A short lecture series on contour integration in the complex plane
NPTEL course me75
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1 Motivation

Let us consider an Euler-Bernoulli beam driven by a point force

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + m'\frac{\partial^2 w(x,t)}{\partial t^2} = F\delta(x-x_0)e^{j\omega t} \tag{1}$$

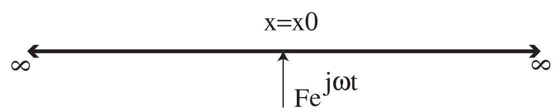


Figure 1: Figure showing an Euler-Bernoulli beam with a concentrated point force.

Since the response is going to be at the same frequency, let us substitute

$$w(x,t) = W(x)e^{j\omega t},$$

which gives

$$EI\frac{d^4 W(x)}{dx^4} - m'\omega^2 W(x) = F\delta(x-x_0). \tag{2}$$

Taking the Fourier transform

$$\begin{aligned}
\int_{-\infty}^{\infty} EI\frac{d^4W(x)}{dx^4}e^{-jkx}dx - \int_{-\infty}^{\infty} m'\omega^2 W(x)e^{-jkx}dx &= \int_{-\infty}^{\infty} F\delta(x-x_0)e^{-jkx}dx \\
EIk^4W(k) - m'\omega^2W(k) = Fe^{-jkx_0} \\
W(k) = \frac{Fe^{-jkx_0}}{EIk^4 - m'\omega^2}.
\end{aligned} \tag{3}$$

From here, in order to find $w(x)$ we must invert the Fourier transform

$$w(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Fe^{-jkx_0}e^{jkx}}{EIk^4 - m'\omega^2}dk.$$

This integral can be computed using contour integrals in the complex domain. I will present an introduction to the theory of complex variables. **A lot of the material is taken from the popular textbook "Complex variables and applications" by J. W. brown and R. V. Churchill**.

2 The beginning of complex variables topic

Most likely it started with the equation

$$x^2 + 1 = 0$$

and then a calculus was built around $\sqrt{-1} = i$. Now a number in the complex domain is $z = x+iy$ and a complex function f(z)=f(x,y)=u(x,y)+i v(x,y). Do such functions follow the real variable calculus rules?

3 Complex numbers and limits

- Let us examine the limit of $\sin(z)$ as $z \to 0$.

$$\lim_{x,y\to 0} \sin(x+iy) = \lim_{x,y\to 0} (\sin(x)\cosh(y) + i\cos(x)\sinh(y))$$

In one case we take $x \to 0$ first and in the other we take $y \to 0$

$$\begin{aligned} \lim_{x\to 0} \sin(x+iy) &= \lim_{x\to 0} (\sin(x)\cosh(y) + i\cos(x)\sinh(y)) \\ &= \lim_{y\to 0} 0 + i\sinh(y) \\ &= 0. \end{aligned}$$

Next taking $y \to 0$ first

$$\begin{aligned} \lim_{y\to 0} \sin(x+iy) &= \lim_{y\to 0} (\sin(x)\cosh(y) + i\cos(x)\sinh(y)) \\ &= \lim_{x\to 0} \sin(x) + i0 \\ &= 0. \end{aligned}$$

- Let us examine $\sin(z)/z$ as $z \to 0$.

$$\begin{aligned} &= \lim_{x,y\to 0} \frac{\sin(x+iy)}{x+iy} = \lim_{x,y\to 0} \frac{(\sin(x)\cosh(y) + i\cos(x)\sinh(y))}{x+iy} \\ &= \lim_{x,y\to 0} \frac{(\sin(x)\cosh(y)(x-iy) + i\cos(x)\sinh(y)(x-iy))}{x^2+y^2} \\ &= \lim_{x,y\to 0} \frac{\sin(x)\cosh(y)x + y\cos(x)\sinh(y) - iy\sin(x)\cosh(y) + i\cos(x)\sinh(y)x}{x^2+y^2} \end{aligned}$$

In one case we take $x \to 0$ first and in the other we take $y \to 0$

$$\begin{aligned} &\lim_{x\to 0} \frac{\sin(x)\cosh(y)x + y\cos(x)\sinh(y) - iy\sin(x)\cosh(y) + i\cos(x)\sinh(y)x}{x^2+y^2} \\ &= \frac{y\sinh(y)}{y^2} \\ &= \lim_{y\to 0} \frac{\sinh(y)}{y} = 1. \end{aligned}$$

And

$$\begin{aligned}
&\lim_{y\to 0} \frac{\sin(x)\cosh(y)x + y\cos(x)\sinh(y) - iy\sin(x)\cosh(y) + i\cos(x)\sinh(y)x}{x^2+y^2} \\
=& \frac{\sin(x)x}{x^2} \\
=& \lim_{x\to 0}\frac{\sin(x)}{x} = 1
\end{aligned}$$

The complex variable z follows the real variables as far as limits are concerned.

lecture 1 ends here.

