ME261 Differential Equations: 2D/ second order ODEs

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NOTE: In these notes, I've used $x(t)$ in place of $y(t)$ to denote the dependent variable. Please make suitable changes to your own notes while working things out. t is always used to denote the independent variable.

We now turn to second order ODEs, both linear and nonlinear. Second order systems are ubiquitous in nature and engineering—spring-mass-damper oscillators, newton's second law etc. come immediately to mind. Just as we did in the 1D or first order case, consider the following prototypical form for a second order ODE:

$$\frac{d^2x}{dt^2} = f\left(\frac{dx}{dt}, x, t\right) \qquad x(t_0) = x_0, \frac{dx}{dt}(t_0) = v_0 \tag{1}$$

The first thing we notice is that complete specification of the system now requires *two* additional conditions instead of just one. This results in a natural dilemma—at what points t_0 do we specify these conditions? Can we specify them independently at two different t values? Do we need derivative information in addition to just the values of x at these t? All of these possibilities are allowed. Consequently, we can classify the nature of the problem depending on the type of conditions we specify:

- **Initial value problem**: Second order ODE + $x(t), \frac{dx}{dt}$ specified at single point $t = t_0$
- **Dirichlet problem**: ODE + $x(t)$ specified at two different points $t = t_i, t = t_f$
- **Neumann problem**: ODE + $\frac{dx}{dt}$ specified at two different points $t = t_i, t = t_f$
- **Mixed problem**: ODE + a combination of $x(t)$ and $\frac{dx}{dt}$ specified at two different points $t = t_i, t = t_f$

The final three options are usually referred to as *boundary value problems* in the theory of ODEs sice our domain is bounded between $t = t_i$ and $t = t_f$. For the remainder of these notes we will restrict ourselves to studying initial value problems.

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1 Linear second order ODEs

Just we did with first order systems, we can obtain exact solutions to the ODE problem if the RHS $f(\frac{dx}{dt}, x, t)$ takes on certain special forms. In the present case, we can only make statements about general solutions for linear second order ODEs, of which there are several sub-classes.

1 Homogeneous: $f(\frac{dx}{dt}, x, t) = -p(t)\frac{dx}{dt} - q(t)x$

Linear homogeneous ODEs are of the form:

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = 0 \tag{2}$$

Any linear second order homogeneous ODE of this form has two *linearly independent* solutions, irrespective of the initial conditions. We will denote these by $x_1(t), x_2(t)$. The general solution $x(t)$ is consequently given by

$$x(t) = c_1x_1(t) + c_2x_2(t) \tag{3}$$

where the constants c_1, c_2 are determined by the initial conditions. It is easy to see that if $x_1(t)$ is a solution to the ODE, and if another linearly independent $x_2(t) \neq x_1(t)$ is also a solution, then any linear combination of x_1 and x_2 is also a solution to the homogeneous linear ODE. So in principle, if we can find two linearly independent solutions to the ODE, we can solve it in closed form for any initial conditions. But first, how do we know if two functions x_1 and x_2 are linearly independent?

Theorem 1. *(Linear independence of functions) Two functions $x_1(t)$ and $x_2(t)$ are linearly independent if and only if the Wronskian $W(x_1, x_2) = x_1\frac{dx_2}{dt} - x_2\frac{dx_1}{dt}$ is non-zero.*

As an example, the functions $\exp(t)$ and $\exp(-t)$ are linearly independent by this measure. This theorem can be proved by considering the linear combination $c_1x_1 + c_2x_2 = 0$ and its derivative $c_1\frac{dx_1}{dt} + c_2\frac{dx_2}{dt} = 0$. The constants are stricly zero only if the determinant of the multiplying matrix—the Wronskian—is non-zero.

For linear homogeneous second order ODEs, determining both linearly independent x_1, x_2 is often not possible. However, we can use a clever technique to generate a linearly independent x_2 if an x_1 is known—this is known as the method of reduction of order. The idea is as follows. First, we write $x_2(t) = \nu(t)x_1(t)$ and substitute this for x_2 in the original ODE.

