Lecture 28a

Topological derivatives
Theory, Method and Application

ME 260 at the Indian Institute of Science, Bengaluru

Structural Optimization: Size, Shape, and Topology

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Outline of the lecture

Introduction to shape and topological derivatives, and the limiting relationship between them.

Topological derivative method in isotropic linear elasticity.

Topological derivative-based topology optimization (a brief overview).

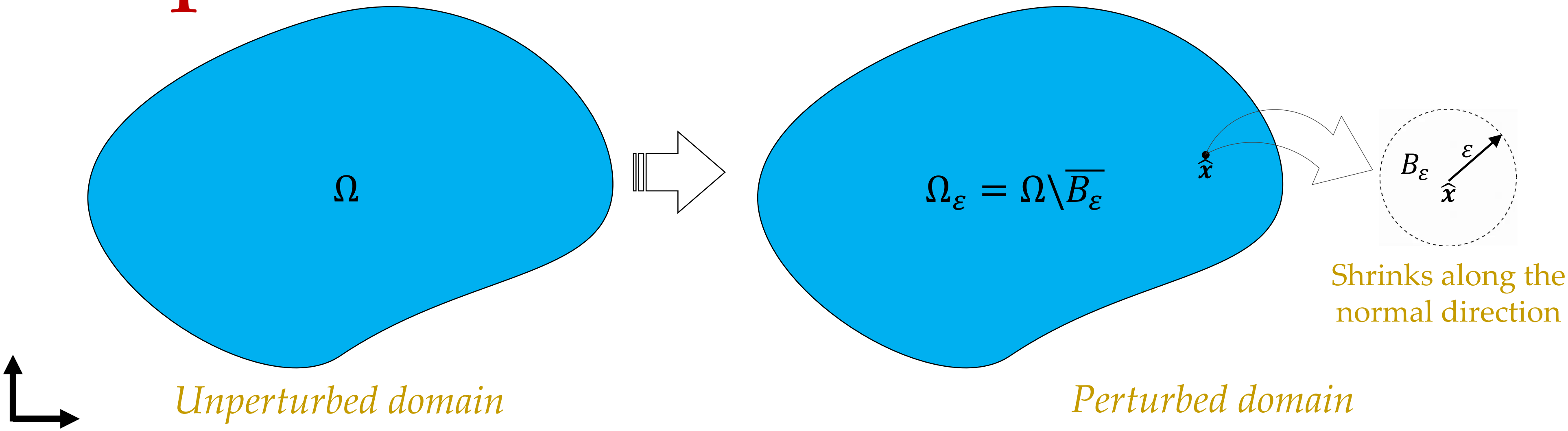
What we will learn:

What is topological derivative and how it is related to shape sensitivity.

How to derive topological derivative for a given performance functional using asymptotic analysis.

Visualization of topological derivative-based optimization approach.

Shape derivative



Let $\psi(\cdot)$ be the performance functional associated with the domains, then

$$\frac{d\psi(\Omega_\varepsilon)}{d\varepsilon} = \lim_{\delta\varepsilon \to 0} \frac{\psi(\Omega_{\varepsilon+\delta\varepsilon}) - \psi(\Omega_\varepsilon)}{\delta\varepsilon}$$

Sokolowski, J. and Zolesio, J.P., Springer, 1992.

$\delta\varepsilon$ is the perturbation of the shape parameter, ε.

Shape derivative determines the sensitivity of the performance functional with respect to perturbation of the boundary of the domain.

Topological derivative

Topological asymptotic expansion: $\psi(\Omega_\varepsilon) = \psi(\Omega) + f(\varepsilon)TD(\hat{\boldsymbol{x}}) + o(f(\varepsilon))$

Here, $f(\varepsilon)TD(\hat{\boldsymbol{x}})$ is the first order term with two parts; (i) $TD(\hat{\boldsymbol{x}})$ is the topological derivative; and (ii) $f(\varepsilon)$ is the positive correction factor such that when $\varepsilon \to 0, f(\varepsilon) = 0$.

Topological derivative:

$$TD(\hat{\boldsymbol{x}}) = \lim_{\varepsilon \to 0} \frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f(\varepsilon)}$$

as $\lim_{\varepsilon \to 0} \frac{o(f(\varepsilon))}{f(\varepsilon)} = 0$

Novotny, A.A. and Sokolowski, J., Springer, 2013.

Differentiating $\psi(\Omega_\varepsilon)$ with respect to ε:

$$TD(\hat{\boldsymbol{x}}) = \lim_{\varepsilon \to 0} \left[\frac{1}{f'(\varepsilon)} \frac{d\psi(\Omega_\varepsilon)}{d\varepsilon} \right]$$

Sokolowski, J. and Zochowski, A.,
SIAM Journal on Control and Optimization, 1999.

