ME261 Differential Equations: 1D/ first order ODEs

Koushik Viswanathan*

Dept. of Mechanical Engineering, IISc, Bangalore

December 7, 2020

NOTE: In these notes, I've used $x(t)$ in place of $y(t)$ to denote the dependent variable. Please make suitable changes to your own notes while working things out. t is always used to denote the independent variable.

Our study of 1D or first order ODEs will set the stage for studying higher-order, nonlinear equations. Consider the following prototypical first order ODE system (IVP)

$$\frac{dx}{dt} = f(x,t) \qquad\qquad x(t_0) = x_0 \tag{1}$$

In general, this system is not solvable in closed form so that we will take one of two approaches—either we will solve explicitly for $x(t; x_0)$ for certain special forms of $f(x,t)$ or we will semi-quantitatively study the behaviour of the solution without actually solving the differential equation.

1 Solutions for special forms of $f(x,t)$

When the RHS $f(x,t)$ takes on certain forms, the first order IVP becomes easier to handle. Here are some examples

1.1 Separable: $f(x,t) = g(x)h(t)$

When the RHS is of the separable form, we can write down a formal solution as follows:

$$\int \frac{dx}{g(x)} = \int h(t)dt + C \tag{2}$$

where the constant C is determined from the initial conditions. This expression is often implicit, i.e., cannot be inverted to obtain an expression $x(t)$, even if the integral on the LHS can be performed in closed form. Several examples may be written down at once—for instance $\frac{dx}{dt} = \frac{xt}{\sqrt{1+x^2}}$ etc.

*Email: koushik@iisc.ac.in

1.2 Homogeneous: $f(x,t) \equiv \tilde{f}\left(\frac{x}{t}\right)$

When the dependence of f on x, t is of the form $f(x,t) = \tilde{f}(\frac{x}{t})$, the ODE may be simplified by making the substitution $y = \frac{x}{t}$ to obtain

$$\frac{dx}{dt} = t\frac{dy}{dt} + y \qquad \Longrightarrow \quad \frac{dy}{dt} = \frac{\tilde{f}(y) - y}{t} \tag{3}$$

which is of the separable form. Now the solution may be written down again in the form of an integral as

$$\int \frac{dy}{\tilde{f}(y) - y} = \int \frac{dt}{t} + C \qquad \text{for } y = \frac{x}{t} \tag{4}$$

Just as in the previous case, even if the LHS can be integrated in closed form, the solution $x(t)$ is obtained in implicit form.

1.3 Linear: $f(x,t) = -p(t)x + q(t)$

This is the easiest of all first order ODEs because it can be solved in explicit form. There are two ways to go about this—either using an integrating factor or using variation of parameters. We will illustrate the former. First, multiply both sides of the ODE by some well-behaved function $r(t)$ to obtain

$$r(t)\frac{dx}{dt} + r(t)p(t)x = r(t)q(t) \tag{5}$$

If we choose a function $r(t)$ such that the LHS of this ODE is reduced to an exact differential of the form $\frac{d}{dt}\left(r(t)x(t)\right)$, then we can make the identification that

$$\frac{dr}{dt} = p(t)r \qquad \Longrightarrow \quad r(t) = e^{\int p(t)dt} \tag{6}$$

This $r(t)$ is called the *integrating factor*. Consequently, the original differential equation now becomes

$$\frac{d\left(e^{\int p(t)dt}x(t)\right)}{dt} = e^{\int p(t)dt}q(t) \quad \Longrightarrow \quad x(t) = e^{-\int p(t)dt}\left(\int dt\ e^{\int p(t)dt}q(t) + C\right) \tag{7}$$

Notice the constant C now appears along with a $t-$dependent term $e^{-\int p(t)dt}$ so that one must be careful in applying the initial conditions.

1.4 Substitutional techniques

There are several additional special substitutional techniques that one can discuss vis-á-vis first order ODEs. These usually intricately depend on the nature of the RHS $f(x,t)$. One prominent example is the equation

$$\frac{dx}{dt} + p(t)x = q(t)x^n \tag{8}$$

which is called the Bernoulli equation. By substituting $y = x^{1-n}$, this ODE can be reduced to the following linear ODE in y

$$\frac{dy}{dt} + (1-n)p(t)y = (1-n)q(t) \tag{9}$$

which can again be solved using an integrating factor.

Several ODEs can be reduced to one of these forms described above by making suitable change of variables. However, this can often be a hit-or-miss procedure, in which case one often has to resort to numerical solution (via quadrature) or semi-quantitative techniques.