4 Differentiation

Let f be a function whose domain of definition contains a neighborhood of a point z_0. The derivative of f at z_0 written $f'(z_0)$ is defined by the equation

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \tag{4}$$

provided this limit exists. The function is said to be differentiable at z_0 when its derivative exists at z_0. Taking $\Delta z = z - z_0$ we get

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \tag{5}$$

Since $f(z)$ is defined in the neighborhood of z_0, $f(z_0+\Delta z)$ is always defined for $|\Delta z|$ sufficiently small.

$$\Delta w = f(z_0+\Delta z) - f(z), \quad \frac{dw}{dz} = \lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} \tag{6}$$

4.1 Example

- Take for example $f(z) = z^2$.

$$\begin{aligned}
\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} &= \lim_{\Delta z\to 0}\frac{(z+\Delta z)^2 - z^2}{\Delta z} \qquad (7)\\
&= \frac{\Delta x^2 - \Delta y^2 + 2iy\Delta x + 2i\Delta x\Delta y + 2ix\Delta y + 2x\Delta x - 2y\Delta y}{\Delta x + i\Delta y}
\end{aligned}$$

Here again Δx and Δy should be taken independently. Taking Δx to zero first we get

$$\lim_{\Delta x\to 0}\frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{-\Delta y^2 + 2ix\Delta y - 2y\Delta y}{i\Delta y} \tag{8}$$

Then taking $\Delta y \to 0$ we get

$$2(x+iy) = 2z.$$

Alternatively taking $\Delta y \to 0$ first we get

$$\lim_{\Delta y\to 0}\frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{\Delta x^2 + 2iy\Delta x + 2x\Delta x}{\Delta x} \tag{9}$$

Then taking $\Delta x \to 0$ we get

$$2(x+iy) = 2z.$$

- Let us look at the derivative of $|z|^2$.

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\bar{z}+\bar{\Delta}z) - z\bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \bar{z} + \bar{\Delta}z + z\frac{\bar{\Delta}z}{\Delta z} \quad (10)$$

Here again if Δx is set to zero first we get

$$\lim_{\Delta x \to 0} \bar{z} + \bar{\Delta}z + z\frac{\bar{\Delta}z}{\Delta z} = \bar{z} - i\Delta y - z \quad (11)$$

And then we set Δy to zero we get

$$\bar{z} - z$$

Alternatively if we set Δy to zero first we get

$$\lim_{\Delta y \to 0} \bar{z} + \bar{\Delta}z + z\frac{\bar{\Delta}z}{\Delta z} = \bar{z} + \Delta x + z \quad (12)$$

which leads to

$$\bar{z} + z$$

when Δx is set to zero. Thus, the answer depends on the path taken. Thus, the real calculus differentiation does not apply directly in case of complex variables. What we mean is that $|x|^2$ is a nice function at every point on the x axis, but not $|z|^2$.

4.2 Cauchy-Reimann Equations

Suppose

$$f(z) = u(x,y) + iv(x,y)$$

and suppose

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

exists. Now

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta z + i\Delta y$$

$$Re[f'(z_0)] = \lim_{\Delta x, \Delta y \to 0} Re(\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z})$$

$$Im[f'(z_0)] = \lim_{\Delta x, \Delta y \to 0} Im(\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z})$$

Now,

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)}{\Delta x + i\Delta y}$$

As Δx and Δy can tend to zero in any manner, we set $\Delta y = 0$ as if it is already zero and send Δx to zero as a limiting process.

$$Re(f'(z_0)) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x}$$

$$Im(f'(z_0)) = \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

Or

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Next if we set $\Delta x = 0$ right away and send Δy to zero as a limiting process,

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$$

And so for a derivative to exist

$$u_x = v_y, \quad \text{and} \quad u_y = -v_x.$$

These are called the Cauchy-Reimann equations. So if $f'(z_0)$ exists, then necessarily the CR conditions must be satisfied. These are necessary conditions but not sufficient, i.e., if CR conditions are satisfied then the derivative may or may not exist. For sufficiency,, the first order partial derivatives of $u(x, y)$ and $v(x, y)$, i.e., u_x, v_x, u_y and v_y must be continuous.
Lecture 2 ends here.

4.3 Examples on CR equations

- Consider the function $f = z^2$. It has the derivative $f' = 2z$ at every point in the complex plane. Taking $z = x + iy$, $f(z) = z^2 = u(x, y) + iv(x, y) = (x^2 - y^2) + i2xy$. Let us check the CR conditions.

  $$u_x = 2x = v_y, \quad \text{and} \quad u_y = -2y = -v_x.$$

  Since f has a derivative, CR conditions are necessarily satisfied.