Topological derivative is obtained from the topological asymptotic expansion of the performance functional.
Topological derivative is the limiting value of the shape derivative.

A simple example

Let us choose an unperturbed performance functional: $\psi(\Omega) = \int_{\Omega} \phi(\boldsymbol{x})\ d\Omega$

Performance functional associated with the perturbed domain: $\psi(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} \phi(\boldsymbol{x})\ d\Omega_\varepsilon$

Topological asymptotic expansion:

$$
\begin{aligned}
\psi(\Omega_\varepsilon) &= \int_{\Omega_\varepsilon} \phi(\boldsymbol{x})\ d\Omega_\varepsilon + \int_{\mathrm{B}_\varepsilon} \phi(\boldsymbol{x})\ d\mathrm{B}_\varepsilon - \int_{\mathrm{B}_\varepsilon} \phi(\boldsymbol{x})\ d\mathrm{B}_\varepsilon \\
&= \int_{\Omega} \phi(\boldsymbol{x})\ d\Omega - \int_{\mathrm{B}_\varepsilon} \phi(\boldsymbol{x})\ d\mathrm{B}_\varepsilon \\
&= \psi(\Omega) - \int_{\mathrm{B}_\varepsilon} \phi(\boldsymbol{x})\ d\mathrm{B}_\varepsilon
\end{aligned}
$$

$$
\psi(\Omega_\varepsilon) = \psi(\Omega) - \underbrace{|B_\varepsilon|\phi(\hat{\boldsymbol{x}})} + o(B_\varepsilon)
$$

$|B_\varepsilon|$ is the area measure of B_ε

$-\phi(\hat{\boldsymbol{x}})$ is the topological derivative

Topological derivative is obtained from the first-order term in the topological asymptotic expansion of the performance functional.

Steps in the TD evaluation

Step 1: Write the strong and weak forms of the governing equation and define the performance functional.

Step 2: Write the set of equations for the perturbed domain.

Step 3: Evaluate the shape derivative using adjoint analysis. The shape derivative turns out to be a surface integral on the inclusion.

Step 4: Perform asymptotic analysis of the perturbed solution to evaluate the shape derivative in closed form.

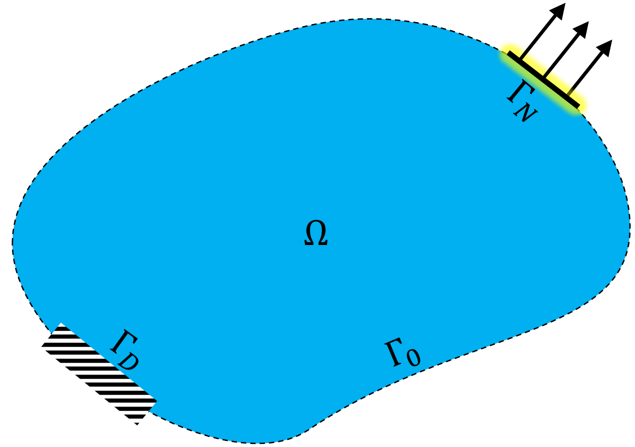
Step 5: By using the limiting relationship between shape and topological derivatives, obtain closed-form topological derivative expression.

Step 6: Using the limiting values of the contrast parameter, evaluate topological derivatives for creating voids, and adding back material.

TD in linear elasticity

Strong form of the governing equation and boundary conditions:

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \mathbf{0} & in\ \Omega \\ \boldsymbol{u} = \mathbf{0} & on\ \Gamma_D \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F} & on\ \Gamma_N \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{0} & on\ \Gamma_0 \end{cases}$$



Constitutive relation: $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u})$

Strain as symmetric gradient: $\boldsymbol{\epsilon}(\boldsymbol{u}) = \nabla^s \boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$

Weak form of the governing equation and boundary conditions:

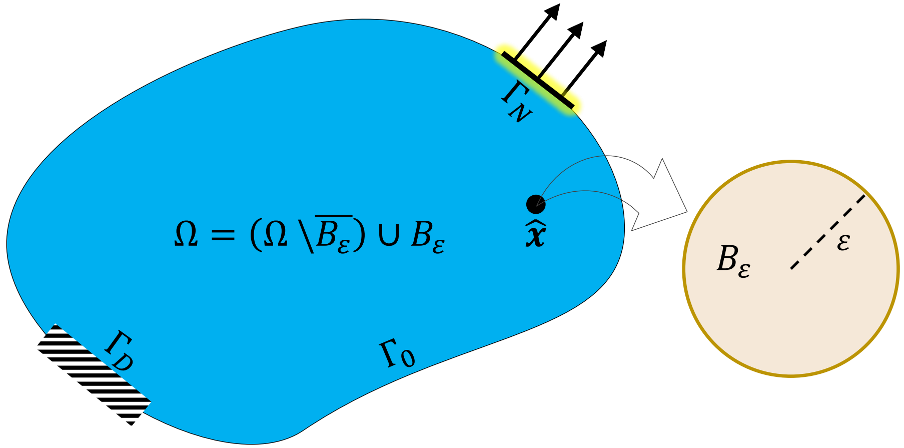
$$\boldsymbol{u} \in \mathcal{U}: \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{\epsilon}(\boldsymbol{\eta}) = \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{\eta}, \qquad \boldsymbol{\eta} \in \mathcal{V}$$