2 Phase space analysis

When the ODE IVP is not easily solved via integration, we resort to semi-quantitative techniques to obtain information about the solution. This is done using the construct of phase space discussed earlier. For the time being, we will restrict ourselves to autonomous 1D ODEs. The non-autonomous case can be treated using analogous techniques when we study second-order systems.

Phase space for a 1D/ first order system is just a line, corresponding to the only independent variable $x(t)$. Furthermore, we define the trajectory taken by the solution curve to an ODE problem as the *integral curve* or *flow* in phase space. The fixed points x^* are determined by the relation $f(x^*) = 0$. These points, being dependent on the form of $f(x)$ alone (and not the initial conditions $x(t_0) = x_0$), determine the behaviour of *all* solution curves for first order ODEs.

If the function $f(x)$ is linear, i.e., $f(x) = px + q$, then there is only one fixed point $x^* = -q/p$. Consequently, any solution curve or trajectory that starts at x_0 near $x^* = -q/p$ must either come towards the fixed point or go away from it, depending on the sign of $f(x)$ at that point. As an example, see Fig. 1(left). The phase space is the horizontal axis, and the function $f(x)$ is shown on the vertical axis for convenience. Consider the only fixed point x^* in this system. For all values of x to the right of x^*, $f(x) > 0$ so that $\frac{dx}{dt} > 0$ and x should increase with t. Hence any trajectory that starts to the right of x^* should go away to $+\infty$. Likewise, for a trajectory starting to the left of x^* in the figure, $f(x) < 0$ always so that all trajectories to the left must eventually go away to $-\infty$. That this is also quantitatively true may be checked at once from the solution of the ODE $x(t) = -\frac{q}{p} + x_0 \exp(pt)$. Hence all trajectories

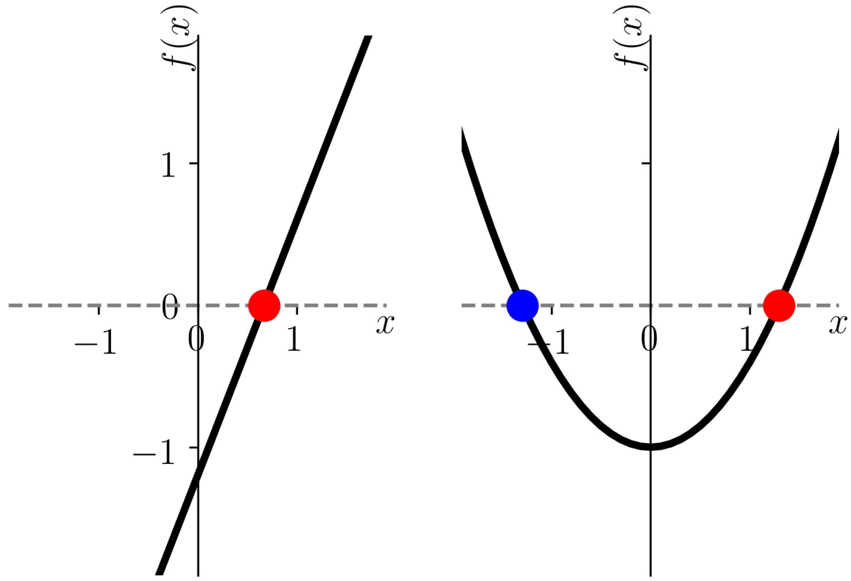

Figure 1: Examples of 1D trajectories or flows. Phase space is the x axis and the sign of $f(x)$ determines the direction that the trajectory takes.

that start 'sufficiently close' to x^* diverge away from it. This fixed point is consequently an *unstable fixed point*.

On the other hand, if $f(x)$ is non-linear, multiple solutions x^* exist for the algebraic equation $f(x^*) = 0$. An example is shown in Fig. 1(right). In this case, there are two fixed points, marked red and blue. The system's behaviour near the red fixed point is the same as in the linear case—any trajectories starting sufficiently close to the red fixed point tend to diverge away. However, the opposite happens in the case of the blue fixed point. Here, any trajectory that starts to its right (between the two fixed points) has $f(x) < 0$ so that $x(t)$ decreases with t. Similarly, for trajectories to the left of the blue fixed point, $f(x) > 0$ so all trajectories should have $x(t)$ increasing with t and eventually approach x^*. Hence the blue fixed point is a *stable fixed point* in that it tends to 'attract' all nearby trajectories.

3 Stability and Linearization

In order to make general statements about trajectories close to a fixed point, we must first define what we mean by 'stability'. Two definitions are often used:

Lyapunov stability: For a fixed point x^*, if we start a trajectory at x_0 such that $|x_0 - x^*| < \delta$ for some small $\delta > 0$, then the fixed point x^* is said to be *Lyapunov stable* if $\forall t > 0$, $|x(t) - x^*| < \epsilon$ for some small $\epsilon > 0$. Lyapunov stability only guarantees that any trajectory that starts sufficiently close to a fixed point stays sufficiently close to it forever.