Second, simplifying this using the fact that $x_1(t)$ obeys the ODE as well, we obtain

$$x_1 \frac{d^2\nu}{dt^2} + \left(2\frac{dx_1}{dt} + p(t)x_1\right)\frac{d\nu}{dt} = 0 \tag{4}$$

Since x_1 and its derivatives are known, we can write $w = \frac{d\nu}{dt}$ to obtain a first order linear equation for $w(t)$

$$\begin{aligned} \frac{dw}{dx}x_1 + \left(2\frac{dx_1}{dt} + p(t)x_1(t)\right)w = 0 \\ \implies w(t) = C\exp\left(-\int \frac{2}{x_1}\frac{dx_1}{dt}dt + p(t)dt\right) \\ = C\frac{1}{x_1^2}\exp\left(-\int p(t)dt\right) \\ \implies \nu(t) = C\int\left[\frac{1}{x_1^2}\exp\left(-\int p(t)dt\right)\right]dt \end{aligned} \tag{5}$$

so that the second linearly independent solution is (constant of integration can be dropped)

$$x_2(t) = x_1\int\frac{1}{x_1^2}\exp\left(-\int p(t)dt\right)dt \tag{6}$$

So given one solution $x_1(t)$ of the ODE in Eq. 2, we can generate a second linearly independent solution $x_2(t)$ via Eq. 6 and hence the general solution $x = c_1x_1 + c_2x_2$. All of this is done without any recourse to the initial conditions being specified in the problem. For a general homogeneous problem, this is all we can say. So either one must guess a solution x_1 or obtain it by some other means initially. However, this can be done systematically when $p(t)$ and $q(t)$ are constants.

2 Homogeneous constant coefficients: $f(\frac{dx}{dt}, x, t) = -\frac{b}{a}\frac{dx}{dt} - \frac{c}{a}x$

When the coefficients are constant, the two linearly independent solutions x_1, x_2 can be individually obtained in a straightforward manner. First, we substitute $x(t) = \exp(rt)$ for some constant r into the ODE to obtain the *characteristic equation*:

$$ar^2 + br + c = 0 \qquad \implies r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

the nature of the roots determines the nature of the solutions to the ODE:

- $r_{1,2}$ are both real $\implies$ $x_1 = \exp(r_1t), x_2 = \exp(r_2t)$ are both exponential functions

- $r_{1,2}$ are complex conjugate $\implies$ $x_1 = \exp(-\frac{b}{a}t)\cos\gamma t, x_2 = \exp(-\frac{b}{a}t)\sin\gamma t$ where $\gamma^2 = \frac{4ac-b^2}{4a^2}$ are both sinusoidal with exponentially varying amplitudes

- $r_{1,2}$ are both real and equal, i.e., $b^2 - 4ac = 0$. In this case, both solutions $x_1, x_2 = \exp(-\frac{b}{a}t)$ are the same; their Wronskian is zero and another linearly independent solution must be generated using Eq. 6.

So for constant coefficient homogeneous equations, we can generate exponential or sinusoidal solutions using this technique.

3 Non-homogeneous: $f(x,t) = -p(t)\frac{dx}{dt} - q(t)x - g(t)$

For an ODE that does not have the homogeneous property, things get a little more complicated. Firstly, the presence of the $g(t)$ term invalidates our linear combination solution since if an $x(t)$ obeys the ODE, $cx(t)$ now does not. However, any such non-homogeneous ODE can be solved by splitting it into two parts. The *complementary equation* is obtained by setting $g(t) = 0$

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = 0 \tag{8}$$

and can be solved as before with two linearly independent solutions x_1, x_2. Now, we define a particular solution $x_p(t)$ as solving the non-homogeneous ODE with $g(t) \neq 0$

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = g(t) \tag{9}$$

so that the general solution $x(t)$ is given by

$$x(t) = c_1x_1(t) + c_2x_2(t) + x_p(t) \tag{10}$$

So far, given an x_1 solving Eq. 8, we know how to obtain x_2. Now, given x_1, x_2, how do we obtain x_p? For this, we use a technique called the *method of variation of parameters*. We define

$$x_p = u_1x_1 + u_2x_2 \tag{11}$$

where u_1, u_2 are *a priori* unknown functions. We substitute this form for x_p into Eq. 9, and add the auxiliary condition

$$x_1\frac{du_1}{dt} + x_2\frac{du_2}{dt} = 0 \tag{12}$$

on u_1, u_2. We can always do this because both functions are arbitrary and independent of each other. This condition only puts a small constraint on the possible functions that we wish to consider. Substituting x_p from Eq. 11 into Eq. 9, and using the fact that x_1, x_2 are solutions of Eq 8, we obtain

$$\frac{dx_1}{dt}\frac{du_1}{dt} + \frac{dx_2}{dt}\frac{du_2}{dt} = g(t) \tag{13}$$

