

ME 260: Structural Optimization: Size, Shape, and Topology		
Assigned 10 AM, Oct. 26, 2021	Midterm Examination Open-everything except human interaction	Due 5 PM, Oct. 26, 2021

This is an open-everything examination. That is, you can refer to books, notes, internet resources, etc., but you cannot interact with any human being in any manner. You must cite the significant sources that you have referred to. Please sign the honor code when you submit the answer script as a single pdf file by email to suresh@iisc.ac.in.

Honor code

"I have neither consulted nor sought help from any living person, directly or indirectly, in answering the questions in this ME 260 midterm examination. I understand the guidelines given by the instructor about the code of conduct of this examination."

Name:

Signature:

Place:

Date:

Question 1 (10 points)

Write the necessary conditions for the following optimization problem. Lowercase symbols denote scalars, lowercase symbols in bold denote vectors, and uppercase symbols in bold denote matrices. Collate all equations and unknowns and give an outline of the steps to solve this problem.

$$\text{Min}_{\mathbf{a}} \mathbf{u}^T \mathbf{u}$$

Subject to

$$\mathbf{K}\mathbf{u} - \mathbf{f} = \mathbf{0}$$

$$\mathbf{a}^T \mathbf{l} - V^* \leq 0$$

Data : $\mathbf{l}$ and V^* are constants; and $\mathbf{f}$ and $\mathbf{K}$ are known functions of $\mathbf{a}$.

Question 2 (10 points)

In the functional $J = \int_a^b (\alpha y''^2 - \beta y) dx + \gamma y^2 \Big|_{x=\frac{a+b}{2}} + \kappa y'^2 \Big|_{x=\frac{3(b-a)}{4}}$, $\alpha(x)$ and $\beta(x)$ are given functions of x while γ and κ are given constants. Assume that $y = y' = 0$ at $x = a$, and $y = 0$ at $x = b$.

- (a) Write the differential equation and all the conditions to evaluate $y(x)$ to minimize J , using the framework of calculus of variations.
- (b) Suggest a physical problem that suits this mathematical modeling and comment on what all symbols and conditions denote.

Question 3 (10 points)

Consider a slender iceberg floating in sea water. Imagine that it has the shape of a solid of revolution with its axis vertical. By assuming that the iceberg has formed such that its mean compliance due to the combined effect of gravity and buoyancy is minimized, find the shape of the cross-section if the total volume of the iceberg is V and axial extent, i.e., the length is l .

- Note that the density of ice is ρ_i and that of sea water is ρ_w and that $\rho_w > \rho_i$.
- Use g to denote the acceleration due to gravity.
- Notice that you have to find the extent of the iceberg inside the sea water as part of the solution.
- Since it is a slender iceberg, using the 1D axial model is tenable for capturing elastic deformation. Towards this, include $A_{\max}$ for the permissible maximum area of cross-section. Assume also $A_{\min}$ for continuity.
- If you make any other assumptions, please state them explicitly.

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Question 1 (10 points)

Solve the following problem analytically as much as possible to comment on which of the constraints might be active at the local optimum.

$$\text{Min}_{a,b} f = \frac{G a b n p d^4}{8 W L D^3}$$

Subject to

$$g_1 = \frac{8 W L D k_w \sqrt{a^2 + b^2}}{\pi a b n d^3} - 345 E 6 \leq 0$$

$$g_2 = \frac{80 W L D^3 \sqrt{a^2 + b^2}}{G a b n p d^4} - 7 \leq 0$$

$$g_3 = \frac{G d^4 a b n}{8 W L D^3} - 100 \leq 0$$

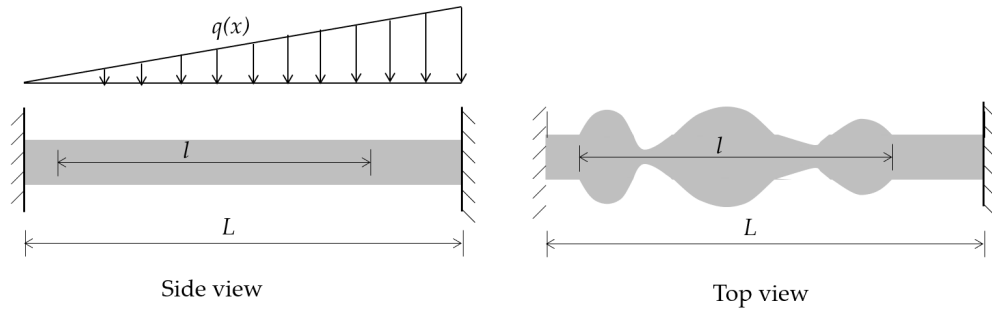
Data: $W = 30 \times 9.81$, $G = 79 E 9$, $L = 0.15$, $n = 3$