Mean compliance as the performance functional: $\psi(\Omega) = \psi_\Omega(\boldsymbol{u}) := \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{u}$

Notation

- $\boldsymbol{u}$: unperturbed displacement
- $\boldsymbol{\eta}$: variation of $\boldsymbol{u}$
- $\boldsymbol{n}$: normal vector
- $\boldsymbol{F}$: prescribed traction vector
- $\boldsymbol{\sigma}$: Cauchy stress tensor
- $\boldsymbol{\epsilon}$: linear strain tensor
- $\mathbb{C}$: fourth-order constitutive tensor
- $\mathcal{U}$: solution space
- $\mathcal{V}$: space of admissible variations

Perturbed domain



Strong form: $$\begin{cases} \nabla \cdot \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) = \boldsymbol{0} & in\ \Omega \\ \boldsymbol{u}_\varepsilon = \boldsymbol{0} & on\ \Gamma_D \\ \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)\boldsymbol{n} = \boldsymbol{F} & on\ \Gamma_N \\ \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)\boldsymbol{n} = \boldsymbol{0} & on\ \Gamma_0 \\ [\![\boldsymbol{u}_\varepsilon]\!] = \boldsymbol{0} & on\ \partial \mathrm{B}_\varepsilon \\ [\![\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)]\!]\boldsymbol{n} = \boldsymbol{0} & on\ \partial \mathrm{B}_\varepsilon \end{cases}$$

Weak form $\boldsymbol{u}_\varepsilon \in \mathcal{U}_\varepsilon$: $\int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{\eta}_\varepsilon) = \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{\eta}_\varepsilon, \quad \boldsymbol{\eta}_\varepsilon \in \mathcal{V}_\varepsilon$

Constitutive relation: $\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) = \mathbb{C}_\varepsilon(\boldsymbol{x})\boldsymbol{\epsilon}(\boldsymbol{u}_\varepsilon)$

$$\mathbb{C}_\varepsilon(\boldsymbol{x}) = \begin{cases} \mathbb{C} & if\ \boldsymbol{x} \in \Omega \setminus \overline{B_\varepsilon} \\ \gamma\mathbb{C} & if\ \boldsymbol{x} \in B_\varepsilon \end{cases}$$

Transmission condition: $[\![\cdot]\!] = (\cdot)_{\Omega\setminus\overline{B_\varepsilon}} - (\cdot)_{B_\varepsilon}$

Perturbed performance functional: $\psi_\varepsilon(\Omega) = \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{u}_\varepsilon$

Notation

- $\boldsymbol{u}_\varepsilon$: perturbed displacement
- $\boldsymbol{\eta}_\varepsilon$: variation of $\boldsymbol{u}_\varepsilon$
- $\boldsymbol{\sigma}_\varepsilon$: perturbed Cauchy stress tensor
- γ: contrast parameter
- $\mathcal{U}_\varepsilon$: space of perturbed solution
- $\mathcal{V}_\varepsilon$: space of perturbed variation

Adjoint analysis

We use adjoint method, where we add the adjoint weak form to the functional

$$\psi_\varepsilon(\Omega) = \underbrace{\int_{\Gamma_N} \boldsymbol{F}\cdot\boldsymbol{u}_\varepsilon}_{W_1} + \underbrace{\int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{\epsilon}(\boldsymbol{\lambda}_\varepsilon) - \int_{\Gamma_N} \boldsymbol{F}\cdot\boldsymbol{\lambda}_\varepsilon}_{W_2}$$

Here, $\boldsymbol{\lambda}_\varepsilon$ is the adjoint displacement. Next, we evaluate the sensitivity of W_1 and W_2, individually, and further add to determine the sensitivity of the functional $\psi_\varepsilon(\Omega)$. Here, we use ***Reynolds transport theorem*** that states:

$$\frac{d}{d\varepsilon}\int_\Omega \phi = \int_\Omega \phi' + \int_{\partial\Omega} \phi(\boldsymbol{v}\cdot\boldsymbol{n})$$

$$= \int_\Omega (\dot{\phi} - \nabla\phi\cdot\boldsymbol{v}) + \int_{\partial(\Omega\setminus\overline{B_\varepsilon})} \phi(\boldsymbol{v}\cdot\boldsymbol{n}) + \int_{\partial B_\varepsilon} [\![\phi]\!](\boldsymbol{v}\cdot\boldsymbol{n})$$