We now have two linear algebraic equations Eq. 12, 13 for $\frac{du_1}{dt}, \frac{du_2}{dt}$ which we can invert to obtain (dot denotes d/dt)

$$\frac{du_1}{dt} = \frac{g(t)x_2}{\dot{x}_1 x_2 - x_1 \dot{x}_2} \qquad \frac{du_2}{dt} = -\frac{g(t)x_1}{\dot{x}_1 x_2 - x_1 \dot{x}_2} \tag{14}$$

which can be integrated to obtain

$$u_1 = \int \frac{g(t)x_2}{\dot{x}_1 x_2 - x_1 \dot{x}_2} dt \qquad u_2 = -\int \frac{g(t)x_1}{\dot{x}_1 x_2 - x_1 \dot{x}_2} dt \tag{15}$$

The sequence of operations is as follows. Given an inhomogeneous linear second order ODE Eq. 9, we first obtain one solution x_1 of the complementary ODE Eq. 8 and the second x_2 using Eq. 6. Then, using the variation of parameters, we obtain u_1, u_2 given by Eq. 15 so that the particular solution x_p, Eq. 11, is determined. Finally, the general solution to Eq. 9 is obtained in the form of Eq. 10.

With this, our solution scheme for any linear second order ODE is complete. We must remember that in the general case, for arbitrary $p(t), q(t)$, one solution x_1 is necessary before we can go ahead with it. However, when $p(t)$ and $q(t)$ are both constants, we can determine the general solution irrespective of the form of $g(t)$.

4 Special techniques for certain $p(t), q(t), g(t)$

We now discuss three specialized techniques for linear second order ODEs, depending on the form of the functions $p(t), q(t), g(t)$ in Eq. 9.

Method of undetermined coefficients

This technique is applicable only to constant coefficient ODEs with a RHS $g(t)$. Even though the method discussed for Eq. 9 applies when $p(t), q(t)$ are constants, it can often be very cumbersome to implement. When the function $g(t)$ is such that its derivatives start repeating after a certain order (e.g., sines/cosines, exponential functions, polynomials), an alternative technique becomes much easier to use.

As an illustration, consider the system

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = \sin\omega t \tag{16}$$

we could as well determine the particular solution x_p using Eq. 11. However, in general we note that the function x_p should have terms like $\sin \omega t$ and $\cos \omega t$ with some coefficients. So we assume a form

$$x_p = a_1 \sin \omega t + a_2 \cos \omega t \tag{17}$$

for the particular solution and put it back into the ODE in Eq. 16. Simplifying and comparing the coefficients of the sine and cosine terms on both sides gives us relations for the constants a_1, a_2.

Now let us apply this to a forced undamped oscillator problem

$$\frac{d^2x}{dt^2} + \alpha x = \frac{F_0}{m} \cos \omega t \tag{18}$$

Assuming $x_p = A \cos \omega t + B \sin \omega t$, the ODE reduces to

$$\cos \omega t \left(A\alpha - A\omega^2 \right) + \sin \omega t \left(\alpha B - B\omega^2 \right) = \cos \omega t \frac{F_0}{m} \tag{19}$$

from which we get

$$A = \frac{F_0/m}{\alpha - \omega^2} \qquad B = 0 \tag{20}$$

This type of *ansatz* for x_p also works when $g(t)$ is of the form $\exp(pt)$ or a polynomial of the form $At + Bt^2 + Ct^3$.

Reduction techniques

For a general second order linear homogeneous system with arbitrary $p(t), q(t)$, we can attempt a reduction to a first order equation by using the substitution $x(t) = \exp(g(t))$.

