ME 260: Structural Optimization: Size, Shape, and Topology		
Start: Nov. 2, '20, 10 AM	Quiz 1: Solution	End: Nov. 2, '20, 1 PM

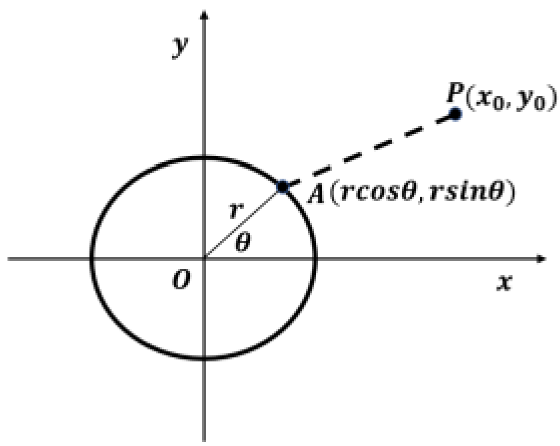
Problem 1 (5 points)

(a) Given a point P with coordinates (8,6) and a circle of radius 5 units with its center at the origin, find a point on the circle that is closest to P.

(b) With the same point and the circle, find a straight line that separates P and the circle so that they lie on either side of the line and the distance from P to the line is the largest.

Solution:

(a)



Let $P(x_0, y_0)$ be a given point and there is a circle of radius r on which we need to find a point $A(x, y)$ such that the distance between the point P and A is minimum. As $A(x, y)$ is a point on the circle, its co-ordinates is given by $A \equiv (r\,\text{Cos}\,\theta, r\,\text{Sin}\,\theta)$.

The distance between A and P is $s = \sqrt{(x_0 - r\,\text{Cos}\,\theta)^2 + (y_0 - r\,\text{Sin}\,\theta)^2}$.

We need to minimize this distance and the variable here is the point A, which is a function of the angle θ. So, the optimization problem is posed as,

$$\underset{\theta}{\text{Min}} \quad s^2 = (x_0 - r\,\text{Cos}\,\theta)^2 + (y_0 - r\,\text{Sin}\,\theta)^2$$

The necessary condition is,

$$\frac{ds^2}{d\theta} = (x_0 - r\,\text{Cos}\,\theta)(r\,\text{Sin}\,\theta) + (y_0 - r\,\text{Sin}\,\theta)(-r\,\text{Cos}\,\theta) = 0 \qquad (1)$$

Solving Eq. (1), we get,

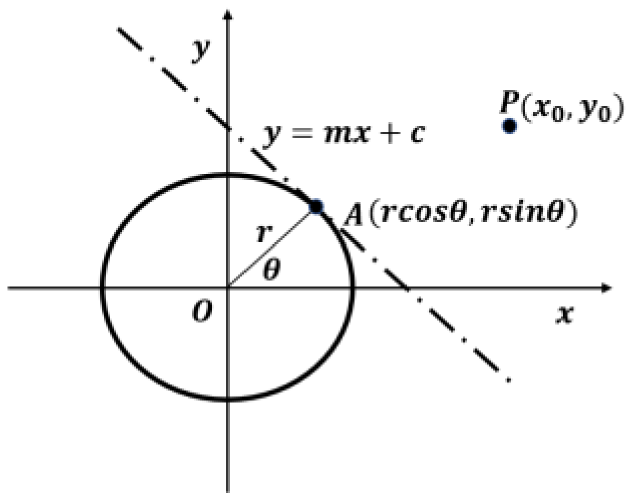
$$\Rightarrow x_0(r\,\text{Sin}\,\theta) - y_0(r\,\text{Cos}\,\theta) = 0$$

$$\Rightarrow \tan\theta = \frac{y_0}{x_0} \qquad (2)$$

For the given problem $P(x_0, y_0) \equiv (8,6)$. Substituting in Eq. (2), we get, $\tan\theta = \frac{3}{4}$. So, for $r = 5$, the co-ordinates of the point A is given by, $A(x, y) \equiv (4,3)$.

So, from the point $A(4,3)$ the distance of the point $P(8,6)$ is the smallest among any other point on or inside the circle.

(b)



We need to find a straight line that separates P and the circle so that they lie on either side of the line and the distance from P to the line is the largest. If we just need to find a straight line such that the distance from P to the line is largest then it would go to infinity, but we have a constraint that the circle and the point P should be either side of the line. So, clearly the line cannot cut or cross the circle. So, to get the maximum distance from the point P from the line, the line at most can be a tangent to the circle at point $A(x, y)$ which co-ordinates is given by $A \equiv (r\,\text{Cos}\,\theta, r\,\text{Sin}\,\theta)$, where r is the radius of the circle.

So, the equation of the tangent passing through point $(r\,\text{Cos}\,\theta, r\,\text{Sin}\,\theta)$ is given by, $y = -\frac{x}{\tan\theta} + \frac{r}{\text{Sin}\,\theta}$.

The distance of point $P(x_0, y_0)$ from the line is given by

$$s = \frac{y_0 + \frac{x_0}{\tan\theta} - \frac{r}{\text{Sin}\,\theta}}{\sqrt{1 + \left(\frac{1}{\tan\theta}\right)^2}} = \tan\theta\,\text{Cos}\,\theta\left(y_0 + \frac{x_0}{\tan\theta} - \frac{r}{\text{Sin}\,\theta}\right) = y_0\,\text{Sin}\,\theta + x_0\,\text{Cos}\,\theta - r \qquad (1)$$

We need to minimize this distance and the variable here are the coordinates of Point A, which are function of the angle θ. So, the optimization problem is posed as

$$\underset{\theta}{\text{Min}} \quad s = y_0 \text{Sin}\,\theta + x_0 \text{Cos}\,\theta - r$$

The necessary condition is,

$$\frac{ds}{d\theta} = y_0 \text{Cos}\,\theta - x_0 \text{Sin}\,\theta = 0 \qquad (2)$$

Solving Eq. (2) we get, $\tan\theta = \dfrac{y_0}{x_0}$.

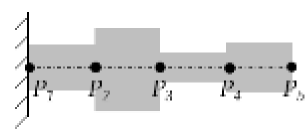
For the given problem, $P(x_0, y_0) \equiv (8,6)$. Substituting in Eq. (2), we get, $\tan\theta = \dfrac{3}{4}$. So, for $r = 5$, the co-ordinates of Point A are given by, $A(x, y) \equiv (4,3)$.

The equation of the line that separate the point P and the circle is given by, $3y = 25 - 4x$.