- Consider $f = |z|^2$.

  $$f(x, y) = x^2 + y^2 + i0, \;\; u(x, y) = x^2 + y^2, \;\; v(x, y) = 0.$$

  Let us check the CR equations.

  $$u_x = 2x \neq v_y, \quad \text{and} \quad u_y = 2y \neq -v_x.$$

  Since, the CR equations are not satisfied, the f' does not exist anywhere in the complex plane.

4.4 Analytic Functions

A function $f(z)$ of the complex variable z is analytic in an open set if it has a derivative at each point in that set. If f is analytic at a point z_0, it is analytic in the neighborhood of z_0.

- A function can be differentiable at a point but not analytic at that point.
- A function is analytic at a point only if it is analytic in the neighbourhood of that point. Hence, the function

  $$|z|^2$$

  which has a derivative only at z=0, is nowhere analytic.
- If a function fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is a singular point of the function.

5 Complex Integrations

We will next investigate integrations on the complex plane. Since the region of integration is 2-D, integrations end up as line integrals on contours. In a contour integral, a function is defined on a contour and hence in general the value of the integral depends on the function as well as the contour.

5.1 A few useful definitions:Curves, Arcs and Contours

- A set of points $z = (x, y)$ is called an arc if $x = x(t)$ and $y = y(t)$ $(a \leq t \leq b)$. $x = x(t)$ and $y = y(t)$ are continuous functions of t. Hence, $z(t) = x(t)+iy(t)$ is an arc. Discrete points do not make an arc. It is a simple arc C if it does not cross itself. When C is simple except that $z(a) = z(b)$, then C is called a simple closed curve.
- If $x'(t)$ and $y'(t)$ are continuous throughout $(a \leq t \leq b)$, then C is called a differentiable arc. It follows that $z'(t) = x'(t) + iy'(t)$ and the real valued function

  $$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

  is integrable over the interval $(a \leq t \leq b)$ and the length of the arc being

  $$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{x'^2 + y'^2}dt = \int_a^b |z'(t)|dt \tag{13}$$

- If $z(t)$ is a differentiable arc, and if $|z'(t)| \neq 0$ anywhere in $(a \leq t \leq b)$, then the tangent turns continuously. Such an arc is called smooth.
- A contour is a piecewise smooth arc, consisting of a finite number of smooth arcs joined end to end. Hence $z'(t)$ is now piecewise continuous. When the two end points $z(a) = z(b)$, it is called a simple closed contour.
- Example of a contour

  $$z = \begin{cases} x + ix & 0 \leq x \leq 1 \\ x + i & 1 \leq x \leq 2 \end{cases} \tag{14}$$

- Example.

  $$z = e^{i\theta}, \;\; 0 \leq \theta \leq 2\pi$$

  This is a simpled closed contour.

Lecture 3 ends here.

5.2 Back to integration

A real function $f(t)$ that is piecewise continuous on an interval $(a \leq t \leq b)$. is integrable. A piecewise continuous function is finite everywhere, but there are points where the left limit does not equal the right limit. Thus,

$$\int f(t)dt$$

exists since the integrand is piecewise continuous.

Integrals of complex valued functions which are defined on contours end up as line integrals or contour integrals written as

$$\int_C f(z)dz \quad or \int_{z_1}^{z_2} f(z)dz.$$

Suppose C is a simple contour given by $z = z(t)$, $(a \leq t \leq b)$. $z(t)$ is a piecewise smooth arc, so $z'(t)$ is not continuous but has finite jumps. Let $f(z)$ be piecewise continuous on C.

$$\int_C f(z)dz = \int_C (u(t) + iv(t))z'(t)dt = \int_C (u(t) + iy(t))(x'(t) + iy'(t))dt.$$

Writing this as two real integrals

$$\int_C (u(t)x'(t) - v(t)y'(t))dt + i\int_C (u(t)y'(t) + v(t)x'(t))dt$$

Both the integrals exist because the integrands are piecewise continuous. Thus, the total integral exists.

5.3 Some more definitions

- The epsilon neighborhood of a point z_0 in the complex plane consists of all points z lying inside but not on the circles centered on z_0. It is represented as

  $$|z - z_0| < \epsilon,$$

  where $\epsilon > 0$.
- A set is a collection of points following a certain rule.
- A point z_0 is said to be the interior point of the set S, whenever there is some neighborhood of z_0 that contains only points of S. z_0 is called the exterior point of S when there exists a neighborhood of it containing no points of S. If z_0 is neither an interior nor an exterior point, it is a boundary point of S.
- A set is open if it contains none of its boundary points. A set is open iff every point is an interior point. A set is closed if it contains all of its boundary points.
- An open set S is connected if each pair of points z_1 and z_2 in it can be joined by a polygonal line consisting of a finite number of line segments joined end to end that lies entirely in S. An open connected set is called a domain. A domain can be simply-connected or multiply-connected.
- A simply-connected domain D is a domain such that every simple closed contour within it encloses only points of D. If a domain is not simply-connected, it is multiply-connected.
- A set is bounded if every point of S lies inside some circle $|z| = R$. Otherwise it is unbounded.