Absolute stability: For a fixed point x^*, if we start a trajectory at x_0 such that $|x_0 - x^*| < \delta$ for some small $\delta > 0$, then the fixed point x^* is said to be *absolutely stable* if for $t \to \infty$, $x(t) \to x^*$. Absolute stability of a fixed point is clearly a more stringent condition than Lyapunov stability. For first order systems, it is easy to see that absolute stability implies Lyapunov stability.

In order to obtain a quantitative measure of stability, we will resort to a technique called *linearization* about the fixed point x^*. Consider a fixed point x^* and let $\eta(t) = x(t) - x^*$ denote a trajectory that is near x^*. Then

$$\frac{d\eta}{dt} = \frac{dx}{dt} = f(x^* + \eta) \tag{10}$$

by expanding the RHS in a Taylor series, and using the fact that $f(x^*) = 0$, $\eta(t)$ obeys the resulting *linearized* equation:

$$\frac{d\eta}{dt} = \eta f'(x^*) + O(\eta^2) \tag{11}$$

where f' denotes df/dx, evaluated at x^*.

The linear Eq. (11) describes the flow in the vicinity of x^*, irrespective of the starting point x_0. It is clear from this that the direction taken by the trajectory is determined by $f'(x^*)$—if $f'(x^*) < 0$, the fixed point x^* is a *stable node* This is the case when all trajectories that are 'near' x^* eventually end up *at* x^*. On the other hand, if $f'(x^*) > 0$, all trajectories starting near x^* eventually go far away from it (unless there are other x^* values nearby which attract the trajectory). In such a case, the fixed point x^* is said to be an *unstable node*

In certain special cases, one may have $f'(x^*) = 0$, which can happen when the fixed point x^* is a saddle node. Visually, this means that $f(x)$ has a minimum at $x = x^*$ simultaneous with $f(x^*) = 0$. Consequently, from one direction, the fixed point is stable $(f'(x) < 0)$ and from another direction it is unstable $(f'(x) > 0)$. There are other situations where x^* is not a saddle but $\frac{df}{dx} = 0$. In these situations, linearization fails only at these points and one must resort to other techniques.

Hence by linearization of the governing ODE near a fixed point x^*, we can determine the stability of the fixed point for trajectories that start sufficiently close to it. This is a very useful technique in understanding the behaviour of nonlinear order ODEs and we will often take recourse to it.

4 Bifurcations

Once the system Eq. (1) is specified, along with x_0, the features of interest of the system are completely determined. However, an interesting situation arises when *families* of such equations are considered instead. For instance, let us consider the case when the function $f(x)$ depends on a real–valued parameter λ. We denote this dependence explicitly as $f(x; \lambda)$. In general, as the value of λ changes, the fixed points x^* can be expected to alter their position. It is possible that at some value $\lambda = \lambda_c$, the fixed points can change stability or even cease to exist altogether. At such a value of λ, a *bifurcation* is said to have occurred. A necessary condition for bifurcations in 1D systems is:

$$\frac{df}{dx}_{|x^*, \lambda_C} = 0 \tag{12}$$

Please note that this condition does not hold for 2D and higher order systems.
Different types of bifurcations exist and some of them will be enumerated shortly, but it must first be mentioned that nonlinearity is an essential condition for them to occur—for linear $f(x) = px + q$, change in stability is often trivial since it only depends on the sign of p and there is only one fixed point $x^* = -q/p$.

4.1 A nonlinear example

Consider the nonlinear example shown in Fig. 2 for a $f(x; \lambda) = \lambda + x^2$. When $\lambda > \lambda_C$, $f(x)$ lies entirely above the horizontal axis so that there are no fixed points at all, see black curve in the figure. As λ is reduced, the function $f(x)$ changes and at a critical value $\lambda = \lambda_C = 0$, a single fixed point appears at $x^* = 0$. For all $\lambda < \lambda_C$, two distinct fixed points occur at $x^* = \pm\sqrt{-\lambda}$, see red curve in the figure. This is a typical example of a bifurcation, with λ_C acting as a *bifurcation point*.

4.2 Saddle-node bifurcation

Another way to visualize this bifurcation is to plot the fixed point x^* as a function of λ, in what is called a *bifurcation diagram*, see Fig. 3. In the figure, a solid line corresponds to an stable fixed point while a dashed line depicts the position of an unstable fixed point. This stability is determined using linearization by evaluating $f'(x)$ at the fixed points x^*.
By just observing the values of x^*, one can conclude that as λ varies, the two fixed points 'come towards each other and annihilate'. This is called a *saddle node bifurcation*. In a saddle node bifurcation, fixed points are either created or destroyed as the parameter λ is varied.

