ME 260: Structural Optimization: Size, Shape, and Topology		
Assigned: Nov. 10, 2026	Homework 6	Due: Nov. 17, 2026

Problem 1 (30 points) "Deriving sensitivity using adjoint method in a multiphysics problem"



Consider the *heatuator* (portmanteaux of heat and actuator) shown here. The long narrow beam has a length of 500 μm and a width of 2 μm. The short narrow beam below it has a length of 50 μm and width of 2 μm. The wide arm has a length of 450 μm and width of 25 μm. The gap between the top and bottom arms is 5 μm. The short connector between the top and bottom arms is 5 μm long and 3 μm wide. The thickness is uniform throughout and is equal to 2 μm. Assume that it is made of polycrystalline silicon that is suitably doped.

When we apply a DC voltage between the two anchored pads, it bends down as shown. This is because the narrow beam at the top expands more than the wide beam at the bottom due to higher current density and higher ensuing heat. This problem requires us to solve governing equations in three energy domains, namely, electrical conduction, heat conduction, and elastic deformation. Let us consider them in the weak form.

(1) $t\int_\Omega \sigma_e \nabla V \cdot \nabla V_v \, d\Omega = 0$ with Dirichlet boundary condition at the end of the beams that connect to the anchored pads between which electric potential of 2 V is applied. Here, t is the uniform thickness of the heatuator, $V(x,y)$ is the voltage at different points in the domain, Ω, V_v is the weak variable corresponding to the state variable, V. And, $\sigma_e = \rho$ 3×10⁴ S/m is electrical conductivity where the fictitious density $\rho = 1$.

(2) $t\int_\Omega (k_t \nabla T \cdot \nabla T_v - \sigma_e \nabla V \cdot \nabla V \, \delta T) \, d\Omega = 0$ where the second term in the integrand is Joule heating; with Dirichlet boundary condition at the ends of the beams that connect to the anchored pads that are maintained at $T_0 = 300$ K. Here, $T(x,y)$ and $T_v(x,y)$ are the temperature state variable and its corresponding weak variable, respectively. And $k_t = \rho$ 30 W/(m K).

(3) $t\int_\Omega \boldsymbol{\epsilon}_v^T \boldsymbol{D} \boldsymbol{\epsilon}_u \, d\Omega - v_y = 0$ with Dirichlet boundary condition at the ends of the beams that connect to the anchored pads with $u_x = u_y = v_x = v_y = 0$ for $\boldsymbol{u}$ and $\boldsymbol{v}$, which are the displacement vector and its weak variable for a unit virtual load applied at the lower-right corner of the wide beam. Here, $\boldsymbol{D}$ is the 3×3 matrix that relates stress and strain of an isotropic material. Note also that

$\boldsymbol{\epsilon}_u = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ and $\boldsymbol{\epsilon}_v = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ are strains due to the displacement and its weak variable. For $\boldsymbol{D}$, take Young's modulus = ρ 158 GPa and Poisson ratio = 0.2 ρ.

By using Maizel's theorem, we can write the y-displacement, u_y, of the lower-right corner of the wide beam (where the unit virtual load is applied), denoted by ψ as

$$\psi = t \int_\Omega \boldsymbol{\epsilon}_v^T \boldsymbol{D} \boldsymbol{\epsilon}_{th} \, d\Omega$$

Here, $\boldsymbol{\epsilon}_{th} = \alpha(T - T_0)\begin{Bmatrix}1\\1\\0\end{Bmatrix}$ is the thermal due to heating. And, α = ρ 2.8×10^{-6} K^{-1} is the thermal expansion coefficient.

Now do the following:

a) Obtain the sensitivity of ψ w.r.t. ρ as an analytical expression using the adjoint method.
b) Simulate the heatuator in COMSOL Multiphysics and plot the sensitivity of ψ. Where is it the smallest and where is it the largest?

Problem 2 (20 points) "Computing and validating sensitivity using COMSOL or some other code"

Using the finite-difference method, verify your analytical expression for ψ at the points where it is the smallest and the largest, and some other third point of your choice. Note that you need to do this y varying $\Delta\rho$ at that point over a large range of values to see "blow-up" and stable regions.

What to submit:

Pictures of your COMSOL solution of electro-thermal-elastic analysis, showing

(i) voltage distribution,
(ii) temperature distribution,
(iii) deformed heatuator,
(iv) plot of ψ as a surface plot on the heatuator,
(v) three plots of sensitivity w.r.t. $\Delta\rho$.

and COMSOL file saved as <yourname>.mph,

and a hand-written derivation for Problem 1 and our interpretation of the results of Problems 1 and 2.

Bonus points 20 if you find the sensitivity of maximum temperature w.r.t. ρ at any point (x, y) in the domain.