Lecture 4 ends here.

6 Theorem 1

Suppose that a function f(z) is continuous on a domain D. If any one of the following statements is true, then all the other statements are true.

1. f has an antiderivative in D.
2. The integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value.
3. Integrals of $f(z)$ around closed contours lying entirely in D all have zero value.

6.1 Proofs

Let 1) be true, i.e., $f(z)$ has an antiderivative $F(z)$ in D. Let z(t) be a contour C, going from z_1 to z_2 $a \leq t \leq b$ then

$$\frac{d}{dt}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t)$$

and

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = F(z(t))|_b^a = F(z(b)) - F(z(a)),$$

thus the integral is independent of the path. And if $a = b$ then the integral is zero. The proof from 3 to 1 is involved and so we take it for granted.

Lecture 5 ends here.

6.2 Examples

1. Let us see
$$\int_C \frac{dz}{z^2},$$
where C is a circle of radius 1 centered on the origin. The domain D is the entire complex plane except the origin. The function is continuous everywhere except at the origin. It has an antiderivative $-1/z$ which means that
$$\frac{d}{dz}(-\frac{1}{z}) = \frac{1}{z^2}$$
in the domain D. Thus,
$$\int_{z1}^{z2} \frac{dz}{z^2} = -\frac{1}{z}|_{z1}^{z2}$$
And if $z1 = z2$ then the value is zero.
2. Let us see
$$\int_C \frac{dz}{z},$$
where C is a circle of radius 1 centered on the origin. The domain D is the entire complex plane except the origin. The function is continuous everywhere except at the origin. It has an antiderivative $\log z$ but not entirely in D, which means that
$$\frac{d}{dz}(\log z) = \frac{1}{z}$$

not everywhere in the domain D. Let us see how this happens. Consider the closed contour $z = re^{i\theta}$ the circle with radius r around the origin. Then on this circle

$$\log z = \log re^{i\theta} = \log r + i\theta$$

If we start at $\theta = 0$ and go round the cirle once then theta increases by 2 π. Thus at the same point $\log z$ has values

$$\log r + i0 \quad \text{and} \quad \log r + i2\pi$$

Thus the function is discontinuous in D. So in D, $f(z)$ does not have an antiderivative.

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = 2i\pi$$

7 Cauchy-Gorsat Theorem. Theorem 2

Let C be a simple closed contour given parametrically as $z = z(t)$, $(a \le t \le b)$. Let f(z) be analytic everywhere interior to and on C. Then

$$\int_C f(z)dz = \int_C (u(x,y) + iv(x,y))(dx + idy) = \int_C (udx - vdy) + i\int_C (vdx + udy).$$

Now, let us use a result from real variable calculus.
Green's Theorem: If two real valued functions $P(x,y)$ and $Q(x,y)$ together with their first order partial derivatives are continuous throughout a region R consisting of all points interior to and on C, then

$$\int_C Pdx + Qdy = \int\int_R (Q_x - P_y)dxdy,$$

the line integral done in the counter clockwise direction.

Now, since $f(z)$ is analytic, it is continuous and it satisfies the CR equations. Also, for now let us assume that $f'(z)$ is continuous, so that u_x, v_x, u_y and v_y are continuous.

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) = \int\int(-v_x - u_y)dxdy + \int\int(u_x - v_y)dxdy,$$

from the theorem above. Using the CR equations we get

$$\int_C f(z)dz = 0.$$

Here we had to assume that $f'(z)$ be continuous. This was proved by Cauchy. It was later proved by Gorsat that the continuity of f' is not required. So we state the Cauchy-Gorsat theorem: If $f(z)$ is analytic interior to and on a simple closed contour C, then

$$\int_C f(z)dz = 0.$$

C is positively oriented. This is also a one-way theorem, if the function is analytic then the integral is zero, not the other way. For example, above we saw

$$\int_C \frac{dz}{z^2} = 0$$

when C is any simple closed contour enclosing the origin. But the integrand here is not analytic at the origin. The integral is zero does not imply analyticity of the function.

Lecture 6 ends here.

7.1 The consequence of this theorem

If

$$\int_C f(z)dz = 0$$

in the Domain enclosed by C, then from theorem 1, $f(z)$ has an antiderivative, i.e., $F'(z) = f(z)$. Hence, since $F'(z)$ exists everywhere inside and on C, $F(z)$ is analytic inside and on C. The theorem then applies to $F(z)$ which again has an antiderivative and so on.

7.2 Alternative forms of the theorem

If $f(z)$ is analytic throughout a simply-connected domain D, then

$$\int_C f(z)dz = 0.$$

for every closed contour C lying in D.