Spatial derivative: $\phi' = {}^{d\phi}/_{d\boldsymbol{x}}$

Material derivative: $\dot{\phi} = {}^{d\phi}/_{d\varepsilon}$

Relation: $\phi' = \dot{\phi} - \nabla\phi\cdot\boldsymbol{v}$

Shape sensitivities of W_1 and W_2

In this slide, we evaluate the shape sensitivities of W_1 and W_2

$$W_1 = \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{u}_\varepsilon$$

$$\frac{dW_1}{d\varepsilon} = \int_{\Gamma_N} \boldsymbol{F} \cdot \dot{\boldsymbol{u}}_\varepsilon$$

The boundary term does not contribute as the velocity at the external boundary Γ_N is **0.**

$$W_2 = \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{\lambda}_\varepsilon) - \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{\lambda}_\varepsilon$$

$$\frac{dW_2}{d\varepsilon} = \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon) \cdot \nabla^s \boldsymbol{u}'_\varepsilon + \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \nabla^s \boldsymbol{\lambda}'_\varepsilon - \int_{\Gamma_N} \boldsymbol{F} \cdot \boldsymbol{\lambda}'_\varepsilon + \int_{\partial\Omega} \big(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{\lambda}_\varepsilon)\big)\, \boldsymbol{v} \cdot \boldsymbol{n}$$

Converting spatial derivatives to the material derivatives will get

$$\frac{dW_2}{d\varepsilon} = \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon) \cdot \nabla^s \dot{\boldsymbol{u}}_\varepsilon + \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \nabla^s \dot{\boldsymbol{\lambda}}_\varepsilon - \int_{\Gamma_N} \boldsymbol{F} \cdot \dot{\boldsymbol{\lambda}}_\varepsilon - \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon) \cdot \nabla(\nabla \boldsymbol{u}_\varepsilon \boldsymbol{v})$$
$$- \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \nabla(\nabla \boldsymbol{\lambda}_\varepsilon \boldsymbol{v}) + \int_{\partial\Omega} \big(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{\lambda}_\varepsilon)\big)\, \boldsymbol{v} \cdot \boldsymbol{n}$$

By using Gauss divergence theorem to the terms in red, we obtain

$$\frac{dW_2}{d\varepsilon} = \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon) \cdot \nabla^s \dot{\boldsymbol{u}}_\varepsilon + \int_\Omega (\nabla \cdot \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon)) \cdot (\nabla \boldsymbol{u}_\varepsilon \boldsymbol{v}) - \int_{\partial\Omega} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon)\boldsymbol{n} \cdot (\nabla \boldsymbol{u}_\varepsilon \boldsymbol{v})$$
$$- \int_{\partial\Omega} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)\boldsymbol{n} \cdot (\nabla \boldsymbol{\lambda}_\varepsilon \boldsymbol{v}) + \int_{\partial\Omega} \big(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{\lambda}_\varepsilon)\big)\, \boldsymbol{v} \cdot \boldsymbol{n}$$

Shape derivative evaluation

$$\frac{d\psi_\varepsilon}{d\varepsilon} = \frac{dW_1}{d\varepsilon} + \frac{dW_2}{d\varepsilon}$$

$$\frac{d\psi_\varepsilon}{d\varepsilon} = \int_{\Gamma_N} \boldsymbol{F} \cdot \dot{\boldsymbol{u}}_\varepsilon + \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon) \cdot \nabla^s \dot{\boldsymbol{u}}_\varepsilon + \int_\Omega (\nabla \cdot \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon)) \cdot (\nabla \boldsymbol{u}_\varepsilon \boldsymbol{v}) - \int_{\partial\Omega} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon)\boldsymbol{n} \cdot (\nabla \boldsymbol{u}_\varepsilon \boldsymbol{v})$$
$$- \int_{\partial\Omega} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)\boldsymbol{n} \cdot (\nabla \boldsymbol{\lambda}_\varepsilon \boldsymbol{v}) + \int_{\partial\Omega} \big(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{\lambda}_\varepsilon)\big)\, \boldsymbol{v} \cdot \boldsymbol{n}$$

Next, we isolate the terms involving $\dot{\boldsymbol{u}}_\varepsilon$ and equate it to 0, to get the weak form for solving adjoint variable, i.e.,

$$\int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\lambda}_\varepsilon) \cdot \nabla^s \dot{\boldsymbol{u}}_\varepsilon = -\int_{\Gamma_N} \boldsymbol{F} \cdot \dot{\boldsymbol{u}}_\varepsilon$$

