Stiff and light statically determinate three-bar truss

An appetizer for ME 260@IISc

Structural optimization follows a hierarchy of topology, shape, and size. Topology is about how many holes there are in a structure. Shape is about what that word implies: the shape. The size refers to dimensions and other parameters that determine the size without changing shape and topology. We often say and think that if we do not have the best topology, we never get the best design no matter how much effort is put into shape and size optimization. Likewise, after the most appropriate topology is chosen, if the shape is not right, size optimization does not lead to the best design. Let us see for ourselves what this means in the simplest setting that we can deal with—a planar truss.

Consider a statically determinate truss consisting of three bars as shown in Fig. 1 wherein the vertices and members are labelled. Vertex 1 is fixed while vertex 2 is constrained in the y-direction. At vertex 3, we have two forces in the x and y directions as shown. Let the lengths of the members be denoted by $l_i, i = 1,2,3$, cross-section areas by A_i, and the internal forces by f_i. Since it is a statically determinate truss, the internal forces in the members do not depend on the Young's modulus, $(E = 70\ GPa)$, of the material and the areas of cross-section of the truss members.

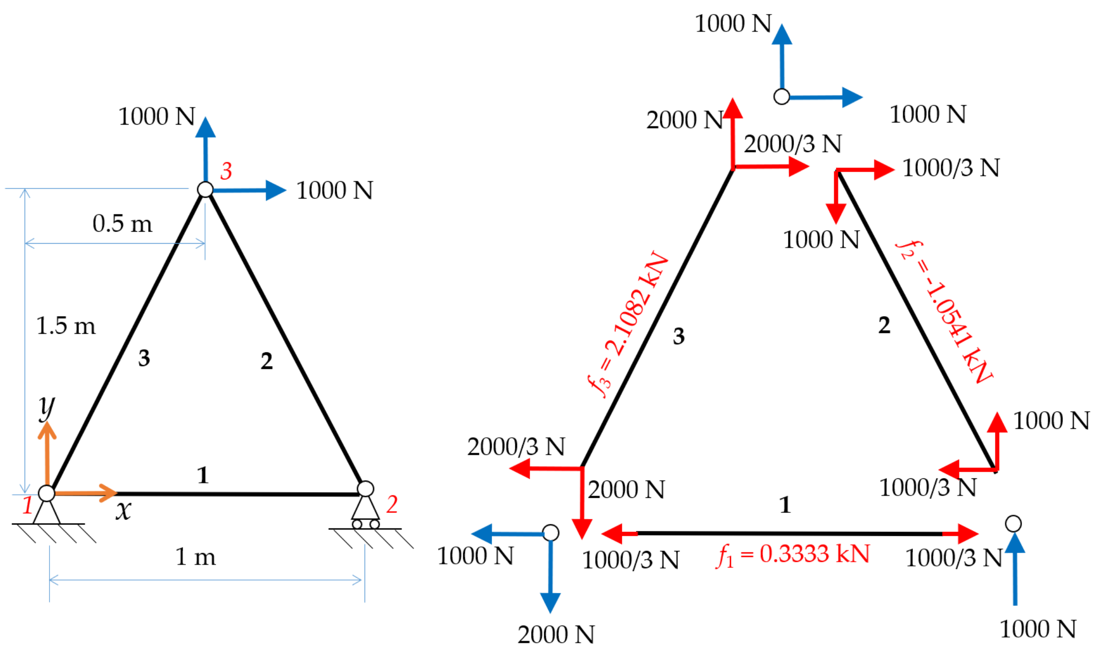

Figure 1. A statically determinate three-bar truss and its free-body diagram

The lengths and forces in the three members are as follows: $l_1 = 1\ m, l_2 = 1.5811\ m, l_3 = 1.5811\ m;$ $f_1 = 0.3333\ kN, f_2 = -1.0541\ kN, f_3 = 2.1082\ kN$. Now, the strain energy (SE) of the truss can be written as

$$SE = \sum_{i=1}^{3} \frac{f_i^2 l_i}{2A_i E} \tag{1}$$

The volume is given by

$$V = \sum_{i=1}^{3} A_i l_i \tag{2}$$

We can pose an optimization problem to minimize SE subject to the volume constraint and by using the areas of cross-section as the optimization variables.

$$\begin{aligned} &\underset{\{A_1,A_2,A_3\}}{\text{Min}} \; SE = \sum_{i=1}^{3} \frac{f_i^2 l_i}{2A_i E} \\ &\text{Subject to} \;\; \sum_{i=1}^{3} A_i l_i - V^* = 0 \end{aligned} \tag{3}$$

This is a constrained minimization problem. But in this simple three-variable problem with a linear constraint, we can eliminate one variable, say A_3, and make it an unconstrained minimization problem.

$$A_3 = (V^* - A_1 l_1 - A_2 l_2)/l_3 \tag{4}$$

$$\underset{\{A_1,A_2\}}{\text{Min}} \; SE = \sum_{i=1}^{3} \frac{f_i^2 l_i}{2A_i E} = \frac{f_1^2 l_1}{2A_1 E} + \frac{f_2^2 l_2}{2A_2 E} + \frac{f_3^2 l_3^2}{2(V^* - A_1 l_1 - A_2 l_2)E} \tag{5}$$