7.3 Extension to multiply-connected domains: Theorem 3

Let C and C_k (k=1,2,3...n) be the simple closed contours. C is described in the counter clockwise direction. C_k are described in the clockwise direction and are interior to C and their interiors have nothing in common. If a function f is analytic throughout the closed region consisting of all points within and on C except for points interior to any C_k then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

7.4 Corollary:

Let C_1 and C_2 denote positively oriented simple closed contours, where C_2 is interior to C_1. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

If C_1 is continuously deformed to C_2, always passing through points at which f is analytic, then the value of the integral of f over C_1 never changes.

8 Cauchy Integral Formula: Theorem 4

Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_o is any point interior to C, then

$$2\pi i f(z_0) = \int_C \frac{f(z)}{z - z_0} dz.$$

Since the function $f(z)$ is analytic and hence continuous inside C, we have

$$|f(z) - f(z_0)| < \epsilon \ \ for \ \ |z - z_0| < \delta.$$

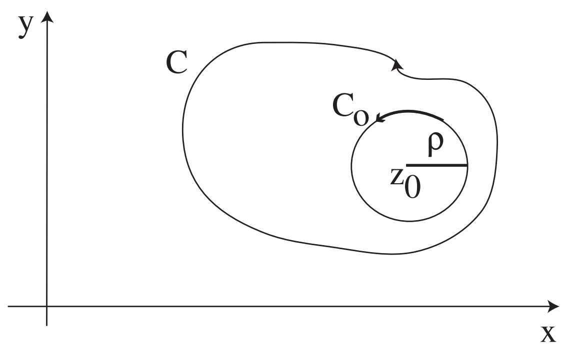


Figure 2: Figure for Cauchy Formula.

This statement will be used later. Now since $f(z)/(z-z_0)$ is analytic in the closed region consisting of the contours C and C_0 and all points between them, we know from the corollary above that

$$\int_C \frac{f(z)}{z-z_0}dz = \int_{C_0} \frac{f(z)}{z-z_0}dz.$$

and C_0 can be as small as possible without actually going to a point till we touch the singularity $z = z_0$. Thus, we have

$$\int_C \frac{f(z)}{z-z_0}dz - f(z_0)\int_{C_0} \frac{dz}{z-z_0} = \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz.$$

And the absolute value of the last integral is bounded as follows. (The magnitude of the integral is less than the integral of the magnitude).

$$|\int_{Co} \frac{f(z)-f(z_0)}{z-z_0}dz| < \int_{Co} |\frac{f(z)-f(z_0)}{z-z_0}|dz < \frac{\epsilon 2\pi\rho}{\rho} = 2\pi\epsilon.$$

As C_0 can be made as small as possible, ϵ can be arbitrarily small, then integral goes to zero. Also,

$$2\pi i = \int_{C_0} \frac{dz}{z-z_0}.$$

and hence

$$2\pi i f(z0) = \int_C \frac{f(z)}{z-z0}dz.$$

8.1 Extensions to Cauchy Integral Formula

Let f be analytical within and on a simple closed contour C, and let z be any interior point, then

$$2\pi i f(z) = \int_C \frac{f(s)}{s-z}ds.$$

Consider

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}ds \qquad (15)$$

$$
\begin{aligned}
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \int_C \left(\frac{f(s)ds}{(s-z-\Delta z)(s-z)} \right) \\
&= \frac{1}{2\pi i} \int_C \frac{f(s)ds}{(s-z)^2} = f'(z).
\end{aligned}
$$

Let us further take another derivative

$$
\begin{aligned}
\lim_{\Delta z \to 0} \frac{f'(z+\Delta z) - f'(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{1}{2\pi i} \int_C \left(\frac{1}{(s-z-\Delta z)^2} - \frac{1}{(s-z)^2} \right) \frac{f(s)}{\Delta z} ds \quad (16) \\
&= \frac{n!}{2\pi i} \int_C \frac{f(s)ds}{(s-z)^{n+1}}, \quad n = 2.
\end{aligned}
$$

Thus, we can continue for ever. So if a function is analytic at a point, then it has derivatives of all orders at that point. This is not true in real variable calculus.

Lecture 7 ends here.

Earlier we said that if a function f is analytic interior to and on a simple closed contour C then

$$\int_C f(z)dz = 0,$$

and not the other way. Now we say, if a function is such that in a domain

$$\int_C f(z)dz = 0,$$

then f has an antiderivative F in that domain and an antiderivative is analytic. Thus, all orders of derivatives exist in that domain and thus f is analytic in that domain. The domain may or may not be simply-connected. If it is simply-connected then we have the exact converse.

9 Cauchy Principal Value

The Reimann integral we know from undergraduate is for continuous and piecewise continuous functions and also on finite intervals (not when the interval is infinite). However, we extend the idea in the following way:

- The improper integral below on the LHS implies the RHS.