$d = 3 E - 3$, $D = 27 E - 3$, $p = 4 E - 3$

$$k_w = \frac{4c - 1}{4c - 4} + \frac{0.615}{c}; c = \frac{D}{d}$$

Question 2 (10 points)

A beam of constant square cross section of side s , length L , and Young's modulus E under distributed linearly-tapering transverse load $q(x) = q_1 x$ (where $x = 0 \cdots L$) needs to be remodeled by adding or subtracting material symmetrically along the breadth direction to make the cross section rectangular with constant depth of s . The objective is to make the beam as stiff as possible using additional material V^* . The condition is that the cross section can be modified to rectangular section over a continuous span of length l that is smaller than the beam length L . See the figure for further clarity. Pose this as an optimization problem. Write the design and adjoint equations along with the boundary conditions. You need not solve the equation analytically to the extent possible by simplifying the design, adjoint, and other equations.



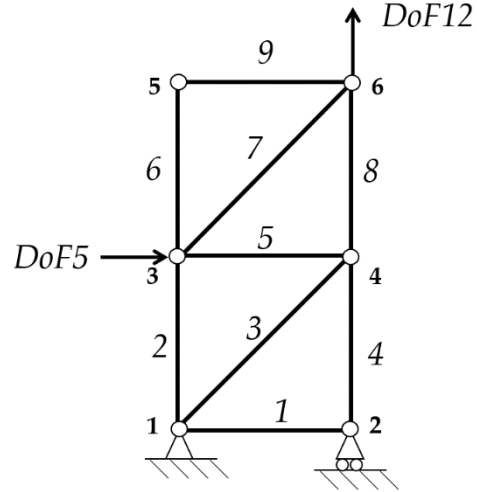
Question 3 (2×5 = 10 points)

- Derive the expression for the lowest critical buckling load for a fixed-free column by posing it as a problem of calculus of variations. Hint: Consider the buckled transverse displacement of the column, write the strain energy and work potential (for the compressive point load at the free end) and minimize it with respect to the transverse displacement.
- Let a functional J be equal to the product of two other functionals K and L , all of which have the same domain of the function $f(x)$. Show from first principles that variation of J is given by $(K \cdot \text{variation of } L + L \cdot \text{variation of } K)$ for an arbitrary vector $h(x)$ that is used to take the variation.

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Question 1 (4 + 8 = 12 points)

(a) Consider the planar truss shown here. The truss has six vertices (numbered in bold, small font) and nine members (numbered in large not-bold font). Two DoFs are highlighted. The areas of cross-section of members are denoted by the vector $\mathbf{a}$ and lengths by $\mathbf{l}$. If only a force of p value is applied at DoF5, the internal forces in all members and displacement vector at all vertices are given by $\mathbf{f}_A$ and $\mathbf{u}_A$. When only a force of $2p$ is applied at DoF12, the internal forces and displacements are given by $\mathbf{f}_B$ and $\mathbf{u}_B$. By assuming that both load cases are in the linear regime, answer the following questions.



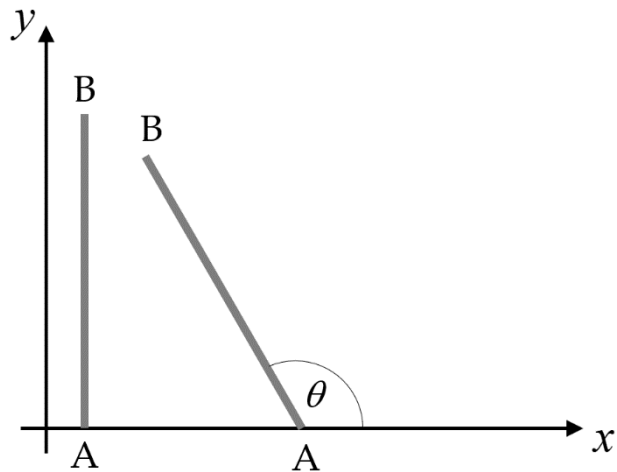
- What is ratio of stresses in members 3 and 7 when there is a load of p only on DoF5?
- What is the strain energy of element 7 when there is a load of p only on DoF5?
- What is the ratio of the contribution of elements 3 to the displacement along DoF12 when there is load of p only on DoF5?
- If we denote the displacement along DoF12 when a load of p is applied along only DoF5 as $u_{12,5}$ and the displacement along DoF5 when a load of $2p$ is applied along only DoF12 as $u_{5,12}$, can you which is larger between $u_{12,5}$ and $u_{5,12}$?

