ME261: Differential Equations
Aug–Dec 2022
ODE Assignment — Due Oct. 24, 2022

Answer the questions below. If appropriate, please box your final expression/ result.

Question 1: Population growth [*10 marks*]

In population dynamics, it is common to assume that the rate of change of population dN/dt (of a state/country etc.) is proportional the current population N, the proportionality constant being the growth rate r. If r were a constant, then all growing populations will eventually explode exponentially. However, since any population can only be sustained if there are enough resources, there is, in reality, a maximum sustainable population for any given state/country etc. To account for this, we introduce a quadratic dependence of r on N, say $r = aN^2 + b$:

(a) If you are given that a certain country (Breat Gritain) had a population 4×10^8 in 1950 with an annual growth rate of 2% and a population of 14×10^8 in 2020 with the annual rate down to 1%. Write down the governing differential equation for the population of Breat Gritain.

(b) Perform a stability analysis of this system and qualitatively show how the population will evolve in phase space.

(c) Can you determine the year when the population will become stable $= N_\infty$? What happens as $N \to N_\infty$?

Question 2: Cooling things [*10 marks*]

You have been asked by your thesis advisor to go and cool a hot object in the institute swimming pool over a period of 20 days and study its temperature variation every hour. You may assume the object to be spherical and the temperature governed by:

$$\frac{dT}{dt} = \beta(T - T_\infty) \tag{1}$$

where T_∞ is the pool temperature and β is a property of the object.

(a) Can you explain how Eq. 1 is obtained?

(b) Since you are at the pool for 20 days, the water temperature T_∞ is not constant but changes with time. Assume $T_\infty = T_m \sin(\omega t)$ and estimate the value of ω to account for ambient temperature changes between day and night.

(c) Solve this system if $T(0) = T_0$ is the initial temperature of the object. Remember that we said that 1D systems do not show any oscillations. Can you comment on your result?

(d) How does the solution change by varying the parameter $\lambda = \frac{\omega}{\beta}$? What does this parameter signify? What are its physical units if T is in $^\circ C$ and t is in seconds?

Question 3: The Leaky Cauldron [*15 marks*]

Consider a real leaky cauldron (no offense to Harry Potter) with a hole at the bottom. Let $h(t)$ be the height of the water remaining in the bucket at time t; a be the area of hole; A cross-sectional area of bucket (assumed constant); and $v(t)$ velocity of the water passing through the hole.

(a) Derive the governing equation for $h(t)$. What physical law are you invoking?

(b) We need additional equation for $v(t)$ to solve the differential equation in part(a). Using conservation of energy, $v = \sqrt{2gh}$. Let H be the initial height of the water in the bucket, can you find the expression for time T when the bucket will be empty? Conversely, if you see an empty bucket, can you figure out when the bucket was full?

(c) If the water is also pumped in the bucket at volume flow rate Q_o = constant, then write the modified differential equation for $h(t)$. Choose the appropriately the values for Q_o, a, A, and H. Sketch the vector field to say something about the equilibrium water level ($t \to \infty$).

Question 4: The Bessel equation [*30 marks*]

As discussed in class, there are often situations where one cannot obtain closed form solutions of a differential equation. A classical example is the heat conduction equation in cylindrical coordinates, where one solves for the temperature as a function of space and time. After some manipulation, we can reduce the original PDE to a Bessel equation

$$x^2 y'' + x y' + x^2 y = 0 \tag{2}$$

This equation is linear, second order, but cannot be solved in closed form using elementary functions. In this exercise, we systematically demonstrate the use of power series to deal with such problems.

(a) Consider the solution of the differential equation (2) of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n} \tag{3}$$

where $a_0 \neq 0$, and r is some constant that needs to be determined. Using the ideas we discussed in class, first determine all possible values of r. Following this, obtain a recursive relation for the constants a_n for any one value of r.

(b) Solving the recursion relations above, obtain one solution in series form for the Bessel equation. We will term this solution $J_0(x)$, it is called the Bessel function of first kind and zero order. Does the series representation converge for all values of x?

(c) Following the same procedure in (b) for the other value of r you obtained in part (a), determine the second solution $y_2(x)$ Do you see any issues? If so, how will you find a second linearly indepenent solution $Y_0(x)$ of the Bessel function? You can leave your final answer in terms of an integral involving $J_0(x)$.

Question 5: Linear ODE [*15 marks*]

Take the following second order non-homogeneous ODE:

$$(t-1)\frac{d^2x}{dt^2} - t\,\frac{dx}{dt} + x = (t-1)^2 \tag{4}$$

(a) Given, $x_1 = t$ is a solution to complementary problem (RHS =0 above), find another linearly independent solution x_2 for the complementary problem

(b) Using x_1 and x_2, determine the general solution to the inhomogeneous ODE.

Question 6: Chandrayaan 3 [*40 marks*]

Let us assume that you are heading the mission control team that has to put a satellite into orbit around the moon. The relativistically corrected equation for the orbit $r(\theta)$ of a satellite around the moon is

$$\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r + \sigma r^2 + \epsilon = 0 \tag{5}$$

where r, θ are the polar coordinates of the satellite from the centre of the moon. $\sigma > 0$ is a constant depending on the mass of the moon and the angular momentum of the satellite. The small positive constant $\epsilon > 0$ is a relativistic correction term. Apart from these facts, no other knowledge of rocket science is necessary.

(a) Use a change of variables $y = 1/r$ to recast this differential equation in terms of $y(\theta)$. Rewrite this transformed ODE as a 2D system in $y, dy/d\theta$.

(b) If we had $\epsilon = 0, \sigma = 0$, what does the system reduce to? For $\sigma \neq 0, \epsilon = 0$, solve this system explicitly for $y(\theta)$ and so for $r(\theta)$.

(c) Draw a trajectory $r(\theta)$ from part (b), assume $\sigma = 5$. What is the physical interpretation for the satellite?

(d) Now consider the full non-linear equation with $\epsilon \neq 0$. Find the fixed points for this system and determine their nature.

(e) Do any of the fixed points have positive determinant for the Jacobian? By comparing your result with that from part (c), what consequence does it have for the satellite?