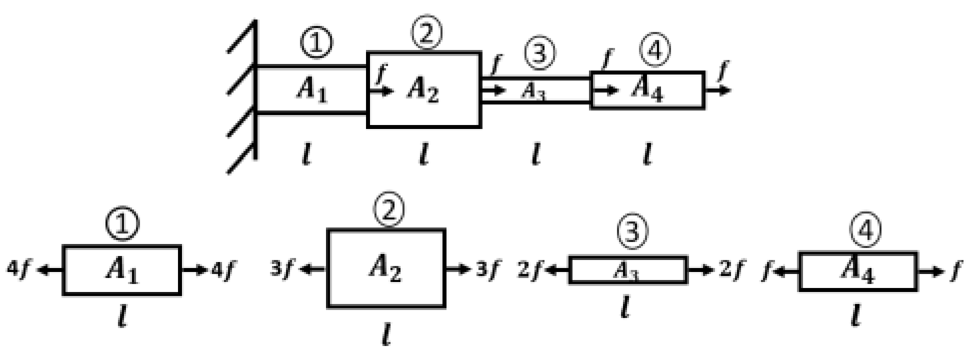
Notice that the two problems in Parts (a) and (b) are dual to each other. With same data (circle and point), we have a minimization problem and a maximization problem. This particular pair of examples go under the name of **Hahn-Banach theorem**.

Problem 2 (5 points)

Consider a stepped bar of four segments. One of its ends is fixed and the other is free. There is an axial force f at each of the points, P_2 to P_5. Find the areas of cross-sections (denoted by $A_{i=1,2,3,4}$) of the four segments to minimize the mean compliance subject to the volume constraint of V^*, upper and lower bounds on areas of cross-section A_U and A_L, given that all four segments are of length of l and the Young's modulus is denoted by E.



Solution:



Free body diagram and force distribution

Let there be four bars with area, length and internal force given by A_i, l_i and P_i. Note that

$$P_1 = 4f;\, P_2 = 3f;\, P_3 = 2f;\, P_4 = f$$

Then, the mean compliance can be written as: $\text{SE} = \sum_{i=1}^{4} P_i u_i = \sum_{i=1}^{4} P_i \left(\frac{P_i l_i}{A_i E} \right) = \sum_{i=1}^{4} \frac{P_i^2 l_i}{A_i E}$.

The volume of the structure is given by $V = \sum_{i=1}^{4} A_i l_i$.

So, according to the given problem, we have to minimize the mean compliance with volume constraint and lower and upper bounds on areas of cross-section, i.e.,

$$\underset{A_i}{\text{Min}} \quad \text{MC} = \sum_{i=1}^{N} \frac{P_i^2 l_i}{A_i E}$$

subject to:

$$\mu: \qquad \sum_{i=1}^{N} A_i l_i - V^* \le 0$$

$$\lambda_{L_i}: \qquad A_L - A_i \le 0 \qquad i = 1, 2, \ldots, N$$

$$\lambda_{U_i}: \qquad A_i - A_U \le 0 \qquad i = 1, 2, \ldots, N$$

$$\mu, \lambda_{L_i}, \lambda_{U_i} \ge 0$$

The Lagrangian is written as

$$L = \sum_{i=1}^{N} \frac{P_i^2 l_i}{A_i E} + \mu \left(\sum_{i=1}^{N} A_i l_i - V^* \right) + \sum_{i=1}^{N} \lambda_{L_i} \left(A_L - A_i \right) + \sum_{i=1}^{N} \lambda_{U_i} \left(A_i - A_U \right)$$

The necessary conditions are:

$$\frac{\partial L}{\partial A_i} = \frac{-P_i^2 l_i}{A_i^2 E} + \mu l_i - \lambda_{L_i} + \lambda_{U_i} = 0 \qquad (1)$$

$$\mu \left(\sum_{i=1}^{N} A_i l_i - V^* \right) = 0 \qquad (2)$$

$$\lambda_{L_i} \left(A_L - A_i \right) = 0 \qquad i = 1, 2, \ldots, N \qquad (3)$$

$$\lambda_{U_i} \left(A_i - A_U \right) = 0 \qquad i = 1, 2, \ldots, N \qquad (4)$$

<u>Case-1</u> ($\lambda_{L_i} = \lambda_{U_i} = 0$)

$$\frac{-P_i^2 l_i}{A_i^2 E} + \mu l_i = 0 \Rightarrow A_i = \frac{P_i}{\sqrt{\mu E}} \qquad (5)$$

Here $A_i \propto P_i$. So, larger the force the should be the area of cross section of the member.

Also, $\mu \neq 0$. So, from Eq. (2), we get

$$\sum_{i=1}^{N} A_i l_i = V^* \Rightarrow \sum_{i=1}^{N} \frac{P_i l_i}{\sqrt{\mu E}} = V^* \Rightarrow \mu = \frac{\left(\sum_{i=1}^{N} P_i l_i \right)^2}{V^{*2} E} \tag{6}$$

By substituting Eq. (6) in Eq. (5) we get

$$A_i = \frac{P_i V^*}{\sum_{i=1}^{N} P_i l_i} \tag{7}$$

From Eq. (7) we get the areas of all the members. If the area exceeds the upper limit we take the area as the upper bound A_U and if area is lower than the lower limit we set the area to the lower bound (A_L), i.e.,

$$\begin{aligned} A_i &= A_U \quad \text{if} \quad A_i > A_U \quad \text{for} \quad i \in \mathbf{U} \\ A_i &= A_L \quad \text{if} \quad A_i < A_L \quad \text{for} \quad i \in \mathbf{L} \end{aligned} \tag{8}$$

where $\mathbf{U}$ and $\mathbf{L}$ are the sets of members that have the areas either upper or lower bound, respectively. Then from Eq. (2) the updated area is calculated as,

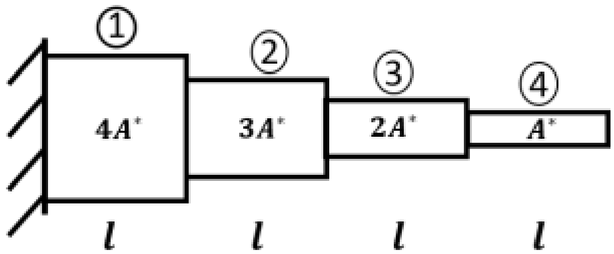
$$\begin{aligned} &\sum_{i \notin (\mathbf{U},\mathbf{L})} A_i l_i + \sum_{i \in \mathbf{U}} A_i l_i + \sum_{i \in \mathbf{L}} A_i l_i = V^* \Rightarrow \sum_{i \notin (\mathbf{U},\mathbf{L})} A_i l_i = V^* - \sum_{i \in \mathbf{U}} A_i l_i - \sum_{i \in \mathbf{L}} A_i l_i \\ &\Rightarrow \sum_{i \notin (\mathbf{U},\mathbf{L})} \frac{P_i l_i}{\sqrt{\mu E}} = V^* - \sum_{i \in \mathbf{U}} A_i l_i - \sum_{i \in \mathbf{L}} A_i l_i \Rightarrow \mu = \frac{\left(\sum_{i \notin (\mathbf{U},\mathbf{L})} P_i l_i \right)^2}{\left(V^* - \sum_{i \in \mathbf{U}} A_i l_i - \sum_{i \in \mathbf{L}} A_i l_i \right)^2 E} \end{aligned} \tag{9}$$

We substitute the value of μ from Eq. (9) in Eq. (5) to get the revised area.