(b) By using the areas of cross-sections as the optimization variables and assuming load of p only along DoF5, pose an optimization problem to minimize the volume of the truss subject to constraints on the strain energy SE^* in the whole truss and a displacement of Δ^* along DoF12. Give the complete solution.

Question 2 (8 points)

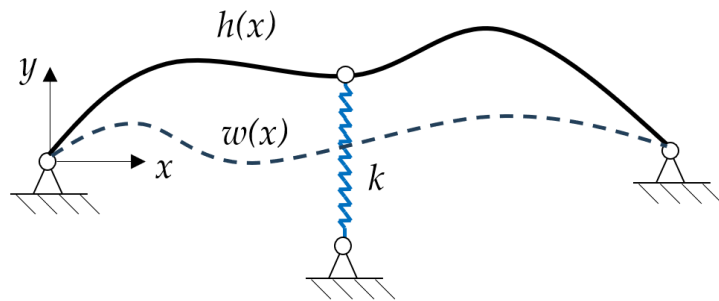
Consider a rigid rod of length l with its ends denoted with A and B. At time $t = 0$, the coordinate of Points A and B are $(x_{A0}, 0)$ and (x_{B0}, y_{B0}) . Point A is sliding along the x -

axis while the rod is rotating with angle θ as shown. Note that $x(t)$ is a known function while $\theta(t)$ is an unknown function. Determine $\theta(t)$ such that $\int_0^T (\dot{x}_B^2 + \dot{y}_B^2) dt$ is minimized over a time period of T starting from $t = 0$. Obtain the differential equation along with the end conditions to solve for $\theta(t)$.



Question 3 (10 points)

An arch is pinned at its ends and has a pin joint midway between end ends. Let the distance between the two ends be L along the x -axis. The undeformed profile of the arch is given by $h(x)$ (which is known) and deformed profile be $w(x)$



(which is unknown) under a transverse load of $q(x)$ (not shown in the figure). There is a transverse spring of stiffness, k , at a distance of $L/2$ from the origin. The arch has constant circular cross-section with radius, R . It is made of material with Young's modulus, E .

- Use the principle of minimum potential energy and obtain the differential equation and end conditions.
- Simplify the equations to the extent possible and comment on the nature of the solution in terms of continuity of slope of the deformed arch. What is continuous at the pin joint at which the spring is attached?

Hint: For ease, consider the approximated axial component of strain energy to be $EA(s - s_0)$ where A is the area of cross-section, s_0 the undeformed arc-length of the arch, and s the deformed arc-length of the arch. The bending component of strain energy is

given by
$$\int_0^L \frac{EI}{2} \left(\frac{d^2 w}{dx^2} - \frac{d^2 h}{dx^2} \right)^2 dx$$

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Question 1 (7 points)

Consider the truss optimization problem where the strain energy is to be minimized subject to the volume constraint along with upper and lower bounds on the design variables, $\mathbf{a}$, the column vector of areas of cross-sections of the truss elements.

$$\underset{\mathbf{a}}{\text{Min}} SE = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u}$$

Subject to

$$\mathbf{K} \mathbf{u} - \mathbf{f} = \mathbf{0}$$

$$a_l \leq a_i \leq a_u \quad i = 1, 2, \dots, n$$

Data : $\mathbf{f}$ is given; and $\mathbf{K}$ is a linear function of $\mathbf{a}$

- Write the optimality criterion for this problem without considering the upper and lower bounds on the areas of cross-section.
- If f_j is increased by Δf_j , give the change in optimized value of SE assuming that the problem is solved. Use superscript $*$ for the optimized quantities when you write the expression for change in SE^* .
- Assume that you have solved the problem without and with the bounds on the areas of cross-section and got SE^* and SE_{bounds}^* as the solutions respectively. Which is likely to be larger and why?