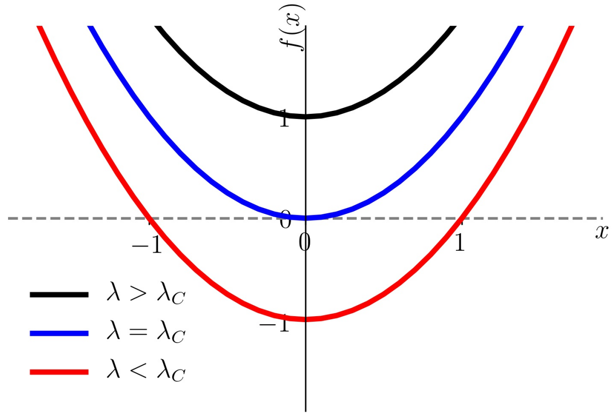

Figure 2: Variation in $f(x)$ with change in λ

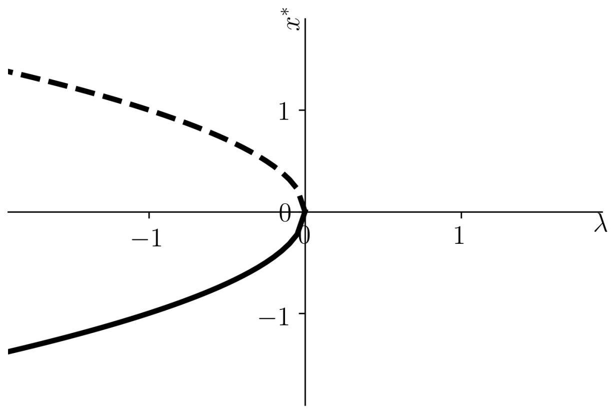

Figure 3: Change in fixed point x^* vs. parameter λ for a saddle–node bifurcation.

The form of the function in the vicinity of a node or critical point can be obtained from a Taylor series expansion and is called the corresponding *normal form* for the function near the critical point. By expanding the function $f(x;\lambda)$ in both x and λ near a fixed point x^* and near a bifurcation point λ_C, one obtains

$$f(x;\lambda) = f(x^*) + \frac{\partial f}{\partial x}_{|(x^*,\lambda_C)}(x - x^*) + \frac{\partial f}{\partial \lambda}_{|(x^*,\lambda_C)}(\lambda - \lambda_C) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}_{|(x^*,\lambda_C)}(x - x^*)^2 + \cdots \quad (13)$$

At a saddle node bifurcation, both $f(x) = 0$ and $f'(x) = 0$ simultaneously so that any function that shows such a bifurcation necessarily has the form:

$$f(x) = a(\lambda - \lambda_C) + b(x - x^*)^2 \quad (14)$$

near (x^*, λ_C), with a, b corresponding to $\frac{\partial f}{\partial \lambda}, \frac{\partial^2 f}{\partial x^2}$, respectively.

As an example, consider the ODE

$$\frac{dx}{dt} = \lambda - x - e^{-x} \tag{15}$$

the fixed points x^* obey $\lambda - x^* = e^{-x^*}$ which may be graphically obtained by plotting the functions $f_1(x) = \lambda - x$ and $f_2(x) = e^{-x}$ and seeing where they intersect. This function is

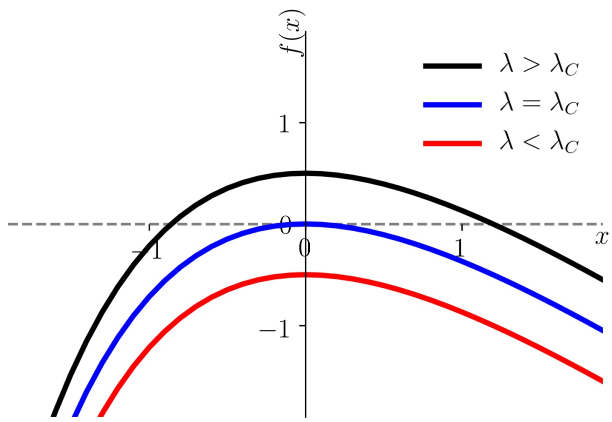

Figure 4: Variation in $f(x) = \lambda - x - \exp(-x)$ with λ

shown in Fig. 4 for three different λ. At $\lambda = \lambda_C = 1$, $f_1(x) = \lambda - x$ becomes tangent to $f_2(x) = \exp(-x)$ so that only one fixed point exists.
This conclusion can be put on firmer ground by expanding $\exp(-x)$ for small x. This gives

$$f(x) \simeq \lambda - 1 - x^2 \tag{16}$$

which is the same as the normal form for a saddle node bifurcation in Eq. 14.