From Eq. (7) we get,

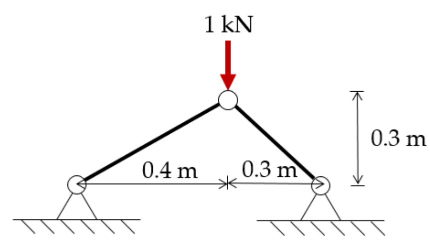
$$\begin{aligned} &A_1 = \frac{4fV^*}{10fl} = \frac{0.4V^*}{l} = 4A^* \\ &A_2 = \frac{0.3V^*}{l} = 3A^*; \quad A_3 = \frac{0.2V^*}{l} = 2A^*; \quad A_4 = \frac{0.1V^*}{l} = A^* \end{aligned} \tag{10}$$

From Eq. (5) we observed that the area is proportional to the force in the member. So, according to the problem we get the area of member-1 as the largest and it gradually decreases and gives the smallest area for member-4 (see Fig. below). It is clear from the Eq. (10) also. If we take more and more members, the jump in the areas of the members will be smoother and we will get a smooth tapering bar, which is the optimum design for a bar having uniformly distributed load and one end fixed. We will see that in the context of a bar optimization later in the course.



Problem 3 (5 points)

A two-bar truss is shown in the adjacent figure. Both bars are made of steel (E = 210 GPa, ρ = 7800 kg/m^3, and S_y = 300 MPa). Design this truss to minimize weight for three different cases:



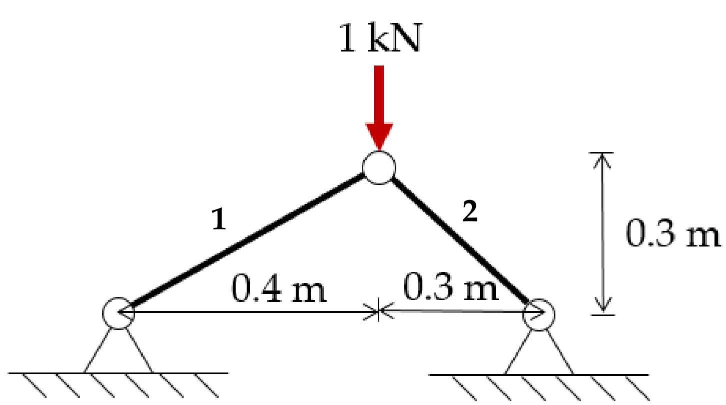
(i) stress limited, i.e., $\sigma_{max} \le S_y$

(ii) stiffness limited, i.e., vertical displacement $= v \le 5\text{mm}$

(iii) stability limited, i.e., both bars do not buckle.

Use circular cross sections. Use unit factor of safety.

Solution:



The force in the members due to the actual load of 1 kN at the node are 0.7143 kN and 0.8081 kN in Member 1 and Member 2, respectively.

So, $(P_r)_1 = 0.7143\text{kN}$; $(P_r)_2 = 0.8081\text{kN}$.

The force in the members due to the unit virtual load at the node are 0.7143 and 0.8081 in member-1 and member-2 respectively.

So, $(P_v)_1 = 0.7143kN$; $(P_v)_2 = 0.8081kN$.

Length of the truss members are, $l_1 = 0.5\,\text{m}$ and $l_2 = 0.3\sqrt{2}\,\text{m}$.

(i) Strength-limited design

Let the Young's modulus, density, area, and length of the members of the truss structures be E, ρ, A_i, and l_i respectively.

According to the question we have to minimize the weight of the structure, i.e., $W = \sum_{i=1}^{N} \rho A_i l_i$.

The problem then posed as,

$$\underset{A_i}{\text{Min}} \quad W = \sum_{i=1}^{N} \rho A_i l_i$$

subject to:

$$\mu_i : \qquad \frac{P_i}{A_i} - S_y \le 0 \qquad \text{for } \; i = 1, 2, \ldots, N$$

$$\mu_i \ge 0$$

The Lagrangian of the above minimization problem is,

$$L = \sum_{i=1}^{N} \rho A_i l_i + \sum_{i=1}^{N} \mu_i \left(\frac{P_i}{A_i} - S_y \right)$$

The necessary conditions are,

$$\frac{\partial L}{\partial A_i} = \rho l_i + \mu_i \left(-\frac{P_i}{A_i^2} \right) = 0 \Rightarrow A_i = \sqrt{\frac{\mu_i P_i}{\rho l_i}} \tag{1}$$

$$\mu_i \left(\frac{P_i}{A_i} - S_y \right) = 0 \qquad \text{for } \; i = 1, 2, \ldots, N \tag{2}$$

In Eq. (2), for this problem $\mu_i \neq 0$. So,

$$\left(\frac{P_i}{A_i} - S_y \right) = 0 \tag{3}$$

Substituting Eq. (1) in Eq. (3), we get,

$$\frac{P_i}{A_i} = S_y \Rightarrow \frac{P_i}{\sqrt{\frac{\mu_i P_i}{\rho l_i}}} = S_y \Rightarrow \mu_i = \frac{P_i \rho l_i}{S_y^2} \tag{4}$$

Substituting Eq. (4) in Eq. (1), we get the optimized area as,

$$A_i^{opt} = \sqrt{\frac{\frac{P_i \rho l_i}{S_y^2} P_i}{\rho l_i}} \Rightarrow A_i^{opt} = \frac{P_i}{S_y} \tag{5}$$

Substituting the values of force (P_i) and the maximum admissible stress (S_y) in Eq. (5), we get the optimized area as,

$$\begin{aligned} A_1^{opt} &= 2.3810\,\text{mm}^2 \\ A_2^{opt} &= 2.6937\,\text{mm}^2 \end{aligned} \tag{6}$$

The optimized weight of the structure is given by,

$$W^{opt} = \sum_{i=1}^{N} \rho A_i^{opt} l_i = 0.0182\,\text{kg}\,.$$

(ii) Stiffness-limited design

Let the Young's modulus, density, area, and length of the members of the truss be E, ρ, A_i, and l_i respectively.

According to the question we have to minimize the weight of the structure, i.e., $W = \sum_{i=1}^{N} \rho A_i l_i$.