Question 2 (8 points)

Write the dual of the following problem. No need to solve the dual problem after you formulate it.

$$\underset{x_1, x_2, x_3}{\text{Min}} f = 5x_1 + 7x_2 - 4x_3 - \sum_{i=1}^3 \ln(x_i)$$

Subject to

$$x_1 + 3x_2 + 12x_3 - 37 = 0$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

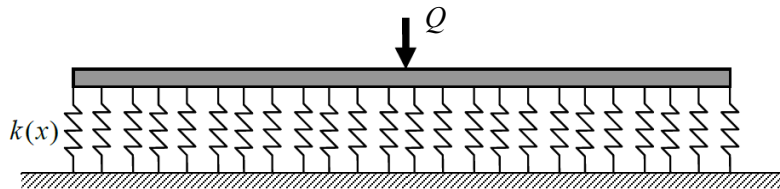
Question 3 (5 points)

Consider

$J = \int_{x_1}^{x_2} F(y, y', y'', y''') dx$ where $y(x)$ is the unknown function. Now, consider a known function $f(x)$. What terms, which are functions of $y(x)$ and $f(x)$ as well as their derivatives, can be added to the integrand of J so that the Euler-Lagrange equations of minimizing the original J and the modified J remain the same? Write all terms that you can add and show your reasoning for each term.

Question 4 (10 points)

The figure below shows a beam of elasticity modulus (i.e., EI) $\phi(x)$, and length l , lying on an elastic foundation of linear stiffness $k(x)$. A point force Q is applied at the midpoint as shown.



- Using the principle of minimum potential energy, derive the governing equation for the elastostatic equilibrium of the beam along with the boundary conditions.
- Pose a structural optimization problem to minimize the work done by the force Q by varying $\phi(x)$ as the design variable. Write the optimality criterion for the problem.

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Question 1 (12 points)

Consider the following two constrained minimization problems.

$$\text{P1: } \min_{x,y} f = x^2 + y^2$$

Subject to

$$(x - 1)^2 - y^2 \leq 0$$

$$\text{P2: } \min_{x,y} f = x^2 + y^2$$

Subject to

$$y^2 - (x - 1)^3 \leq 0$$

- a) For P1 and P2, find all minimizing values of x and y if only real values are allowed.
- b) Check if your solutions satisfy the necessary and sufficient conditions to ensure that they are indeed minima.
- c) If you are not able to find minima that satisfy the necessary conditions for either P1 or P2 or both, explain why.
- d) Check if f at $(x = 1; y = 0)$ has a minimum for P1 and P2 by considering feasible perturbations around $(x = 1; y = 0)$.

Hint: Think graphically.

Question 2 (4 points)

Find the Gateaux variation of $\mathbf{f}^T \mathbf{u}$ and $\mathbf{u}^T \mathbf{K} \mathbf{u}$ where $\mathbf{f}$ and $\mathbf{u}$ are $(n \times 1)$ vectors, and $\mathbf{K}$ is an $(n \times n)$ matrix. Assume that $\mathbf{f}$ and $\mathbf{K}$ are constants while $\mathbf{u}$ is variable.

Question 3 (10 points)

In the functional $J = \int_a^b (\alpha y''^2 - \beta y) dx + \lambda y^2 \Big|_{x=\frac{3(b-a)}{4}} + \mu y'^2 \Big|_{x=\frac{a+b}{2}}$, $\alpha(x)$ and $\beta(x)$ are known

functions of x while λ and μ are known constants. Assume that $y = y' = 0$ at $x = a$, and $y = 0$ at $x = b$.

- (a) Write the differential equation and all the conditions to evaluate $y(x)$ to minimize J , using the framework of calculus of variations.
- (b) Suggest a physical problem that suits this mathematical modeling and comment on what J , $\alpha(x)$, $\beta(x)$, λ , μ , a , and b denote. And mention what the boundary values indicate. Draw a figure that represents the physical problem.

Question 4 (4 points)

You have played *New York Times Connections* puzzle after almost each lecture. With that experience, now, group the 16 words, phrases, and statements listed below into **four groups of four items** and mention the **connections** among the items of each group.

1. Chain rule
2. Shape
3. Material
4. Need for an arbitrary vector belonging to the same function space.
5. $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for any non-zero $\mathbf{x}$.
6. Topology
7. Positive eigenvalues.
8. Quotient rule
9. The addition of two vectors belongs to the same vector space.
10. One
11. Closed under scalar multiplication
12. $\mathbf{R}^T \mathbf{R}$ for any rectangular or square matrix $\mathbf{R}$ that has independent columns.
13. Size
14. Zero
15. Product rule
16. Positive pivots.

Group (insert numerals 1-16) in each box				Connection among the four items of the row