For instance, consider the equation

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = 0 \tag{21}$$

which, with $x = \exp(g)$ reduces to

$$\frac{d^2g}{dt^2} + \left(\frac{dg}{dt}\right)^2 + p\frac{dg}{dt} + q = 0 \tag{22}$$

Now setting $\frac{dg}{dt} = u(t)$, we obtain a first order nonlinear equation for $u(t)$

$$\frac{du}{dt} + u^2 + pu + q = 0 \tag{23}$$

which we can analyze using the techniques we studied for 1D/first order nonlinear systems. Incidentally, Eq. 23 is called the *Ricatti equation*. Granted, the amount of information we can glean from it is, in general, not much. However, for certain forms of $p(t)$ and $q(t)$, this technique may be useful.

Exact equations

The final special technique we will study pertains to what are called *exact equations*. These linear second order ODEs have the general form

$$P(t)\frac{d^2x}{dt^2} + Q(t)\frac{dx}{dt} + R(t)x = 0 \tag{24}$$

with the condition that

$$\frac{d^2P}{dt^2} - \frac{dQ}{dt} + R(t) = 0 \tag{25}$$

If this were so, Eq. 24 can be reduced to a linear first order ODE, which can always be solved using an integrating factor. First, we subtract the condition in Eq. 25, multiplied by $x(t)$ from Eq. 24 and add and subtract an $\frac{dP}{dt}\frac{dx}{dt}$ to obtain

$$\begin{aligned} P\frac{d^2x}{dt^2} + Q\frac{dx}{dt} + Rx - \frac{d^2P}{dt^2}x + \frac{dQ}{dt}x - Rx + \frac{dP}{dt}\frac{dx}{dt} - \frac{dP}{dt}\frac{dx}{dt} &= 0 \\ \implies \frac{d}{dt}\left(P\frac{dx}{dt}\right) + \frac{d}{dt}\left(Qx - \frac{dP}{dt}x\right) &= 0 \\ \implies P\frac{dx}{dt} + \left(Q - \frac{dP}{dt}\right)x = \text{const.} &= C \end{aligned} \tag{26}$$

The final linear ODE is of first order and can be solved in terms of an integrating factor. Hence, whenever we have an exact ODE, i.e., whose coefficients obey Eq. 25, we can reduce it to a first order linear ODE and obtain an explicit solution.

2 Phase space analysis

We have already noted some important features of second order systems. Firstly, comparing linear first and second order ODEs, we see that the latter are, in general, much more complex. Furthermore, we realized that even the most general second order linear system cannot be solved in closed form (upto one quadrature step), in contrast to its first order cousin. These are symptomatic—we will see that the nature of fixed points, parameter dependence, bifurcations etc. are all much more complex in the second order case.

From a phase space point of view, this complexity arises because second order systems can

be reduced to a 2D first order system, with two dependent variables, say x_1, x_2 [1]. This two dimensionality of phase space makes all the difference—we can now have closed orbits, isolated orbits, degenerate fixed points, Lyapunov stable points, spirals and centres about which all trajectories revolve.
Analyzing linear systems in phase space is thus much more instructive in the second order or 2D case. We will use these ideas when we talk about linearization of nonlinear systems, just as we did for the first order case. Again, we will restrict ourselves to autonomous systems where there is no explicit t-dependence.

1 Types of fixed points

Before we start considering phase space, we first notice that, for linear second order autonomous or for 2D first order ODEs in general, of the form

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = 2 \times 2 \text{ matrix} \tag{27}$$

the origin $\vec{x} = (0, 0)$ is always a fixed point, since the LHS is zero here. However, in contrast to 1D systems where only $f'(x)$ was necessary to determine stability, we now have to deal with a matrix A.
Let us look at two concrete examples. Consider the system

$$\frac{dx_1}{dt} = \alpha x_1 \qquad \frac{dx_2}{dt} = -x_2 \tag{28}$$

which consists of uncoupled linear first order ODEs and can be solved individually. Consequently, the general solution is

$$x_1 = x_1^0 \exp(\alpha t) \qquad x_2 = x_2^0 \exp(-t) \tag{29}$$

Depending on α, we can now draw trajectories in phase space, just as we did for the case of 1D systems. The various possibilities for $-\infty < \alpha < \infty$ are shown in Fig. 1. we can interpret these diagrams as follows. Since the equations are decoupled, if we start with an initial point $(0, x_2^0)$, our trajectory will always lie on the x_2 axis, decaying towards the origin as $x_2^0 \exp(-t)$. On the other hand, if we start at $(x_1^0, 0)$ our trajectory will lie on the x_1 axis while either decaying towards the origin or going away from it, depending on α.
Now consider a general trajectory starting at an arbitrary initial point (x_1^0, x_2^0). For $\alpha < -1$, the x_2 component of the trajectory decays slower than the x_1 component, leading to a curve