4.3 Transcritical bifurcation

In addition to saddle-node bifurcations, first order nonlinear ODEs can also exhibit other types of bifurcations. These have their own normal forms, and any $f(x)$ that exhibits such bifurcations can be brought into the corresponding normal form, just as we did in the saddle node case.
Consider the ODE

$$\frac{dx}{dt} = \lambda x - x^2 \tag{17}$$

which has two fixed points $x^* = 0$ and $x^* = \lambda$. The bevahiour of this $f(x)$ is shown in Fig. 5 Just as in the saddle node case, we now see that there are two fixed points for $\lambda < \lambda_C$. However, now these exist for all values of λ, the only change being their stability. It can be

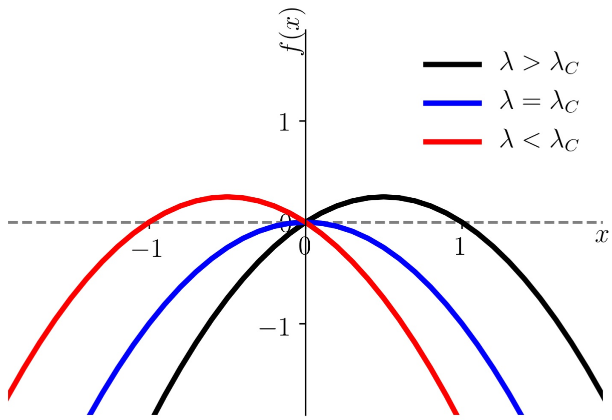

Figure 5: Variation in $f(x) = \lambda x - x^2$ with λ

checked from linearization of $f(x)$ that the fixed point $x^* = 0$ is stable for $\lambda < \lambda_C = 0$ and becomes unstable for other λ. On the contrary, the point $x^* = \lambda$ is unstable for $\lambda < \lambda_C$ and stable otherwise. The value $\lambda_C = 0$ is known as a *transcritical bifurcation* point.

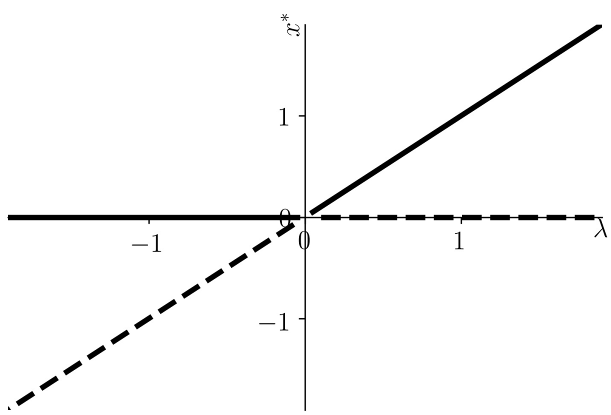

Figure 6: Variation in $f(x) = \lambda x - x^2$ with λ

The corresponding bifurcation diagram for a transcritical bifurcation is shown in Fig. 6. Again, the stable branches are marked in solid lines and the unstable ones using dashed lines. If we look solely at the branches, we can see that near a transcritical bifurcation point, two fixed points of opposing stability approach each other. At λ_C, they ‘contact’ each other and ‘exchange’ stabilities. This is typical for a transcritical bifurcation.

4.4 Pitchfork bifurcation

A third type of bifurcation in 1D systems occurs often when the ODE has a symmetry. Unlike the transcritical type, this bifurcation involves the appearance of two fixed points (both stable or unstable) when the parameter λ is changed. Two cases must be distinguished

Supercritical case

Consider the normal form:

$$\frac{dx}{dt} = \lambda x - x^3 \tag{18}$$

where the RHS is as in Fig. 7 for three different values of λ. First notice that $x^* = 0$ is always a fixed point $\forall\lambda$. When $\lambda \leq 0$, this is the only fixed point possible. It can be seen by linearization that this fixed point is stable for $\lambda < 0$ and unstable for $\lambda > 0$. However, at $\lambda = \lambda_C = 0$, a change occurs. Firstly, the function $f(x^*)$ and $\frac{df}{dx}$ both become zero. Secondly, two additional fixed points become possible for small $\lambda > \lambda_C$, and they occur at $x^* = \pm\sqrt{\lambda}$. For $\lambda = 1$, these are shown in Fig. 7 (black curve). Their stability is again determined by linearlization, and both fixed points are stable.