Thus, it is observed that $\boldsymbol{\lambda}_\varepsilon = -\boldsymbol{u}_\varepsilon$. Therefore, the shape derivative evaluates to

$$\frac{d\psi_\varepsilon}{d\varepsilon} = 2\int_{\partial\Omega} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)\boldsymbol{n} \cdot (\nabla \boldsymbol{u}_\varepsilon \boldsymbol{v}) - \int_{\partial\Omega} \big(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{u}_\varepsilon)\big)\, \boldsymbol{v} \cdot \boldsymbol{n}$$

Also, we have $\partial\Omega = \partial(\Omega \backslash \overline{B_\varepsilon}) \cup \partial B_\varepsilon$, $\boldsymbol{v} = \begin{cases} \boldsymbol{0} & on\ \partial(\Omega\backslash\overline{B_\varepsilon}) \\ -\boldsymbol{n} & on\ \partial B_\varepsilon \end{cases}$, and $[\![\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)]\!]\boldsymbol{n} = \boldsymbol{0}\quad on\ \ \partial B_\varepsilon$.

Hence, the shape derivative is given by: $\quad \dfrac{d\psi_\varepsilon}{d\varepsilon} = \displaystyle\int_{\partial B_\varepsilon} [\![\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \cdot \boldsymbol{\epsilon}(\boldsymbol{u}_\varepsilon)]\!]$

Shape derivative is an integral on the inclusion boundary.

Asymptotic analysis

In order to solve the boundary integral in the shape derivative expression, asymptotic expansion of the perturbed solution in the vicinity of the inclusion is analyzed. The ansätz is proposed as:

$$\boldsymbol{u}_\varepsilon(\boldsymbol{x}) = \underbrace{\boldsymbol{u}(\boldsymbol{x})}_{\text{Unperturbed solution}} + \underbrace{\boldsymbol{w}_\varepsilon(\boldsymbol{x})}_{\text{First-order boundary term}} + \underbrace{\tilde{\boldsymbol{u}}_\varepsilon(\boldsymbol{x})}_{\text{Remainder term}}$$

Applying the stress operator: $\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon(\boldsymbol{x})) = \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}(\boldsymbol{x})) + \boldsymbol{\sigma}_\varepsilon(\boldsymbol{w}_\varepsilon(\boldsymbol{x})) + \boldsymbol{\sigma}_\varepsilon(\tilde{\boldsymbol{u}}_\varepsilon(\boldsymbol{x}))$

Using Taylor series expansion: $\boldsymbol{\sigma}_\varepsilon\big(\boldsymbol{u}_\varepsilon(\boldsymbol{x})\big) = \boldsymbol{\sigma}_\varepsilon\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big) + \nabla\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}(\boldsymbol{y}))(\boldsymbol{x} - \hat{\boldsymbol{x}}) + \boldsymbol{\sigma}_\varepsilon(\boldsymbol{w}_\varepsilon(\boldsymbol{x})) + \boldsymbol{\sigma}_\varepsilon(\tilde{\boldsymbol{u}}_\varepsilon(\boldsymbol{x}))$

Here, $\boldsymbol{y}$ is the intermediate point between $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$. Next, the asymptotic expansion is substituted in the governing equation and boundary conditions of the perturbed domain to obtain boundary value problem for solving $\boldsymbol{w}_\varepsilon$. On the boundary of the inclusion, we have

$$[\![\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon(\boldsymbol{x}))]\!]\boldsymbol{n} = \boldsymbol{0} \quad \Rightarrow \left(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon(\boldsymbol{x}))_{\Omega\setminus\overline{B_\varepsilon}} - \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon(\boldsymbol{x}))_{B_\varepsilon}\right)\boldsymbol{n} = \boldsymbol{0} \quad on\ \partial B_\varepsilon$$

$$\Rightarrow (1-\gamma)\boldsymbol{\sigma}_\varepsilon\left(\boldsymbol{u}(\hat{\boldsymbol{x}})\right)\boldsymbol{n} - \varepsilon(1-\gamma)(\nabla\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}(\boldsymbol{y}))\boldsymbol{n})\boldsymbol{n} + [\![\boldsymbol{\sigma}_\varepsilon(\boldsymbol{w}_\varepsilon(\boldsymbol{x}))]\!]\boldsymbol{n} + [\![\boldsymbol{\sigma}_\varepsilon(\tilde{\boldsymbol{u}}_\varepsilon(\boldsymbol{x}))]\!]\boldsymbol{n} = \boldsymbol{0}$$

Thus, the boundary value problem for solving $\boldsymbol{w}_\varepsilon$ is:

$$\begin{cases} \nabla\cdot\boldsymbol{\sigma}_\varepsilon(\boldsymbol{w}_\varepsilon(\boldsymbol{x})) = \boldsymbol{0} \quad in\ \mathbb{R}^2 \\ \boldsymbol{w}_\varepsilon(\boldsymbol{x}) \to \boldsymbol{0} \quad at\ \boldsymbol{x}\to\infty \\ [\![\boldsymbol{\sigma}_\varepsilon(\boldsymbol{w}_\varepsilon(\boldsymbol{x}))]\!]\boldsymbol{n} = -(1-\gamma)\boldsymbol{\sigma}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big)\boldsymbol{n} \quad on\ \partial B_\varepsilon \end{cases}$$

Asymptotic analysis (Cont.)

