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Understanding vector manipulation in taking variation

First understand this:

$$J = \int_{\Omega} \frac{1}{2}\left(\frac{\partial u_x}{\partial x}\right)^2 d\Omega$$

$$\delta_{u_x} J = \frac{d}{d\varepsilon}\left[\int_{\Omega} \frac{1}{2}\left(\frac{\partial (u_x + \varepsilon \delta u_x)}{\partial x}\right)^2 d\Omega\right]$$

$$= \int_{\Omega} \frac{1}{2}\frac{d}{d\varepsilon}\left(\frac{\partial (u_x + \varepsilon \delta u_x)}{\partial x}\right)^2 d\Omega$$

$$= \int_{\Omega} \frac{1}{2} \cdot 2 \cdot \frac{\partial (u_x + \varepsilon \delta u_x)}{\partial x} \cdot \frac{\partial \delta u_x}{\partial x} \, d\Omega$$

with $\varepsilon = 0$

$$\delta_{u_x} J = \int \frac{\partial u_x}{\partial x} \cdot \frac{\partial \delta u_x}{\partial x} \, d\Omega$$

Now, let us take the compact notation for strain energy and expand it.

$$\bar{\varepsilon} = \begin{Bmatrix} \partial u_x/\partial x \\ \partial u_y/\partial y \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{Bmatrix}$$

$$\frac{1}{2}\bar{\varepsilon}^T \bar{\bar{D}} \bar{\varepsilon} = \frac{1}{2}\left\{ \frac{\partial u_x}{\partial x} \quad \frac{\partial u_y}{\partial y} \quad \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right\}$$

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{Bmatrix} \partial u_x/\partial x \\ \partial u_y/\partial y \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{Bmatrix}$$

$$= \frac{1}{2}\left\{ \frac{\partial u_x}{\partial x} \quad \frac{\partial u_y}{\partial y} \quad \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\right\} .$$

$$\left\{ \begin{array}{l} D_{11}\frac{\partial u_x}{\partial x} + D_{12}\frac{\partial u_y}{\partial y} + D_{13}\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \\ D_{12}\frac{\partial u_x}{\partial x} + D_{22}\frac{\partial u_y}{\partial y} + D_{23}\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \\ D_{31}\frac{\partial u_x}{\partial x} + D_{32}\frac{\partial u_y}{\partial y} + D_{33}\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{array} \right\}$$

$$= \frac{1}{2}\left[D_{11}\left(\frac{\partial u_x}{\partial x}\right)^2 + D_{12}\frac{\partial u_y}{\partial y}\frac{\partial u_x}{\partial x} + D_{13}\frac{1}{2}\frac{\partial u_x}{\partial x}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \right.$$

$$D_{12}\frac{\partial u_y}{\partial x}\frac{\partial u_x}{\partial x} + D_{22}\left(\frac{\partial u_y}{\partial y}\right)^2 + D_{23}\frac{\partial u_y}{\partial x}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) +$$

$$\left. \frac{1}{2}D_{31}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\frac{\partial u_x}{\partial x} + \frac{D_{22}}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\frac{\partial u_y}{\partial y} + \frac{D_{33}}{4}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2 \right]$$

$$= F$$

Now,

$$\mathcal{L}_1 = \int_\Omega F \, d\Omega = \int_\Omega \frac{1}{2}\bar{\varepsilon}^T \bar{\bar{D}} \bar{\varepsilon}$$

$$\delta_{\bar{u}} \mathcal{L}_1 = \frac{1}{2}\int_\Omega 2D_{11}\frac{\partial u_x}{\partial x}\frac{\partial \delta u_x}{\partial x} + D_{12}\frac{\partial u_y}{\partial y}\frac{\partial \delta u_x}{\partial x}$$

$$+ D_{12}\frac{\partial \delta u_y}{\partial y} \cdot \frac{\partial u_x}{\partial x} + D_{13}\frac{1}{2}\frac{\partial \delta u_x}{\partial x}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$+ D_{13}\frac{1}{2}\frac{\partial u_x}{\partial x}\left(\frac{\partial \delta u_x}{\partial y} + \frac{\partial \delta u_y}{\partial x}\right)$$

$$+ D_{12}\frac{\partial \delta u_y}{\partial x}\frac{\partial u_x}{\partial x} + D_{12}\frac{\partial u_y}{\partial x}\frac{\partial \delta u_x}{\partial x} + \text{and so on} \ldots$$

The hilighted terms add up and become $2 D_{12} \frac{\partial u_y}{\partial y} \frac{\partial \delta u_x}{\partial x}$.

Thus, all terms get a 2. This "2" gets cancelled with "$\frac{1}{2}$" outside.

Finally, we get

$$\delta_{\bar{u}} \mathcal{L} = \int_{\Omega} \bar{\varepsilon}^T \bar{\bar{D}} \, \delta \bar{\varepsilon} \, d\Omega .$$