Also, we have a constraint at the point of application of the load in the vertical direction that the displacement should be less than or equal to v. We will pose this constraint using the virtual work method by applying a unit dummy load at that point in the y-direction. Then the displacement constraint is expressed as,

$$\begin{aligned} &\sum_{i=1}^{N} (u_r)_i (P_v)_i - v \le 0 \\ &\Rightarrow \sum_{i=1}^{N} \frac{(P_r)_i l_i}{A_i E} (P_v)_i - v \le 0 \end{aligned} \tag{1}$$

where

$(u_r)_i$ is the actual displacement of i$^{\text{th}}$ the member due to the actual (real) load

$(P_r)_i$ is the actual load in i$^{\text{th}}$ the member due to the actual (real) load

$(P_v)_i$ is the vertual load in i$^{\text{th}}$ the member due to the unit dummy load

The problem then is posed as,

$$\underset{A_i}{\text{Min}} \;\; W = \sum_{i=1}^{N} \rho A_i l_i$$

Subjected to:

$$\mu: \qquad \sum_{i=1}^{N} \frac{(P_r)_i \, l_i}{A_i E} (P_v)_i - v \le 0$$

The Lagrangian is given by,

$$L = \sum_{i=1}^{N} \rho A_i l_i + \mu \left(\sum_{i=1}^{N} \frac{(P_r)_i \, l_i}{A_i E} (P_v)_i - v \right)$$

The necessary conditions we get by differentiating the Lagrangian w.r.t. A_i as

$$\frac{\partial L}{\partial A_i} = \rho l_i + \mu \left(- \frac{(P_r)_i \, l_i}{A_i^2 E} (P_v)_i \right) = 0 \tag{2}$$

$$\mu \left(\sum_{i=1}^{N} \frac{(P_r)_i \, l_i}{A_i E} (P_v)_i - v \right) = 0 \tag{3}$$

Eq. (2) is simplified to get the optimized area A_i^* of each truss member as,

$$\mu \left(\frac{(P_r)_i \, l_i}{A_i^2 E} (P_v)_i \right) = \rho l_i \Rightarrow A_i^* = \sqrt{\frac{\mu (P_r)_i (P_v)_i}{\rho E}} \tag{4}$$

In Eq. (3) for this problem $\mu \neq 0$. So,

$$\sum_{i=1}^{N} \frac{(P_r)_i \, l_i}{A_i E} (P_v)_i - v = 0 \tag{5}$$

Substituting Eq. (4) in Eq. (5) we get μ as,

$$\sum_{i=1}^{N} \frac{(P_r)_i \, l_i}{\sqrt{\frac{\mu (P_r)_i (P_v)_i}{\rho E}} E} (P_v)_i = v$$

$$\Rightarrow \sum_{i=1}^{N} \sqrt{\frac{\rho (P_r)_i (P_v)_i}{\mu E}} l_i = v$$

$$\Rightarrow \mu = \left(\frac{\rho}{E v^2} \right) \left(\sum_{i=1}^{N} \sqrt{(P_r)_i (P_v)_i} \, l_i \right)^2 \tag{6}$$

Substituting Eq. (6) in Eq. (4), we get the optimized area values as,

$$A_i^{opt} = \sqrt{\frac{\left(\frac{\rho}{Ev^2}\right)\left(\sum_{i=1}^{N}\sqrt{(P_r)_i(P_v)_i}l_i\right)^2 (P_r)_i(P_v)_i}{\rho E}} = \frac{\left(\sum_{i=1}^{N}\sqrt{(P_r)_i(P_v)_i}l_i\right)}{Ev}\sqrt{(P_r)_i(P_v)_i} \qquad (7)$$

Substituting the values of $(P_r)_i$, $(P_v)_i$, l_i, E and v in Eq. (7), we get the optimized areas as,

$$A_1^{opt} = 0.4762\,\mathrm{mm}^2$$
$$A_2^{opt} = 0.5387\,\mathrm{mm}^2$$

The optimized weight of the structure is given by,

$$W^{opt} = \sum_{i=1}^{N} \rho A_i^{opt} l_i = 0.003639\,\mathrm{kg}\,.$$

(iii) Stability-limited design

Let the Young's modulus, density, area, and length of the members of the truss structures be E, ρ, A_i, and l_i respectively.

According to the question we have to minimize the weight of the structure, i.e., $W = \sum_{i=1}^{N} \rho A_i l_i$.

Also, we have a constraint of critical buckling load, i.e., the load on the truss member should not exceed the critical buckling load value. So,

$$P_i - \frac{\pi^2 E I_i}{l_i^2} \le 0$$

As we are considering the area profile to be circular, we rewrite the above equation as,

$$P - \frac{\pi^2 E\left(\pi r^4\right)}{4 l_i^2} \le 0 \Rightarrow P_i - \frac{\pi^2 E A_i^2}{4\pi l_i^2} \le 0$$
$$\Rightarrow P_i - \frac{\pi E A_i^2}{4 l_i^2} \le 0 \qquad (1)$$

The problem then is posed as,

$$\underset{A_i}{\text{Min}}\; W = \sum_{i=1}^{N} \rho A_i l_i$$

Subjected to:

$$\mu_i: \qquad P_i - \frac{\pi E A_i^2}{4 l_i^2} \le 0 \qquad \text{for} \quad i = 1, 2, \ldots, N$$

The Lagrangian of the above optimization problem is,

$$L = \sum_{i=1}^{N} \rho A_i l_i + \sum_{i=1}^{N} \mu_i \left(P_i - \frac{\pi E A_i^2}{4 l_i^2} \right)$$

The necessary conditions are,

$$\frac{\partial L}{\partial A_i} = \rho l_i + \mu_i \left(-\frac{\pi E A_i}{2 l_i^2} \right) = 0$$

$$\Rightarrow A_i = \frac{2 \rho l_i^3}{\mu_i \pi E} \quad (2)$$

$$\mu_i \left(P_i - \frac{\pi E A_i^2}{4 l_i^2} \right) = 0 \qquad \text{for} \quad i = 1, 2, \ldots, N \quad (3)$$

In Eq. (3), $\mu_i \neq 0$, So,

$$\left(P_i - \frac{\pi E A_i^2}{4 l_i^2} \right) = 0 \qquad \text{for} \quad i = 1, 2, \ldots, N \quad (4)$$

Substituting the Eq. (2) in Eq. (4), we get the values of μ_i as,

$$\frac{\pi E A_i^2}{4 l_i^2} = P_i \Rightarrow \frac{\pi E \left(\frac{2 \rho l_i^3}{\mu_i \pi E} \right)^2}{4 l_i^2} = P_i \Rightarrow \frac{\rho^2 l_i^4}{\mu_i^2 \pi E} = P_i$$

$$\Rightarrow \mu_i = \frac{\rho l_i^2}{\sqrt{P_i \pi E}} \quad (5)$$

Substituting Eq. (5) in Eq. (2), we get the optimized areas as,

$$A_i^{opt} = \frac{2 \rho l_i^3}{\left(\frac{\rho l_i^2}{\sqrt{P_i \pi E}} \right) \pi E} = \frac{2 l_i \sqrt{P_i}}{\sqrt{\pi E}} \quad (6)$$

Substituting values of P_i, l_i, and E in Eq. (6), we get the optimized areas as,

$$A_1^{opt} = 32.9045 \, \text{mm}^2$$
$$A_2^{opt} = 29.6971 \, \text{mm}^2$$

The optimized weight of the structure is given by,

$$W^{opt} = \sum_{i=1}^{N} \rho A_i^{opt} l_i = 0.2266 \, \text{kg} .$$

Let us compare all three designs.