For $r \geq \varepsilon$
(outside the inclusion)

$$\sigma_\varepsilon^{rr}(\boldsymbol{u}_\varepsilon(r,\theta)) = \phi_1\left(1 - \frac{1-\gamma}{1+\gamma\alpha}\frac{\varepsilon^2}{r^2}\right) + \phi_2\left(1 - 4\frac{1-\gamma}{1+\gamma\beta}\frac{\varepsilon^2}{r^2} + 3\frac{1-\gamma}{1+\gamma\beta}\frac{\varepsilon^4}{r^4}\right)\cos 2\theta + O(\varepsilon^2)$$

$$\sigma_\varepsilon^{\theta\theta}(\boldsymbol{u}_\varepsilon(r,\theta)) = \phi_1\left(1 + \frac{1-\gamma}{1+\gamma\alpha}\frac{\varepsilon^2}{r^2}\right) - \phi_2\left(1 + 3\frac{1-\gamma}{1+\gamma\beta}\frac{\varepsilon^4}{r^4}\right)\cos 2\theta + O(\varepsilon^2)$$

$$\sigma_\varepsilon^{r\theta}(\boldsymbol{u}_\varepsilon(r,\theta)) = -\phi_2\left(1 + 2\frac{1-\gamma}{1+\gamma\beta}\frac{\varepsilon^2}{r^2} - 3\frac{1-\gamma}{1+\gamma\beta}\frac{\varepsilon^4}{r^4}\right)\sin 2\theta + O(\varepsilon^2)$$

For $0 > r < \varepsilon$
(inside the inclusion)

$$\sigma_\varepsilon^{rr}(\boldsymbol{u}_\varepsilon(r,\theta)) = \phi_1\left(\frac{2}{1-\nu}\frac{\gamma}{1+\gamma\alpha}\right) + \phi_2\left(\frac{4}{1+\nu}\frac{\gamma}{1+\gamma\beta}\right)\cos 2\theta + O(\varepsilon^2)$$

$$\sigma_\varepsilon^{\theta\theta}(\boldsymbol{u}_\varepsilon(r,\theta)) = \phi_1\left(\frac{2}{1-\nu}\frac{\gamma}{1+\gamma\alpha}\right) - \phi_2\left(\frac{4}{1+\nu}\frac{\gamma}{1+\gamma\beta}\right)\cos 2\theta + O(\varepsilon^2)$$

$$\sigma_\varepsilon^{r\theta}(\boldsymbol{u}_\varepsilon(r,\theta)) = -\phi_2\left(\frac{4}{1+\nu}\frac{\gamma}{1+\gamma\beta}\right)\sin 2\theta + O(\varepsilon^2)$$

$$\phi_1 = \frac{1}{2}(\Lambda_1 - \Lambda_2) \qquad \phi_2 = \frac{1}{2}(\Lambda_1 - \Lambda_2)$$

Eigenvalues: $$\Lambda_{1,2} = \frac{1}{2}\left((tr(\boldsymbol{\sigma}(\boldsymbol{u}(\hat{\boldsymbol{x}}))) \pm \sqrt{2\boldsymbol{\sigma}_D(\boldsymbol{u}(\hat{\boldsymbol{x}})) \cdot \boldsymbol{\sigma}_D(\boldsymbol{u}(\hat{\boldsymbol{x}}))}\right)$$

Deviatoric stress: $$\boldsymbol{\sigma}_D(\boldsymbol{u}(\hat{\boldsymbol{x}})) = \boldsymbol{\sigma}(\boldsymbol{u}(\hat{\boldsymbol{x}})) - \frac{1}{2}tr(\boldsymbol{\sigma}(\boldsymbol{u}(\hat{\boldsymbol{x}})))\boldsymbol{I}$$

Kozlov, V.A., Mazya, V.G. and Movchan, A.B. Clarendon Press Oxford, 1999.
Novotny, A.A. and Sokolowski, J., Springer, 2013.

