update $\partial\Omega_{di}^{(k+1)} = \partial\Omega_{di}^{(k)} \mathrm{Ratio}^{\beta}$ or $\partial\Omega_{di}^{(k+1)} = \partial\Omega_{di}^{(k)} \left(1\text{-}(1\text{-}\mathrm{Ratio})\beta\right)$

Check if $\partial\Omega_{di}$ has exceeded bounds and equate to the bounds if they did.
Update Λ until $\partial\Omega_{di}$ does not exceed bounds anymore.

Inner loop

Outer loop

k

$k = k + 1$

Continue until $\partial\Omega_d^{(k+1)} = \partial\Omega_d^{(k)}$

$Ratio = 1$

or

$\partial\Omega_{di} = \partial\Omega_{d\min}$ or $\partial\Omega_{d\max}$

What we need to achieve for all elements, $i = 1, 2, \cdots, N$

The end note

shape optimization of 2D elastic stiffness

- Observe shape derivative used to find the sensitivity for stiffness problem. Think how the formulation will change when the loading is on the boundary which is variable.
- We follow six steps to solve the discretized (or finite-variable optimization) problem.
- Identify the optimality criterion.
- Interpret the optimality criterion.
- Iterative numerical solution, when it is needed, remains the same.

Thanks