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx.$$

  Similarly, below

$$\int_{-\infty}^a f(x)dx = \lim_{c \to -\infty} \int_c^a f(x)dx.$$

- For example consider

$$\int_{-\infty}^\infty xdx = \lim_{-a,b \to \infty} xdx = \lim_{-a,b \to \infty} \frac{x^2}{2}|_{-a}^b = \infty - \infty$$

- However, there is another way to compute the above integral which is a restricted version. The integral with infinity as limits can be computed in one shot as follows

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} f(x)dx$$

  Even if the function is not integrable in the strict sense, the above integral may exist, which is called the Cauchy Principal Value. Given a function continuous on $(-\infty, \infty)$, the limit above is called the CPV. When the improper integral exists, it must equal the CPV. Not the other way.

- Next consider, $f(x) = x$.

$$\lim_{\rho\to\infty} \int_{-\rho}^{\rho} xdx = \lim_{\rho\to\infty} \frac{x^2}{2}|_{-\rho}^{\rho} = 0,$$

  exists. So by taking the two limits approaching simultaneously we can make the integral converge.

  So, we will be seeing integrals in our future lectures and they will be done in the Cauchy Principle value sense. That is to be noted.

10 Cauchy Residue Theorem

We need one more last theorem which is the very famous Cauchy Residue Theorem. Many of you should have seen in your earlier classes. The theorem says, let C be a positively oriented simple closed contour. If a function f is analytic inside and on C, except for a finite number of singular points $z(k), k = 1....n$ inside C. Then

$$\oint_C F(z)dz = 2\pi i \sum_k Res\,(f(z))\,|_{(z(k),k=1...n)}, \tag{17}$$

where the contour is taken in the positive sense. Now, I will not prove it; we will accept it as it is and this is very useful when doing our contour integrations. So, now, we are ready to take a look at our first example in contour integration.

11 Problem 1

Consider the problem

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

If we take the substitution

$$x = tan\,\theta \;\; then \;\; dx = sec^2\theta d\theta, \;\; I = \int \frac{sec^2\theta}{tan^2\theta + 1} d\theta$$

which leads to

$$I = \int d\theta = \pi$$

Lecture 8 ends here.

Next we consider the same problem using complex variables and a contour integration in the complex plane. We can recall from our earlier introduction to complex variables that if we have the given integral on a closed contour

$$\oint_C f(z)dz = 2\pi i \sum_n Res(f(z_n))$$

where $z_n$are the poles of f(z). Now, suppose we have,

$$J = \oint_C \frac{dz}{z^2+1}$$

where the contour C is shown in figure 1 with the curved portion being a semi-circle with an infinite radius. The integrand has poles at $Z = \pm i$. Then

$$J = \oint_C \frac{dz}{z^2+1} = \int_{-\infty}^{\infty} \frac{dz}{z^2+1} + \int_C \frac{dz}{z^2+1} = 2\pi i Res(f(z=i)) = 2\pi \frac{i}{z+i}|_{z=i} = \pi$$

and

$$J = I$$

provided

$$\int_C \frac{dz}{z^2+1} = 0$$

which we have to prove.

Considering the integral

$$\int_C \frac{dz}{z^2+1}$$

at $R = \infty$ we substitute $z = Re^{i\theta}$ and get

$$|\lim_{R\to\infty} \int_0^{\pi} \frac{iRe^{i\theta}d\theta}{R^2e^{2i\theta}+1}| \leq \lim_{R\to\infty} \int_0^{\pi} |\frac{iRe^{i\theta}d\theta}{R^2e^{2i\theta}+1}| \leq \lim_{R\to\infty} \int_0^{\pi} |\frac{Rd\theta}{R^2-1}| = \lim_{R\to\infty} \frac{\pi R}{R^2-1} \to 0$$

where we have used the inequality

$$|Z1 + Z2| \geq ||Z_1| - |Z_2||$$

And so we find that using contour integration and the residue theorem,

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$$

12 Problem 2

Consider

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$$

Again we replace this with

$$J = \oint_C \frac{z^2}{z^4+1} dz = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx + \int_C \frac{z^2}{z^4+1} dz = 2\pi i \sum_n Res(f(z=z_n)$$

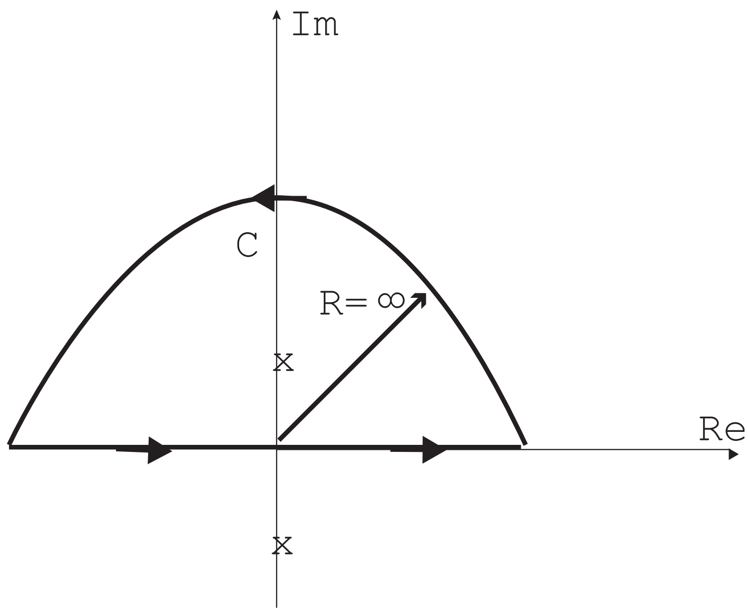


Figure 3: Contour for problem 1.