	Strength-limited design	*Stiffness-limited design*	*Stability-limited design*
Areas of cross section of two members (mm²)	$A_1 = 2.3810$ $A_2 = 2.6937$	$A_1 = 0.4762$ $A_2 = 0.5387$	$A_1 = 32.9045$ $A_2 = 29.6971$
Axial loads in the two members (N)	$P_1 = 714.3$ $P_2 = 808.1$	$P_1 = 714.3$ $P_2 = 808.1$	$P_1 = 714.3$ $P_2 = 808.1$
Axial stresses (MPa)	$\sigma_1 = 300$ $\sigma_2 = 300$	$\sigma_1 = 1500$ $\sigma_2 = 1500$	$\sigma_1 = 21.7$ $\sigma_2 = 27.2$
Displacements (u, v) *of the moving vertex (mm)*	$u = 300$ $v = 300$	$u = 0.7$ $v = 5.0$	$u = 0.0036$ $v = 0.0813$
Critical buckling loads (N) $P_{cri} = \dfrac{\pi E A_i^2}{4 l_i^2}$	$P_{cr1} = 3.7401$ $P_{cr2} = 6.6487$	$P_{cr1} = 0.1496$ $P_{cr2} = 0.2659$	$P_{cr1} = 714.3$ $P_{cr2} = 808.1$
Weight (kg)	0.0182	0.0036	0.2266
Mean compliance (J)	1.0000	5.0000	0.0813
Stress factor of safety $\left(\sigma_i / S \right)$	1, 1	0.2, 0.2	13.8,11.0
Downward vertical displacement (mm)	1.00	5.00	0.08
Buckling factor of safety $\left(P_i / P_{cr} \right)$	0.0052, 0.0082	0.2094E-3, 0.3290E-3	1,1

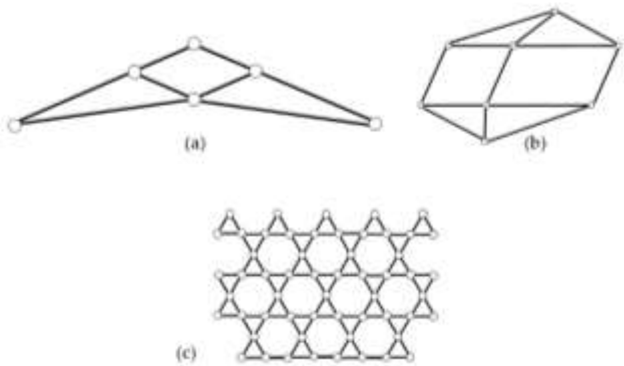
Notice that the vertical displacement of the strength-limited design is 1 mm. Since we asked for 5 mm vertical displacement, the stiffness-limited design is exactly five times the strength-limited one. It is worth noting that we got uniformly stressed design for stiffness-limited design, as to be expected.

The colored boxes indicate that those designs do not satisfy the corresponding criteria.

Notice also that minimum weight for the three cases are: 0.0182 kg for strength-limited design, 0.0036 kg for stiffness-limited, and 0.2266 kg for stability-limited design. So, which is the most conservative approach to design? That is, which criterion if you take, the other two criteria are automatically satisfied? Think about it.

Problem 4 (5 points)

Find the states of self-stress (SoSS) and degrees of freedom (DoF) of the following three configurations of trusses. Show your work in computing SoSS and DoF. If you conclude that they have instantaneous DoF, please suggest how many additional bars would you add to make them stiff and where.



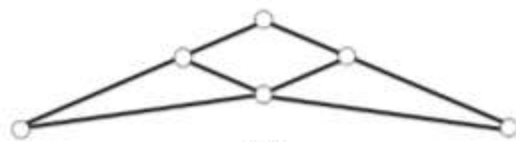
Solution:

(a) Let us count:

Number of bars = $b = 8$

Number of vertices = $v = 6$

So,
$2v-3-b=12-3-8=1=DoF-SoSS=1-0$

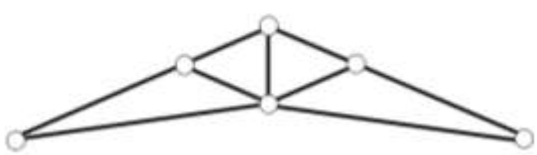

(a)

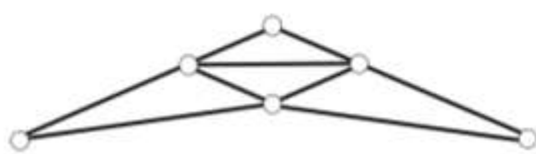
It is easy to see that there is a rhombus at the top that has one DoF. So, the number of SoSS is zero and it is apparent from the two triangles that it is true.

With one more bar, this truss would satisfy Maxwell's rule. Then DoF and SoSS will both be zero.

$2v-3-b=12-3-9=0=DoF-SoSS=0-0$

One more bar can be added in two ways to create four triangle in the truss, as shown next.



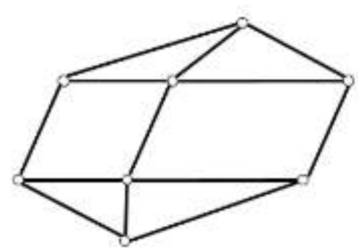


(b) Let us count for this too.

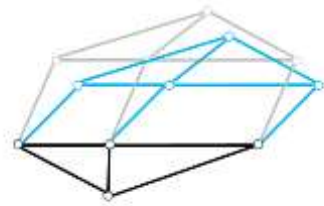
Number of bars = $b = 13$

Number of vertices = $v = 8$

So, $2v-3-b=16-3-13=0=DoF-SoSS=0-0$

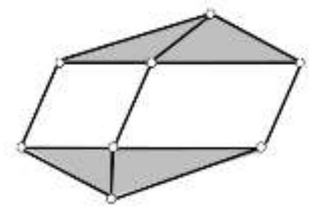


If we look at the connectivity of the truss, it looks to be satisfying Maxwell's rule and has zero DoF and SoFF. But, in reality, you can see that the two parallelograms collapse in tandem, as shown here in cyan lines.