Eshelby and Polarization tensors

Eshelby's theorem states that the *stresses and strains are uniform inside the inclusion*. The stress tensor on the boundary of the inclusion is mapped to the stress tensor at the center of the inclusion by the following uniform fourth-order linear transformation:

$$\boldsymbol{\sigma}_\varepsilon(\boldsymbol{w}_\varepsilon)|_{B_\varepsilon} = \mathbb{T}_\gamma \boldsymbol{\sigma}(\boldsymbol{u}(\hat{\boldsymbol{x}}))$$

Eshelby, J.D., Royal Society, 1957.

Here, $\mathbb{T}_\gamma$ is the fourth-order Eshelby tensor, which is given by

$$\mathbb{T}_\gamma = \frac{\gamma(1-\gamma)}{2(1+\gamma\alpha_2)}\left[2\alpha_2\mathbb{I} + \frac{(\alpha_1-\alpha_2)}{1+\gamma\alpha_1}\boldsymbol{I}\otimes\boldsymbol{I}\right],$$ where

- $\alpha_1 = \dfrac{1+\nu}{1-\nu}$ and $\alpha_2 = \dfrac{3-\nu}{1+\nu}$.
- $\mathbb{I}$ is fourth-order identity tensor.
- $\boldsymbol{I}$ is second-order identity tensor.

The fourth-order tensor $\mathbb{T}_\gamma$ is presented in its closed form for isotropic constitutive behavior. It maps the stress state inside the inclusion.

This fourth-order tensor $\mathbb{T}_\gamma$ contributes to the **Polarization tensor** $\mathbb{P}_\gamma$ that comes from the solution to the boundary value problem that solves $\boldsymbol{w}_\varepsilon$. The **Polarization tensor** *represents the state of the stress inside the domain due to the presence of the inclusion*, and is given by:

$$\mathbb{P}_\gamma = \frac{1-\gamma}{2\gamma}\left[\gamma\mathbb{I} + \mathbb{T}_\gamma\right]$$

Ammari, H. and Kang, H., Applied Mathematical Sciences, 2007.

Topological derivative evaluation

The topological derivative is given by $TD(\hat{\boldsymbol{x}}) = \mathbb{P}_\gamma \boldsymbol{\sigma}(\boldsymbol{u}(\hat{\boldsymbol{x}})) \cdot \boldsymbol{\epsilon}(\boldsymbol{u}(\hat{\boldsymbol{x}}))$

The fourth-order Polarization tensor plays a central role in the topological derivative expression, and is expressed in the tensorial notation as:

$$\mathbb{P}_\gamma = \frac{1-\gamma}{2(1+\gamma\alpha_2)}\left[(1+\alpha_2)\mathbb{I} + \frac{1-\gamma}{2(1+\gamma\alpha_1)}(\alpha_1-\alpha_2)\boldsymbol{I}\otimes\boldsymbol{I}\right]$$

On further simplification, the closed-form expression of topological derivative is given by:

$$TD(\hat{\boldsymbol{x}}) = \frac{1-\gamma}{2(1+\gamma\alpha_2)}\left[(1+\alpha_2)\boldsymbol{\sigma}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big)\cdot\boldsymbol{\epsilon}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big) + \frac{1-\gamma}{2(1+\gamma\alpha_1)}(\alpha_1-\alpha_2)tr\left(\boldsymbol{\sigma}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big)\right)tr\left(\boldsymbol{\epsilon}\big(\boldsymbol{u}(\hat{\boldsymbol{x}})\big)\right)\right]$$

Giusti, S.M., Novotny, A.A. and Padra, C., Engineering Analysis with Boundary Elements Press Oxford, 2008.

On substituting $\gamma \to 0$, voids are created in the material region. → $TD_{I\to V}(\hat{\boldsymbol{x}}) = \frac{2}{1+\nu}\boldsymbol{\sigma}(\hat{\boldsymbol{x}})\cdot\boldsymbol{\epsilon}(\hat{\boldsymbol{x}}) - \frac{1-3\nu}{2(1-\nu^2)}tr\big(\boldsymbol{\sigma}(\hat{\boldsymbol{x}})\big)tr\big(\boldsymbol{\epsilon}(\hat{\boldsymbol{x}})\big)$

On substituting $\gamma \to \infty$, material is added back in the void region. → $TD_{V\to I}(\hat{\boldsymbol{x}}) = -\frac{2}{3-\nu}\boldsymbol{\sigma}(\hat{\boldsymbol{x}})\cdot\boldsymbol{\epsilon}(\hat{\boldsymbol{x}}) - \frac{1-3\nu}{2(1+\nu)(3-\nu)}tr\big(\boldsymbol{\sigma}(\hat{\boldsymbol{x}})\big)tr\big(\boldsymbol{\epsilon}(\hat{\boldsymbol{x}})\big)$

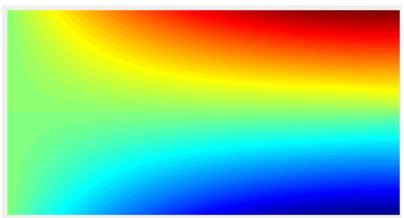
Polarization tensor is operated over the stress tensor in the topological derivative expression.