Here the integrand has the following poles, $Z = e^{i\frac{\pi}{4}}, e^{3i\frac{\pi}{4}}$ and $Z = e^{-i\frac{\pi}{4}}, e^{-3i\frac{\pi}{4}}$. The contour remains the same, except that now there are two poles within the contour as given above. Thus, the solution is

$$J = 2\pi i \frac{(e^{i\frac{\pi}{4}})^2}{(e^{i\frac{\pi}{4}} - e^{3i\frac{\pi}{4}})(e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})(e^{i\frac{\pi}{4}} - e^{-3i\frac{\pi}{4}})} + 2\pi i \frac{(e^{3i\frac{\pi}{4}})^2}{(e^{3i\frac{\pi}{4}} - e^{i\frac{\pi}{4}})(e^{3i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})(e^{3i\frac{\pi}{4}} - e^{-3i\frac{\pi}{4}})} =$$

$$2\pi i \left(\frac{i}{(e^{i\frac{\pi}{4}} - e^{3i\frac{\pi}{4}})(i\sqrt{2})(e^{i\frac{\pi}{4}} - e^{-3i\frac{\pi}{4}})} + \frac{-i}{(e^{3i\frac{\pi}{4}} - e^{i\frac{\pi}{4}})(e^{3i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})(i\sqrt{2})} \right) =$$

$$2\pi i \left(\frac{i}{i\sqrt{2}} \frac{1}{(e^{i\frac{\pi}{4}} - e^{3i\frac{\pi}{4}})} \left(\frac{1}{(e^{i\frac{\pi}{4}} - e^{-3i\frac{\pi}{4}})} + \frac{1}{(e^{3i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}})} \right) \right) =$$

$$\frac{\sqrt{2}\pi i}{(e^{i\frac{\pi}{4}} - e^{3i\frac{\pi}{4}})} \left(\frac{(e^{3i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}) + (e^{i\frac{\pi}{4}} - e^{-3i\frac{\pi}{4}})}{-1-1-1-1} \right) =$$

$$\frac{\sqrt{2}\pi i\sqrt{2}}{1+i-(-1+i)} \left(\frac{2isin(3\pi/4) + 2isin(\pi/4)}{-4} \right) =$$

$$i\pi \left(\frac{2\sqrt{2}i}{-4} \right) = \frac{\pi}{\sqrt{2}}$$

13 Problem 3: $sinx/x$

Integration of $sin\ x/x$ from $-\infty$ to ∞ is an interesting problem. We will do it in four different ways.

13.1 Method 1

In the first method let us consider

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx = \int_{-\infty}^{\infty} \frac{cos(ax)}{x} dx + i \int_{-\infty}^{\infty} \frac{sin(ax)}{x} dx$$

So when Z is real the integral is the imaginary part of the above integral. So if we select a contour as shown below, then on the real axis, $Z = x$ and if we can find the values of the other integrals we can find the answer as the imaginary part of the real line integral. Thus, we replace the integral with

$$\oint_C \frac{e^{iaz}}{z} dz = \int_{-\infty}^{-\epsilon} \frac{e^{iaz}}{z} dz + \int_{C\epsilon} \frac{e^{iaz}}{z} dz + \int_{\epsilon}^{\infty} \frac{e^{iaz}}{z} dz + \int_{C_R} \frac{e^{iaz}}{z} dz$$

As ϵ tends to zero we find that the first and the third part give us the real line integral

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx,$$

in the Cauchy Principal Value sense. As for C_R integral, using Jordan's lemma, it goes to zero for $a > 0$. With regard to C_ϵ we have for

$$\int_{C\epsilon} \frac{e^{iaz}}{z} dz$$

$z = 0$ is a simple pole. Thus, one of the theorems (given below) says

$$\int_{C\epsilon} \frac{e^{iaz}}{z} dz = i\phi C_{-1} = -i\pi$$

where C_{-1} is the residue at the pole. $\phi = -\pi$ because we go in the CW direction.
Lecture 9 ends here
Thus we have

$$\oint_C \frac{e^{iaz}}{z} dz = \int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx - i\pi = 0$$