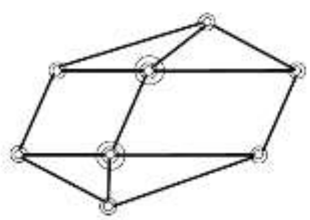


So, it has one hidden DoF because of special geometry. This is a well-known example in kinematics when you learnt about Grübler's formula for computing DoF.

As per Grübler's formula, $DoF = 3(n-1) - 2j_1 = 3(5-1) - 2(6) = 0$ if notice that the top and bottom triangles are rigid and hence they are to be treated as ternary bodies (i.e., bodies connected to two other bodies).



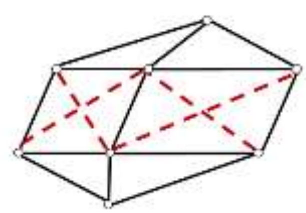
Alternatively, even we look at the bars as they are (instead of imagining ternary or triangular bodes), we still get the same result. But now we need to count the joints properly. That is if three bars share a point, we need to have two joints there (a double joint shown with two circles). Likewise, if four bars share a point, we need to have three joints (a triple joint shown with three circles). Therefore, we have



$$DoF = 3(n-1) - 2j_1 = 3(13-1) - 2(18) = 0.$$

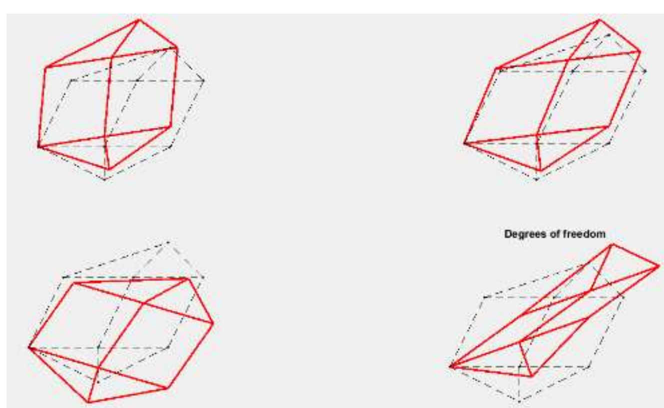
So, Grübler's formula too fails to detect the hidden degree of freedom. This means that we need to look at the rank deficiency of the compatibility matrix. That is always the fool-proof approach.

To restrict this hidden DoF (which is due to special geometry of the parallelograms in tandem), we can add an extra bar as anyone of the four dashed bars indicated in the adjacent figure. Of course, it does not satisfy Maxwell's rule or even Maxwell's rule as modified by Calladine.

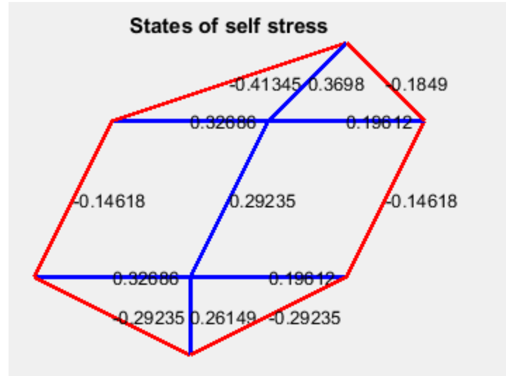


Compatibility matrix approach

The size of the **C** (compatibility matrix) here is 13×16. The rank is 12. So, it has a rank deficiency of 4. So, the null(C) gives four mode shapes. Out of which three are rigid-body modes. The fourth one shows the collapsing of the parallelograms. See adjacent figures.

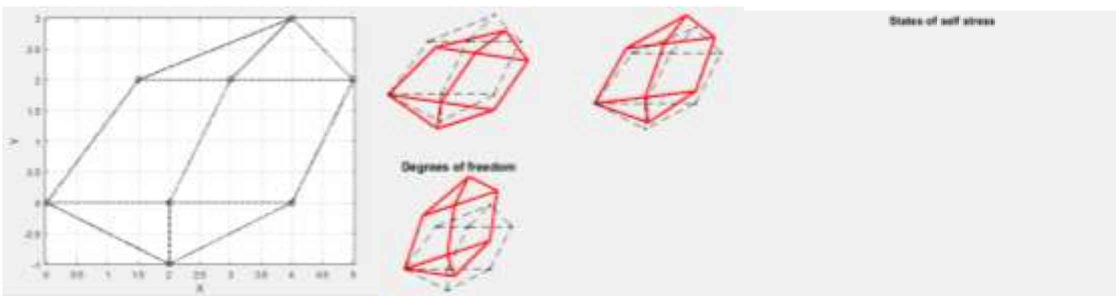


This also has one SoSS mode as can be seen in the next figure that has numbers next to each truss member.

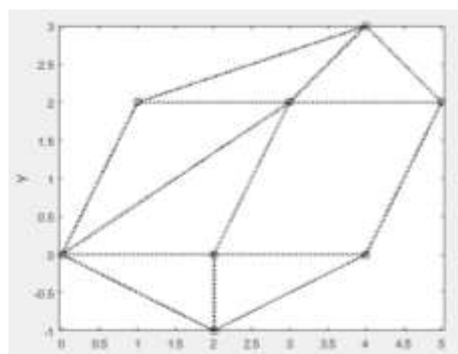


How to remove DoF?

One of the two ways to reduce rank-deficiency of the compatibility matrix is to shift one of the nodes to make one of the parallelograms a quadrilateral. Here one node is shifted to make the rank of **C** equal to 13. Now, only rigid-body modes remain in null(C), and there are no states of self-stress as can be seen in figures that appear next. This truss satisfies Maxwell's rule perfectly.