The two limiting values of the contrast parameter provide topological derivative for interchanging material.

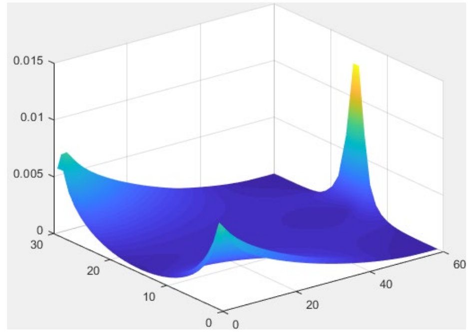
TD-based topology optimization

1 **Initial design** with volume fraction of material $v = 1$.

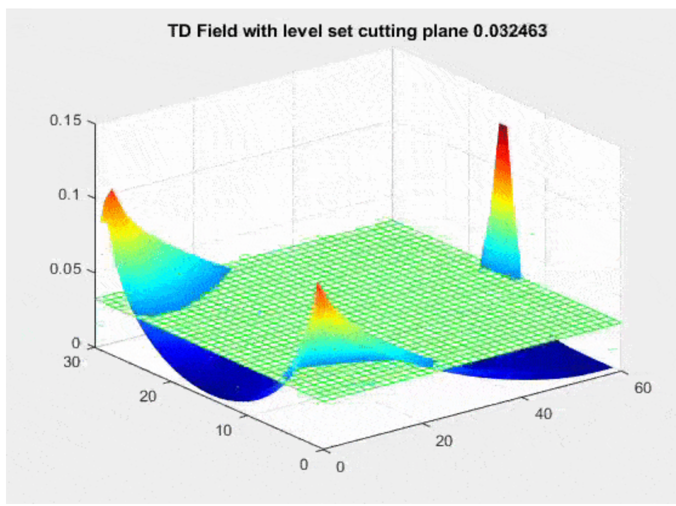
2 Displacements using **Finite Element Analysis.**



3 We analytically derive **topological derivative** and plot the field.

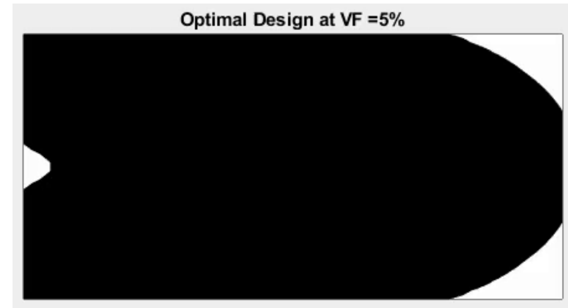


4 **Level-set plane** moves in the topological sensitivity field, such that the volume constraint is satisfied.

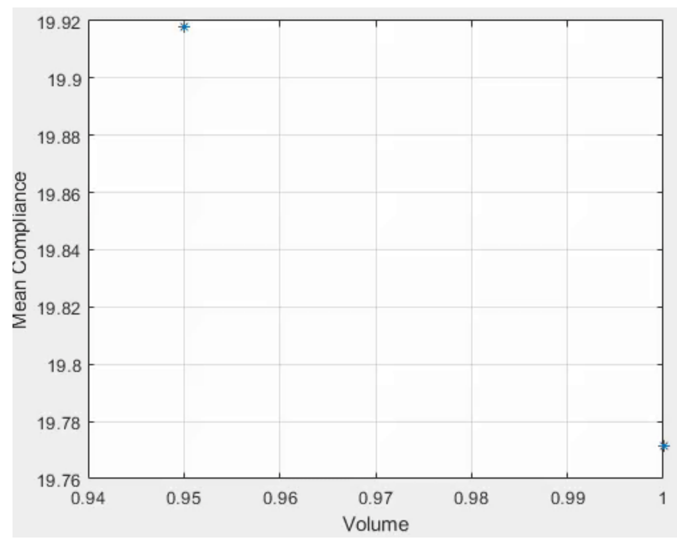


5 **Topology optimization**

■ - Material □ - Void



6 **Pareto front** showing stiffness at all the volume fractions.



The end note

Topological Derivatives
Theory, Method and Application

- The concept of topological and shape derivatives. The relationship between topological sensitivity with the classical shape optimization.
- Obtain the closed-form expression of topological derivative by interpreting topological asymptotic expansion.

  or

- Adjoint analysis to evaluate shape derivative. The final expression of shape derivative turns out to be a surface integral on the boundary of the inclusion.
- Analyze the asymptotes of the perturbed solution to obtain the shape derivative in its closed-form.
- Exploit the limiting relationship to obtain topological derivative.

Thanks