[1] For this section and onward, x_1, x_2 will represent coordinates in phase space. These should not be confused with the two linearly independent solutions we discussed for homogeneous ODEs

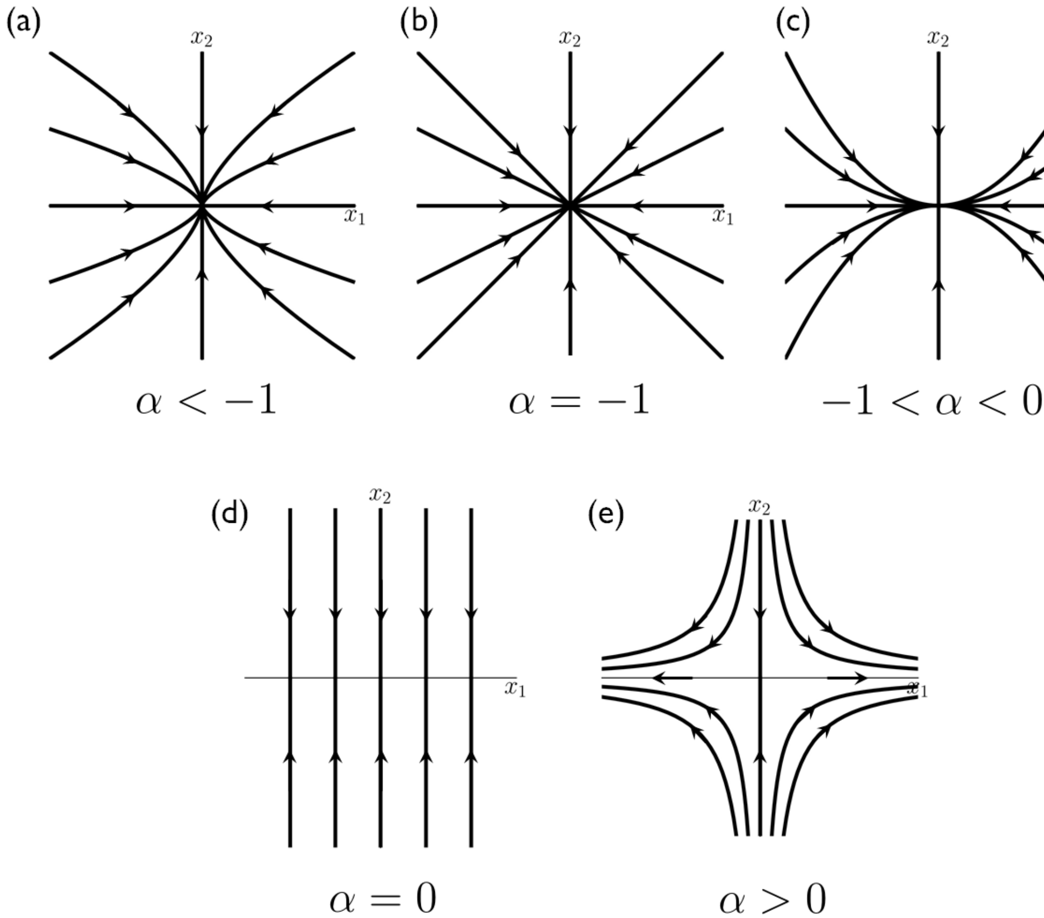

Figure 1: Types of fixed points for varying α

like in Fig. 1(a). For $\alpha = -1$, the decay along both axes is the same so that the trajectory reaches the origin along a straight line $x_2(t)/x_1(t) = x_2^0/x_1^0$. Likewise, for $-1 < \alpha < 0$, the decay along the x_1 axis is slower than along the x_2 axis, leading to the curve in part (c). In all three cases, the origin is absolutely stable—any trajectory that starts in the vicinity of the origin eventually reaches it as $t \to \infty$.