The necessary conditions for this minimization problem are $\frac{\partial SE}{\partial A_1} = 0$ and $\frac{\partial SE}{\partial A_2} = 0$. Thus, we have

$$\frac{\partial SE}{\partial A_1} = -\frac{f_1^2 l_1}{2A_1^2 E} + \frac{f_3^2 l_3^2 l_1}{2(V^* - A_1 l_1 - A_2 l_2)^2 E} = 0 \Rightarrow \frac{f_1^2}{A_1^2} = \frac{f_3^2 l_3^2}{(V^* - A_1 l_1 - A_2 l_2)^2} \Rightarrow \frac{|f_1|}{A_1} = \frac{|f_3| l_3}{(V^* - A_1 l_1 - A_2 l_2)} \tag{6a}$$

and

$$\frac{\partial SE}{\partial A_2} = -\frac{f_2^2 l_2}{2A_2^2 E} + \frac{f_3^2 l_3^2 l_2}{2(V^* - A_1 l_1 - A_2 l_2)^2 E} = 0 \Rightarrow \frac{f_2^2}{A_2^2} = \frac{f_3^2 l_3^2}{(V^* - A_1 l_1 - A_2 l_2)^2} \Rightarrow \frac{|f_2|}{A_2} = \frac{|f_3| l_3}{(V^* - A_1 l_1 - A_2 l_2)} \tag{6a}$$

From Eqs. (6a-6b), we have

$$\frac{|f_1|}{A_1} = \frac{|f_2|}{A_2} = \frac{|f_3| l_3}{(V^* - A_1 l_1 - A_2 l_2)} \tag{7}$$

From the preceding equation, we observe that the stresses in members 1 and 2 are equal. We will come to this important point later after we find the stress in member 3 also. For now, let us proceed to find A_1 using Eq. (6a) by substituting $A_2 = \frac{|f_2|}{|f_1|} A_1$ from Eq. (7).

$$\frac{|f_1|}{A_1} = \frac{|f_3| l_3}{\left(V^* - A_1 l_1 - \frac{f_2}{f_1} A_1 l_2\right)} \Rightarrow |f_1| \left(V^* - A_1 l_1 - \left|\frac{f_2}{f_1}\right| A_1 l_2\right) = A_1 |f_3| l_3$$

$$\Rightarrow A_1 = \frac{|f_1| V^*}{(|f_1| l_1 + |f_2| l_2 + |f_3| l_3)} = 2.6014\text{E-}5 \text{ m}^2 \tag{8a}$$

where we have taken V^* = 4.1623E-4 m³. Similarly, by using Eq. (6b) and $A_1 = \frac{|f_1|}{|f_2|} A_2$ from Eq. (7), we get

$$A_2 = \frac{f_2 V^*}{(|f_1|l_1 + |f_2|l_2 + |f_3|l_3)} = 8.2264\text{E-5 m}^2 \tag{8b}$$

Now, from Eq. (4), we get

$$A_3 = (V^* - A_1 l_1 - A_2 l_2)/l_3 \Rightarrow A_3 = \frac{|f_3| V^*}{(|f_1|l_1 + |f_2|l_2 + |f_3|l_3)} = 1.6453\text{E-4 m}^2 \tag{8c}$$

From the now known values of areas of cross-section, we can find that minimum SE is 0.4881 J. Furthermore, from Eqs. (8a) to (8c), we note that

$$\frac{|f_1|}{A_1} = \frac{|f_2|}{A_2} = \frac{|f_3|}{A_3} = \frac{(|f_1|l_1 + |f_2|l_2 + |f_3|l_3)}{V^*} = 1.2813\text{E7 Pa} \tag{9}$$

We notice that the stresses in all three members are the same. Thus, the stiffest truss for a given volume has the feature that each member is stressed to the same extent. And, this uniform stress depends only on the given data: the volume, member forces, and member lengths.

Let us define an optimal constant, $C^* = (|f_1|l_1 + |f_2|l_2 + |f_3|l_3)$, which in this case is equal to 5.3333E3 J. This constant divided by the volume gives the uniform stress in the truss members. That is, from Eq. (9) we infer that

$$\frac{|f_1|}{A_1} = \frac{|f_2|}{A_2} = \frac{|f_3|}{A_3} = \frac{(|f_1|l_1 + |f_2|l_2 + |f_3|l_3)}{V^*} = \frac{C^*}{V^*} = \frac{5.3333E3}{4.1623E-4} = 1.2813\text{E7 Pa} \tag{10}$$

An important point to note here is that optimal areas of cross-section of a statically determinate truss are readily determined with the help of optimal constant for a given volume.

$$A_i^* = \frac{|f_i|\, V^*}{C^*} \tag{11}$$

wherein we note that it is customary to use the superscript * to indicate optimal quantities.