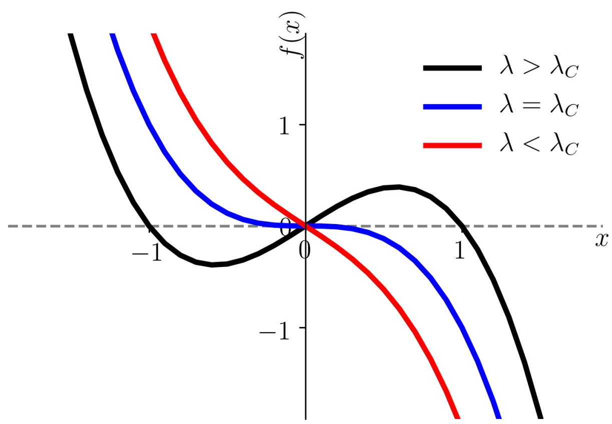

Figure 7: Variation in $f(x) = \lambda x - x^3$ with change in λ

The corresponding bifurcation diagram is shown in Fig. 8. Firstly, the fixed point $x^* = 0$ changes stability at $\lambda = \lambda_C = 0$. Additionally as λ crosses λ_C from below, two new fixed points, both stable, appear alongside $x^* = 0$. This type of bifurcation is called a *supercritical pitchfork bifurcation*, the name being derived from its obvious shape in Fig. 8. The term supercritical indicates that this type of bifurcation does not show any hysteresis. We will understand the full meaning of this term when we study the subcritical case next.

One other feature of the supercritical pitchfork is the fact that $\frac{df}{dx} = 0$ at the critical point λ_C. However, contrary to the saddle node case, the point $x^* = 0$ does not become a saddle here but retains its stability (recall that $x^* = 0$ is stable for $\lambda < \lambda_C$). However, this problem demonstrates a shortcoming of the linearization technique in that it does not provide a necessary condition for the stability of fixed points. Furthermore, it is worth noting that in the normal form Eq. 18, the x^3 term 'stabilizes' the linear term, for it is this term that determines the stability of $x^* = \pm\sqrt{\lambda}$

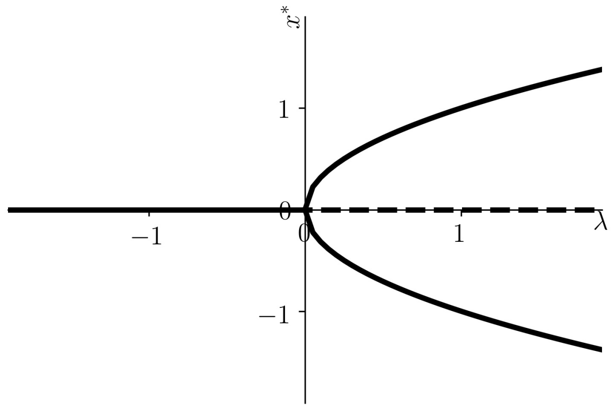

Figure 8: Bifurcation diagram for supercritical pitchfork

Subcritical case

In contrast to the supercritical case, a different type of pitchfork bifurcation occurs when the ODE has the following normal form

$$\frac{dx}{dt} = \lambda x + x^3 \tag{19}$$

The corresponding behaviour of the RHS is shown in Fig. 9. This is a mirror image of Fig. 7. Again, $x^* = 0$ is a fixed point $\forall\lambda$, and its stability is the same as in the supercritical case. This may be checked easily by linearization. However, now this is the only fixed point for $\lambda > 0$ so that at $\lambda < 0$, two additional fixed points appear at $x^* = \pm\sqrt{-\lambda}$. In contrast to the supercritical case, both these fixed points are now unstable.

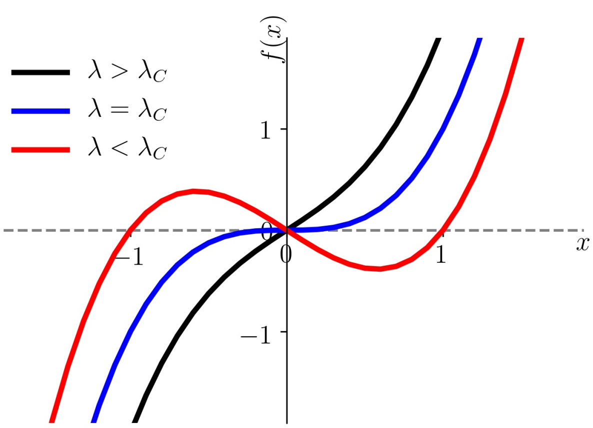

Figure 9: Variation in $f(x) = \lambda x + x^3$ with change in λ

The bifurcation diagram corresponding to this normal form in shown in Fig. 10. Firstly, the location of the symmetric fixed points $\pm\sqrt{-\lambda}$ has changed compared to Fig. 8. Secondly, they are now both unstable, even though the qualitative shape of the diagram is unchanged. This type of a bifurcation at $\lambda = \lambda_C = 0$ is called a *subcritical pitchfork bifurcation.*