For $\alpha = 0$, something interesting happens. The x_2 component decays as $\exp(-t)$ but the x_1 component remains unchanged. Now, any trajectory that starts at (x_1^0, x_2^0) ends up at $(x_1^0, 0)$ as $t \to \infty$. Consequently an infinite number of *non isolated fixed points* arise all along the horizontal x_1 axis. The origin now, is no longer stable—it is only Lyapunov stable. Any trajectory that starts close to the origin always remains in its vicinity but never reaches the origin as $t \to \infty$.

For positive α, the situation changes yet again. Even though the x_2 component of any trajetory reduces with t, the x_1 component increases continuously. Therefore, the origin is stable for some directions (along positive and negative x_2) but unstable for others—it now

becomes a saddle fixed point. The trajectories for any other point not on this axis diverge towards $x_1 = \pm\infty$, as shown in Fig. 1(e).
The same situation, but in reverse, could be envisaged if the ODE system were

$$\frac{dx_1}{dt} = \alpha t \qquad \frac{dx_2}{dt} = +x_2 \tag{30}$$

now we'd have a saddle for $\alpha < 0$ and fixed points of similar kind, but unstable, as in Fig. 1 for $\alpha > 0$.
As a matter of nomenclature, the axis along which the solution decays to the origin is called the *stable manifold* and the axis along which it goes away to $\pm\infty$ is called the *unstable manifold*. When both axes are stable, the entire phase space around the fixed point becomes the stable manifold.
Let us now consider as a second example, the system

$$\frac{d^2x}{dt^2} + 2\zeta\omega\frac{dx}{dt} + \omega^2 x = 0 \tag{31}$$

which, when cast as a 2D system, becomes

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = -\omega^2 x_1 - 2\zeta\omega x_2 \tag{32}$$

These two ODEs are fully coupled unlike the previous case. We know how to solve this ODE system in general form since it has constant coefficients.
Let us see how the solution curve looks in phase space. Consider the $\zeta = 0$ case first. If we pick a point (x_1^0, x_2^0) in phase space, the trajectory will be a closed ellipse passing through this point. Consequently, the origin, around which the ellipse is centered, is called a *centre*. When $\zeta > 0$, the solution will be an exponentially decaying term modulating the amplitude of the oscillating terms so that the trajectory will be a spiral towards the origin. The origin is now a *stable spiral* since it is absolutely stable. If $\zeta > 0$, the trajectory spirals away from the origin making it an *unstable spiral*. The centre, however, is only Lyapunov stable.

2 Classification of linear systems

We have seen two examples already—one where the two first order ODEs are decoupled leading to fixed points and the other where they are completely coupled leading to centres and spirals about the origin. But what if have a general 2D ODE system as below

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{33}$$

If $b = c = 0$, we have fixed points, but if $a = d = 0$, we have centres/ spirals. When $a, b, c, d \neq 0$, we have to obtain some invariants of the matrix to say anything about the behaviour of the LHS. The two invariants are the trace $\tau = a + d$ and the determinant $\Delta = ad - bc$.

Generally speaking, any reasonably well-behaved matrix A as above can be diagonalized by suitable change of coordinate axes. Formally, these axes correspond to the eigenvectors of the matrix A. Motivated by this fact, we attempt to find two linearly independent, not necessarily orthogonal, directions along which the coupled ODE system becomes decoupled. We do this by finding the eigenvectors of A for, by definition, for eigenvectors $\vec{v}$,

$$\frac{d\vec{v}}{dt} = A\vec{v} = \alpha\vec{v} \qquad \Longrightarrow \qquad \vec{v} = \vec{v_0}\exp(\alpha t) \tag{34}$$

with corresponding eigenvalues α. So the two linearly independent eigenvectors of the matrix A give two directions along which the trajectory either decays (corresponding $\alpha < 0$) or grows (corresponding $\alpha > 0$). Hence the information we need about the stability of fixed points lies in the eigenvalues $\alpha_{1,2}$ given by:

$$\alpha_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \tag{35}$$

where τ, Δ are the trace and determinant of the matrix A, respectively.

If the eigenvalues are distinct, we can write the general solution as

$$\vec{x} = c_1 \exp(\alpha_1 t)\vec{v}_1 + c_2 \exp(\alpha_2 t)\vec{v}_2 \tag{36}$$

So by analogy with the problem we introduced in Eq. 28, we can determine the nature of the fixed point at the origin for a linear system by looking at the eigenvalues and corresponding eigenvectors of the matrix A. But wait, what happens if the eigenvalues are complex conjugate? This happens when $4\Delta > \tau^2$ and leads to complex conjugate eigenvectors! How do we interpret the results now?