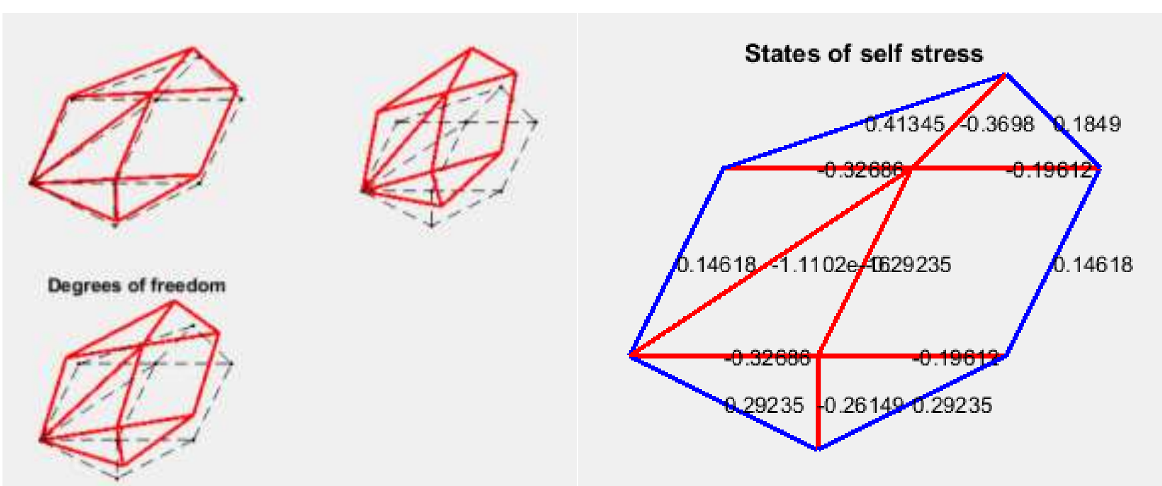


The other way is to reduce the rank-deficiency of the compatibility matrix is to add an extra bar in one of the parallelograms. Now, we have the truss shown in the adjacent figure. For this truss, Maxwell-Calladine rule gives:

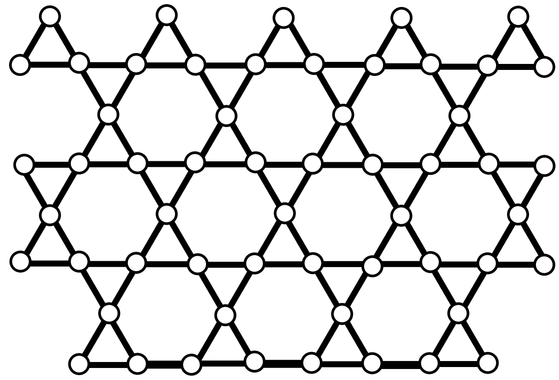


$$2v-3-b=16-3-14=-1=DoF-SoSS=0-1.$$

From the figures of this we see that there are no DoF but there is one SoSS.



(c) The truss configuration considered here is known as a Kagome truss. It is usually infinitely large but in practice it is finitely sized and hence is truncated as shown in the adjacent figure. By applying Maxwell's rule, we get



$b = 96$

$v = 56$

$2v - 3 - b = 112 - 3 - 96 = 13 = DoF - SoSS = ? - ?$

This is too complicated to see intuitively if there are any SoSSs or to visualize DoF modes. All we can conclude from the Maxwell-Calladine rule is that $DoF - SoSS = 13$. We can interpret that the triangles at the top-left and top-right are free to rotate. They count fro 2 DoF. The only other intuition we have is that the six edges of the hexagons could have some DoF. If we add extra bars with the hexagons, we could certainly bring down the DoF. One easy way is to fill up the hexagons with triangles. We have 10 hexagons and each needs one extra vertex and six extra bars. So, we would then have:

$b = 96 + 60 = 156$

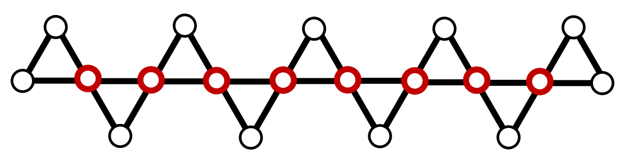
$v = 56 + 10 = 76$

$2v - 3 - b = 152 - 3 - 156 = -7 = DoF - SoSS = ? - ?$

This is not conclusive in terms of splitting DoF and SoSS. Two triangles at the top-right and top-left still have one DoF. Therefore, we need to analyze it by taking sections of this truss.

More importantly, like in part (b), this truss has special geometry. Therefore, it can have hidden DoF (hidden SoFF too).

Simple case 1



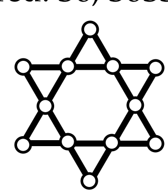
$b = 27$

$v = 19$

$2v - 3 - b = 38 - 3 - 27 = 8 = DoF - SoSS = 8 - 0$

This is really clear here that each triangle can move related to its neighbors. So, we have 8 DoF with nine triangles joined with 8 joints that are colored in red. So, SoSS = 0.

Simple case 2



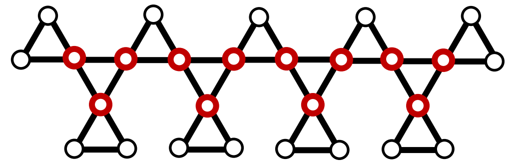
$b = 18$

$v = 12$

$2v - 3 - b = 24 - 3 - 18 = 3 = DoF - SoSS = 3 - 0$

This is basically a six-bar linkage. So, it gives 3 DoF. So, SoSS = 0.

Simple case 3



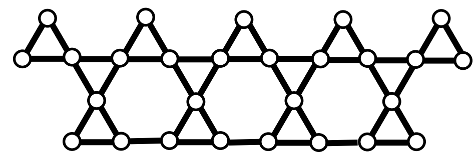
$b = 39$

$v = 27$

$2v - 3 - b = 54 - 3 - 39 = 12 = DoF - SoSS = 12 - 0$

It is easy to see that there is one rotation possible about each red-colored joint. So, there are 12 DoF. As a result, SoSS = 0.

Nor-so-simple case 1



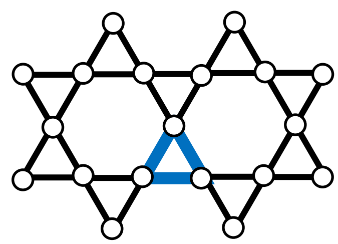
$b = 42$

$v = 27$

$2v - 3 - b = 54 - 3 - 42 = 9 = DoF - SoSS = ? - ?$

Apart from the rotations of the two triangles at the top-left and top-right, the rest of the DoF are not obvious. But see that three bars added here as compared to the previous case has reduced 3 DoF.

Simple case 4



$b = 30$

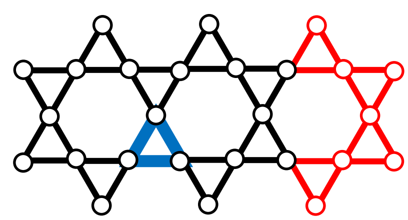
$v = 19$

$2v - 3 - b = 38 - 3 - 30 = 5 = DoF - SoSS = 5 - 0$

It is easy to interpret the 5 DoF here. Imagine that the blue-colored triangle is fixed. Then, the triangle above it can be rotated relative to the blue triangle. Then, the two hexagons will be left with two DoF each. So, connecting two star configurations reduces 1 DoF even though individually, they would have 3 DoF each (as per Simple Case 2).