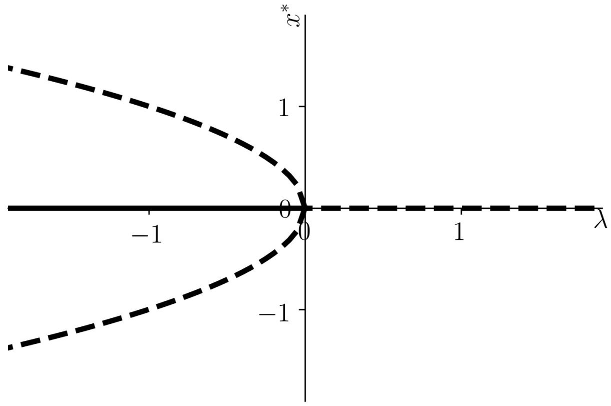

Figure 10: Bifurcation diagram for subcritical pitchfork

It will often be seen that subcritical pitchfork bifurcations have a higher order stabilizing term. When this higher order term is taken into account, a phenomenon known as hysteresis appears, giving subcritical pitchfork bifurcations their practical importance, as illustrated in the following example.

Consider the ODE:

$$\frac{df}{dx} = \lambda x + x^3 - x^5 \tag{20}$$

For $x \sim 0$, the higher order term does not contribute any fixed points and the ODE shows a subcritical pitchfork bifurcation as in Eq. 19. However, for larger $|x|$, the higher order term becomes more dominant. Evaluating the fixed points for this system gives the following solutions:

$$\begin{aligned} \lambda > 0 \quad &\implies x^* = 0, \pm\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda}\right)^{1/2} \\ \lambda = 0 \quad &\implies x^* = 0, \pm 1 \\ -1/4 \leq \lambda < 0 \quad &\implies x^* = 0, \pm\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda}\right)^{1/2} \\ \lambda < -1/4 \quad &\implies x^* = 0 \end{aligned} \tag{21}$$

By linearization, one again obtains stability criteria for each of these fixed points, from which the bifurcation diagram in Fig. 11 may be derived.

This shows a subcritical pitchfork bifurcation at $\lambda = \lambda_C = 0$, However, there are two distinct differences. Firstly, for $\lambda > 0$, apart from the unstable $x^* = 0$, there are two additional stable

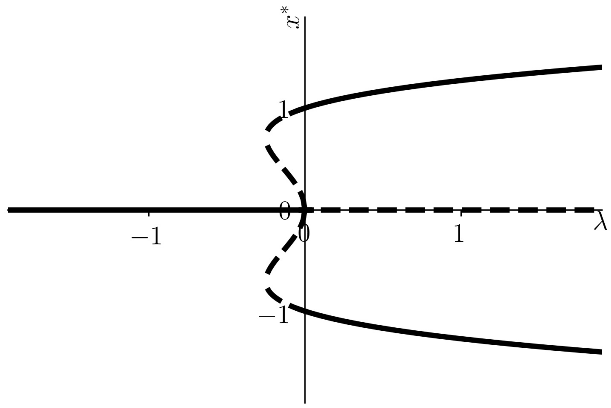

Figure 11: Bifurcation diagram for subcritical pitchfork

fixed points far awar from $x^* = 0$. Secondly, as we lower $\lambda \to \lambda_C$, these two stable points approach each other but they do not meet even at $\lambda = \lambda_C$. At $\lambda = \lambda_C = 0$, as subcritical pitchfork bifurcation occurs—the $x^* = 0$ fixed point becomes unstable and two additional unstable fixed points appear. All this while , the two new fixed points continue to exist without any change in their stability. Finally, as $\lambda \to \lambda'_C = -1/4$, the two pairs of fixed points (two positive and two negative) approach each other so that at $\lambda = -1/4$, *two* saddle node bifurcations occur and all four fixed points annihilate each other. For all $\lambda < -1/4$, there is only one fixed point remaining at $x^* = 0$, which remains stable.

This example illustrates two phenomena—existence of multiple types of bifurcations in the same system (here 2 saddle-node bifurcations at $\lambda = -1/4$ and a subcritical pitchfork at $\lambda = 0$), and the occurrence of hysteresis. The latter may be observed as follows. Consider a system with an initial $\lambda > 0$. If we study the nature of solutions of this system, we will observe that all solutions $x(t) \to \pm \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda^2}\right)^{1/2}$ as $t \to \infty$. These are the two outer fixed points for $\lambda > 0$ in Fig. 11. Depending on whether x_0 is positive or negative, $x(t)$ tends to the positive or negative fixed point.