To understand this situation, recall the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{37}$$

which has complex conjugate eigenvalues $\alpha_{1,2} = \exp(\pm i\theta)$. What does acting on a vector $\vec{v}$ by the matrix $R(\theta)$ do to it? It rotates $\vec{v}$ by an angle θ, irrespective of what $\vec{v}$ is. So we can never find a direction along which vectors remain unchanged except in magnitude. Consequently,

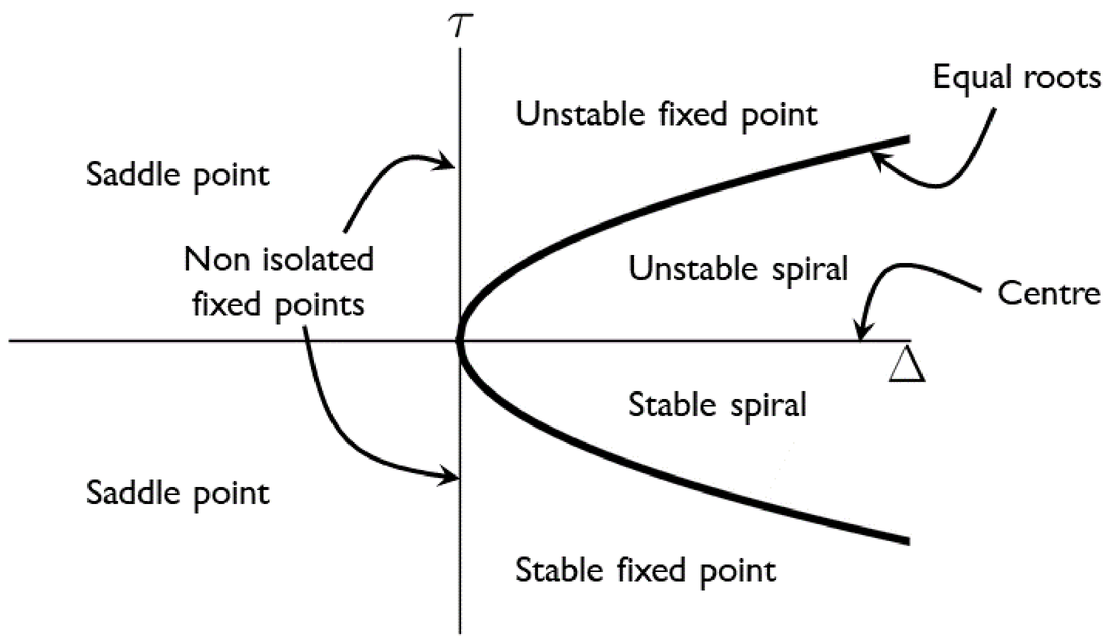

Figure 2: Classification of nature of origin as a fixed point depending on τ, Δ of A

we realise that any such matrix A in Eq. 33 that has complex conjugate eigenvalues cannot have a real direction along which trajectories decay towards or grow away from the origin. From this, we see that when $4\Delta \geq \tau^2$, the behaviour is more like Eq. 31, which also had complex conjugate eigenvalues, resulting in either centres or stable/unstable spirals at the origin.
We can summarize our understanding of fixed points at the origin for a general linear 2D system with RHS matrix A as follows, depending on the τ and Δ of A, see Fig. 2.
The features in this figure may be explained as follows:

- First, consider the parabola $\tau^2 = 4\Delta$ shown in the figure. When $\tau^2 \leq 4\Delta$, we are 'within' this curve, leading to either spirals ($\tau \neq 0$) or a centre $\tau = 0, \Delta > 0$. Remember that Δ must be positive for centres to exist—this means that the product of the eigenvalues $\alpha_1\alpha_2$ is positive—which is true for complex conjugate roots.

- Second, the regions for $\Delta > 0$ but outside $\tau^2 = 4\Delta$ have real α_1, α_2. The roots are of the same sign, both positive (so their sum $\tau > 0$, unstable fixed point) or both negative ($\tau < 0$, stable fixed point).