It is also pertinent to note that if we solve the constrained minimization problem (Eq. (3)) directly, the optimal constant falls out of that calculation as the Lagrange multiplier. We will discuss the concept of Lagrange multiplier and Lagrangian later, but for now consider the solution of the problem in Eq. (3).

$$\left.\begin{array}{l} \underset{\{A_1, A_2, A_3\}}{\text{Min}} \; SE = \sum_{i=1}^{3} \frac{f_i^2 l_i}{2 A_i E} \\ \text{Subject to } \; \sum_{i=1}^{3} A_i l_i - V^* = 0 \\ \quad Data: l_i, f_i, V^*, E \end{array}\right. \Rightarrow L = \text{Lagrangian} = \sum_{i=1}^{3} \frac{f_i^2 l_i}{2 A_i E} + \lambda\left(\sum_{i=1}^{3} A_i l_i - V^*\right) \tag{12a}$$

Now, the necessary condition for the minimum of the constrained minimization problem is

$$\frac{\partial L}{\partial A_i} = -\frac{f_i^2 l_i}{2 A_i^2 E} + \lambda l_i = 0 \Rightarrow \frac{|f_i|}{A_i} = \sqrt{2E\lambda} \tag{13}$$

By substituting A_i into the volume constraint, we can determine λ.

$$\sum_{i=1}^{3} A_i l_i - V^* = 0 \Rightarrow \sum_{i=1}^{3} \frac{|f_i|}{\sqrt{2E\lambda}} l_i - V^* = 0 \Rightarrow \lambda = 1.1726\text{E}3 \text{ J/m}^3 \tag{14}$$

From Eqs. (10) and (13), we also see that

$$\frac{|f_i|}{A_i} = \frac{C^*}{V^*} = \sqrt{2E\lambda} \tag{15}$$

Thus, the optimal constant and the Lagrange multiplier are related in stiff-light trusses that are statically determinate.

There is more intrigue here. There is a famous Maxwell's theorem for statically determinate trusses. Maxwell had stated the following theorem for a statically determinate truss by accounting for the sign of the member forces.

> For a statically determinate truss, $\sum_t |f_i| l_i - \sum_c |f_j| l_j = C$ where the constant C depends on the loads (including reactions at the supports) and their points of application and not on the form of the truss. The first sum is taken for all members in tension and the second sum is taken over all members in compression.
>
> (*Maxwell, J. C., Scientific Papers, Vol. II, (1864), pp 175-177*)

Here, we see that Maxwell's constant, $C = (f_1 l_1 + f_2 l_2 + f_3 l_3)$, keeps the signs of the forces unlike the optimal constant defined earlier. The Maxwell's constant for this case is 2000 Nm. As stated in the aforementioned theorem, this constant does not depend on the form of the truss provided that the forces and points of their application (including reaction forces) do not change. A. G. M. Michell, a great Australian engineer, had used this theorem as a starting point to derive optimal topologies [Michell, A.G.M., "The limits of economy of material in frame-structures," The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 8(47), (1904), pp.589-597]. It will be an interesting exercise to create some other statically determinate trusses and verify Maxwell's theorem. It is instructive to note that C = 2000 Nm in this problem because this is sum of the dot products of the all the external (applied and reaction) forces with the position vectors from the fixed vertex. That is,

$\boldsymbol{p}_1 = \boldsymbol{0}$ because the position vector from vertex 1 to itself is a zero vector

$\boldsymbol{p}_2 = \hat{\imath}$

$\boldsymbol{p}_3 = 0.5\,\hat{\imath} + 1.5\hat{\jmath}$

$(-1000\,\hat{\imath} - 2000\,\hat{\jmath}) \cdot \boldsymbol{p}_1 + (1000\,\hat{\jmath}) \cdot \boldsymbol{p}_2 + (1000\,\hat{\imath} + 1000\,\hat{\jmath}) \cdot \boldsymbol{p}_3$

= 0 + 0 + 2000 = 2000 Nm.

We have illustrated a few important concepts using a simple three-bar statically determinate truss. These concepts hold true for trusses of any size if they are statically determinate. Some of these concepts extend to any truss, frame, or continuum structures (e.g., uniformly stressed designs for stiffest structures for a given volume). This underscores an important point that optimization is not merely a numerical search or an algebraic manipulation; it is an exercise that provides much insight and makes finding an optimum straightforward once we understand the underlying optimality. Here, we saw how to find optimal areas of cross-section without having to do numerical optimization for any general statically determinate truss.

Now, try alternative topologies and shapes and do size optimization to see if we can get a better design for the same volume of material (V^*= 4.1623E-4 m^3) with strain energy lower than 0.4881 J, and without disturbing the fixed pin support, the point where the load is applied, the values of the loads, the material property, but you can locate the roller support anywhere on the x-axis. If you are adventurous, you may also consider statically indeterminate trusses that do not satisfy Maxwell's static determinacy condition:

$$2v - b - 3 = 0 \tag{16}$$

where v = number of vertices and b = number of bars (i.e., the truss members).