As we lower λ, the solutions continue to decay to these fixed points as before, even when we cross the pitchfork bifurcation point $\lambda = \lambda_C$! If we *a priori* knew of this bifurcation point, we would expect that $x^* = 0$ should change its stability and attract all trajectories towards it, but this does not happen. The reason for this is the existence of the two unstable fixed points between the outer fixed points and $x^* = 0$ which repels any trajectories. However, when we reach $\lambda = \lambda'_C$, these unstable trajectories go away and the solution now starts decaying to $x^* = 0$ as we expect. So the switch to the stable branch $x^* = 0$ occurs not at $\lambda = 0$ as we'd expect but at $\lambda'_C = -1/4$.

On the other hand, if we went the other way, i.e., increased λ from $-\infty \to +\infty$, we would see a different picture. Solutions $x(t)$ that start close to $x^* = 0$ continue to converge to it as

$t \to \infty$ as $\lambda \to \lambda_C = 0$. For all $\lambda \geq \lambda_C$, all solutions $x(t)$ would then start diverging towards the two outer fixed points. The reverse transition, remember, occured only at $\lambda = -1/4$ in the other direction. This phenomenon is known as hysteresis and is common with all subcritical bifurcations.

5 Importance of normal forms

So far we have considered four normal forms:

1. **Saddle node**: $\frac{dx}{dt} = \lambda + x^2$
2. **Transcritical**: $\frac{dx}{dt} = \lambda x + x^2$
3. **Supercritical pitchfork**: $\frac{dx}{dt} = \lambda x - x^3$
4. **Subcritical pitchfork**: $\frac{dx}{dt} = \lambda x + x^3$

It turns out that most bifurcations in first-order nonlinear autonomous ODEs can be put in one of these forms.
For example, consider the system

$$\frac{dx}{dt} = \lambda + x + x^2 \tag{22}$$

The system has two fixed points, corresponding to $x^* = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}$. These fixed points only exist for $\lambda < 1/4$. From linearlization, we see that

$$\frac{df}{dx}_{|x^*} = \pm\sqrt{1 - 4\lambda} \tag{23}$$

It is clear from this that for $\lambda < 1/4$, one of the fixed points is stable and the other is unstable. The corresponding bifurcation diagram is shown in Fig. 12. This is again a saddle-node, with the bifurcation point ($\lambda_C = 1/4$) displaced from that for the normal form ($\lambda_C = 0$). Additionally, instead of the two fixed points meeting at a saddle at $x^* = 0$, they meet at $x^* = -1/2$.
That a saddle node bifurcation occurs at these points can be seen by reducing the ODE to the following

$$\frac{dx}{dt} = \lambda + x + x^2 = \left(\lambda - \frac{1}{4}\right) + \left(x + \frac{1}{2}\right)^2 \tag{24}$$

and when we compare this with the normal form of the saddle node, we immediately see that it is of the form $\frac{d\tilde{x}}{dt} = \lambda' + \tilde{x}^2$ with $\lambda' = \lambda - \frac{1}{4}$ and $\tilde{x} = x + \frac{1}{2}$. This explains the behaviour of the present system in terms of the normal form variant.

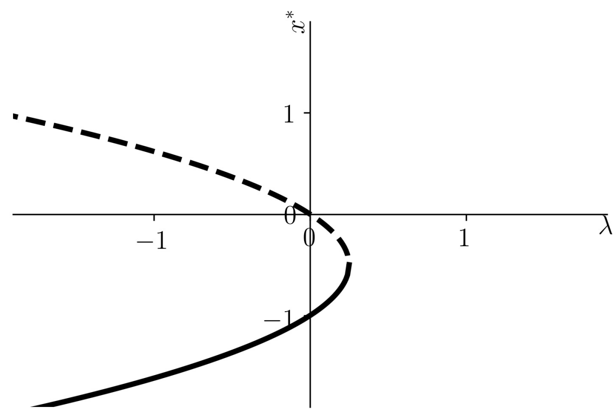

Figure 12: Bifurcation diagram for $\frac{dx}{dt} = \lambda + x + x^2$

Similar conclusions can be drawn for the other normal forms, which is why they are significant. In addition to these, there do exist other 1D bifurcations (such as imperfect pitchforks) which we will not study for the present. A very nice description is provided in the book by Strogatz [1]. These basic bifurcations will show up with a richer variety when we study second order nonlinear ODE systems.

References

[1] Strogatz SH. Nonlinear Dynamics and Chaos. CRC Press; 2018.