- All along the parabola $\tau^2 = 4\Delta$ have equal and real roots since the square root becomes zero. Depending on whether $\tau > 0$ or $\tau < 0$, these are stable or unstable fixed points.

- Whenever $\Delta < 0$, the eigenvalues α_1, α_2 are of opposite sign, irrespective of the sign of

τ—so all points to the left of the τ axis correspond to saddles.

- Finally, when $\Delta = 0$, all along the τ axis, we have non-isolated fixed points since atleast one of the roots is 0. This corresponds to the Lyapunov stability case we disucssed in Fig. 1.

To conclude, given any linear second order autonomous system, we first note two things—the origin is always the only fixed point, and that the nature of trajectories near the origin is determined by evaluating τ, Δ for the RHS matrix A and mapping it to the corresponding zone in Fig. 2.

3 Linearization about a fixed point

Finally, given our understanding of linear second order systems, we can start saying something about the fixed points for nonlinear ODEs. Just as we performed linearization for the 1D case, we can do so again in the vicinity of fixed points for a 2D nonlinear system. However, instead of merely looking at $\frac{df}{dx}$ at $x = x^*$ as in the 1D case, we presently have to evaluate the RHS matrix and it's properties τ, Δ to say anything about the nature of the fixed point about which we've linearized. As one can perhaps also expect, second order systems can show other interesting features such as isolated or *limit cycles* that 1D systems cannot. Consequently, the types of bifurcations we can expect are also manifold. In the interest of time, we will restrict ourselves to just fixed points and their behaviour using linearization.
Consider a 2D nonlinear ODE system given by

$$\frac{dx_1}{dt} = f(x_1, x_2) \qquad \frac{dx_2}{dt} = g(x_1, x_2) \tag{38}$$

where f, g are two arbitrary nonlinear functions. To obtain the coordinates (x_1^*, x_2^*) of the fixed points, we have to solve the simultaneous nonlinear equations

$$g(x_1^*, x_2^*) = f(x_1^*, x_2^*) = 0 \tag{39}$$

Now consider a trajectory $(x_1(t), x_2(t))$ very close to one such fixed point (x_1^*, x_2^*) so that $x_1 - x_1^* = \eta_1(t)$ and $x_2 - x_2^* = \eta_2(t)$ are both very small. The derivatives become

$$\begin{aligned}\frac{d\eta_1}{dt} &= f(x_1^* + \eta_1, x_2^* + \eta_2) \simeq f(x_1^*, x_2^*) + \frac{\partial f}{\partial x_1}_{|(x_1^*, x_2^*)}(\eta_1) + \frac{\partial f}{\partial x_2}_{|(x_1^*, x_2^*)}(\eta_2) \\ \frac{d\eta_2}{dt} &= g(x_1^* + \eta_1, x_2^* + \eta_2) \simeq g(x_1^*, x_2^*) + \frac{\partial g}{\partial x_1}_{|(x_1^*, x_2^*)}(\eta_1) + \frac{\partial g}{\partial x_2}_{|(x_1^*, x_2^*)}(\eta_2)\end{aligned} \tag{40}$$

which can be recast as

$$\frac{d\vec{\eta}}{dt} = J_{|_{(x_1^*, x_2^*)}}\vec{\eta} \tag{41}$$

a linear system in η with the *Jacobain matrix* playing the role of the RHS matrix A. We have defined J as

$$J = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix} \tag{42}$$

evaluated at the fixed point (x_1^*, x_2^*). J therefore has constant entries once the fixed points are known. Consequently, this is the linearized problem equivalent to Eq. 38 about the fixed point (x_1^*, x_2^*). All we have to now do is evaluate the trace τ and determinant Δ of J and map it onto Fig. 2 to obtain the nature of the fixed point we have linearized about.

4 Other features

Just as we had bifurcations in 1D systems, we can envisage changes in stability of fixed points even for the 2D case. However, as we have emphasized before, the number of possibilities is now larger. Additionally, the possibility of isolated closed orbits (limit cycles) now means that fixed points can not only change their stability but can also lead to emergence of limit cycles as a parameter λ is varied.

In general, linearization cannot give us any information about limit cycles (which incidentally can have their own notions of stability just like fixed points). So we will restrict our attention to fixed points alone for now.