Simple case 5

If we connect three star configurations, we would have



$b = 30 + 12 = 42$

$v = 19 + 7 = 26$

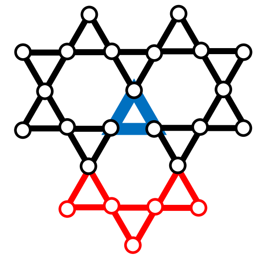
$2v - 3 - b = 52 - 3 - 42 = 7 = DoF - SoSS = 7 - 0$

We had 5 DoF in Simple Case 4. Now, with the new hexagon, we will have 2 more DoF for the red portion, totaling to 7. As we concatenate more star configurations in any direction, we will have 2 extra DoF added.

Let now return to the aforementioned not-so-simple case 1. We can see that by adding two triangle at top-left and top-right to the Simple Case 5, we get the not-so-simple case 1. So, we can now interpret all 9 DoF.

We still have not interpreted how SoSSs come about here. Consider another case.

Simple case 6



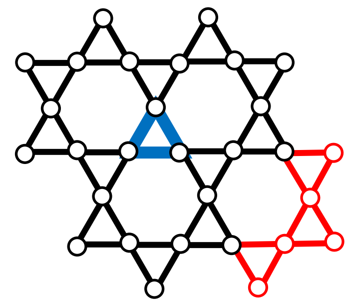
$b = 39$

$v = 24$

$2v-3-b = 48-3-39 = 6 = DoF - SoSS = 6-0$

Here, once we fix the blue triangle, the three hexagons would have two DoF each, totaling to 6 DoF. There are no SoFF.

Simple case 7



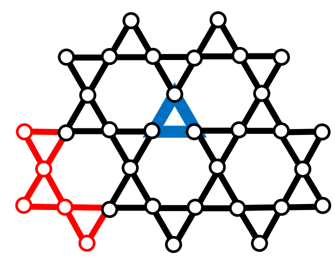
$b = 39+9 = 48$

$v = 24+5 = 29$

$2v-3-b = 58-3-48 = 7 = DoF - SoSS = 7-0$

Here, once we exercise 6 Dof of the preceding case, the red part in the adjacent figure would have the seventh DoF. Still, there are no SoFF.

Simple case 8



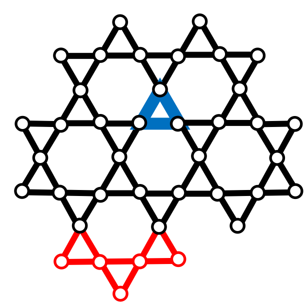
$b = 48+9 = 57$

$v = 29+5 = 34$

$2v-3-b = 68-3-57 = 8 = DoF - SoSS = 8-0$

We now see that with one more hexagon completed, we have only one extra DoF.

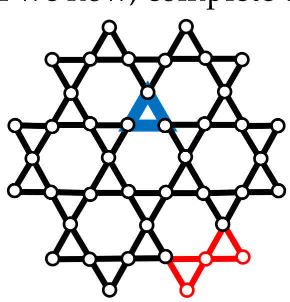
Simple case 9



With one more hexagon completed, we would get

$2v-3-b = 78-3-66 = 9 = DoF - SoSS = 9-0$

If we now, complete the picture by adding the seventh hexagon,

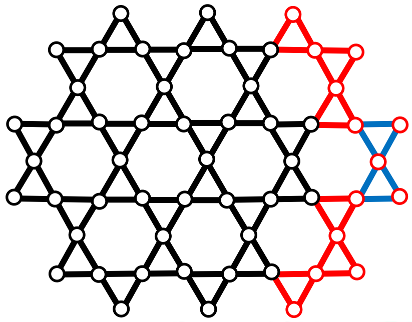


we get

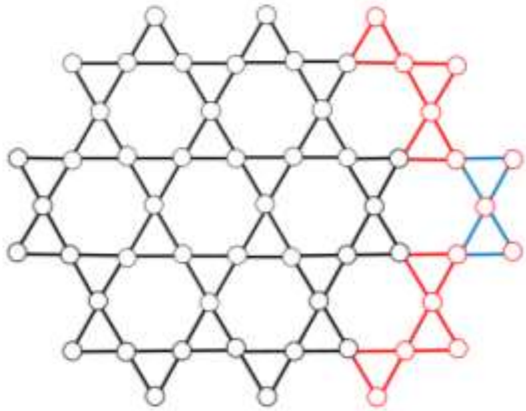
$2v-3-b = 84-3-72 = 9 = DoF - SoSS = 9-0$

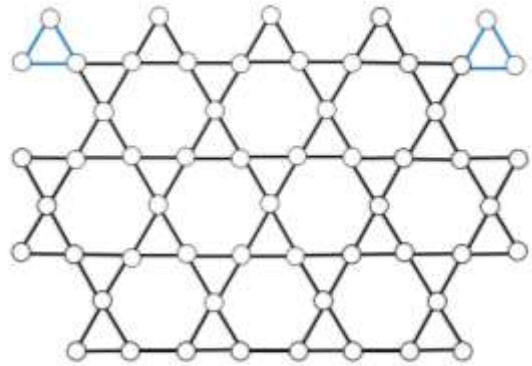
with no more DoF added.

By adding two more three-triangle pieces (red) and one two-tringle piece, we raise DoF to 11 because red ones add one DoF each and blue one none. See the next figure.



Now, we remove two edges and one vertex from the three triangles at the bottom (which does not change DoF as per Maxwell's formula. So, we would have 11 DoF and SoSS. Having confirmed, we add two triangles (i.e., six bars and four vertices, adding 2 DoF) to get what we needed for this problem. See the next two figures. Thus, we confirm that our truncated Kagome truss has 13 DoF.



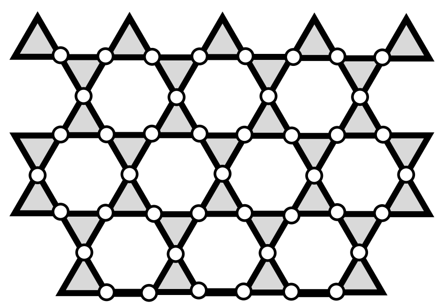


Is the special geometry playing any role here? Will there be more DoF and SoSS if we use the rank-deficiency method? Please find out. Also explore what happens if there is no special geometry. That is, what if none of the triangles are equilateral? Explore.

Check this too:

What if we apply Grübler's formula?

There are 31 triangles and three bars in the bottom row. And there are 43 hinges.



$$n = 34;\ j_1 = 43$$

$$DoF = 3(n-1) - 2j_1 = 3(35-1) - 2(43) = 13$$

So, Grübler's formula too predicts the same DoF. And this it always true. You can check it out with other examples. Just make sure that you count the joints and bodies correctly.