since no pole is enclosed. Thus,

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx = i\pi$$

Matching real to real etc., we have

$$\int_{-\infty}^{\infty} \frac{sinax}{x} dx = \pi$$

14 Some useful theorems

Theorem 4.2.1
Let $f(z) = N(z)/D(z)$ be a rational function such that the degree of $D(z)$ exceeds the degree of $N(z)$ by at least two. Then

$$\lim_{R\to\infty} \int_{C_R} f(z) dz = 0$$

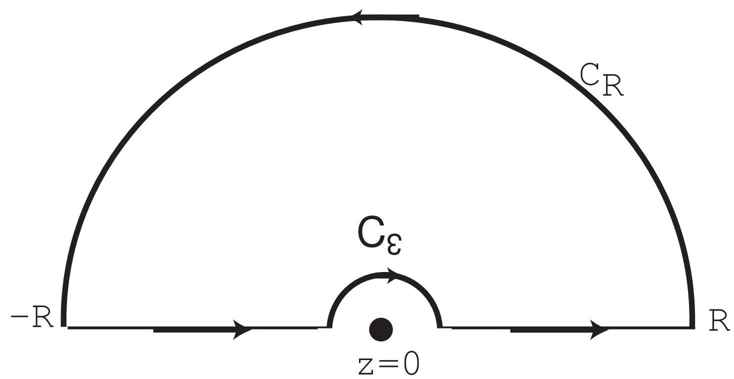


Figure 4: Contour for problem 3 sin x by x.

Theorem 4.2.2: Jordan Lemma
Suppose on a circular arc C_R that is in the upper half of the complex plane, $f(z) \to 0$ uniformly as $R \to \infty$ then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{ikz}dz = 0, \quad k > 0$$

Alternatively

Suppose on a circular arc C_R that is in the lower half of the complex plane, $f(z) \to 0$ uniformly as $R \to \infty$ then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{-ikz}dz = 0, \quad k > 0$$

Theorem 4.3.1a
Suppose on a contour C_ϵ, as shown in the figure below,

$$(z - z_0)f(z) \to 0 \ \ uniformly \ \ as \ \ \epsilon \ \to 0, \ \ then \ \ \lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)dz = 0$$

Theorem 4.3.1b
Suppose $f(z)$ has a simple pole at $z = z_0$ with a residue= C_{-1} then

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z)dz = i\phi C_{-1}$$

where ϕ is the angle subtended by the arc and is taken positive in the anti-clockwise sense.

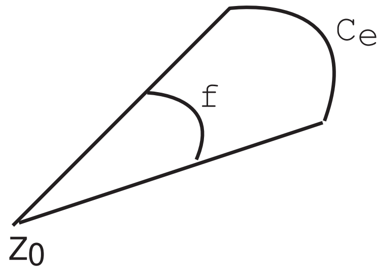


Figure 5: Figure for theorem 4.3.1a.

14.1 Method 2 $\frac{sinx}{x}$

In the second method, we will include the pole within the contour. Thus,

$$\oint_C \frac{e^{iaz}}{z}dz = \int_{-\infty}^{-\epsilon} \frac{e^{iaz}}{z}dz + \int_{C\epsilon CC} \frac{e^{iaz}}{z}dz + \int_{\epsilon}^{\infty} \frac{e^{iaz}}{z}dz + \int_{C_R} \frac{e^{iaz}}{z}dz$$

Or

$$\oint_C \frac{e^{iaz}}{z}dz = \int_{-\infty}^{\infty} \frac{e^{iaz}}{z}dz + \int_{C\epsilon\ CC} \frac{e^{iaz}}{z}dz = 2\pi i(ResF(z=0))$$

Or

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{z}dz + i\pi = 2\pi i$$

Or

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{z}dz = \pi i$$

And the result follows.
Lecture 10 ends here.

14.2 Method 3

$$\int_{-\infty}^{\infty} \frac{sin(ax)}{x}dx = \int_{-\infty}^{\infty} \frac{e^{iax} - e^{-iax}}{2ix}dx \to \int_{-\infty}^{\infty} \frac{e^{iaz} - e^{-iaz}}{2iz}dz$$

The logic here is this: The first integral (what we want) is exactly equal to the second representation. We make it into a complex integral (with real limits) such that the complex integral will equal the two prior integrals when $z = x$. Hence the complex integral (with real limits) must be a branch of any contour we choose. Next, since the complex integral is a sum then it has to be split into its components and then each integral has a singularity at $z = 0$. We must then calculate the penalty for integrating through the singularity.
The first term is

$$J_1 = \int_{-\infty}^{\infty} \frac{e^{iaz}}{2iz}dz$$

If this is integrated over the same contour as in figure 4,

$$\oint_C \frac{e^{iaz}}{2iz}dz = \int_{-\infty}^{\infty} \frac{e^{iaz}}{2iz}dz + \int_{C\epsilon\ Cw} \frac{e^{iaz}}{2iz}dz + \int_{CR}$$

The CR integral is zero by Jordan lemma. Or

$$\oint_C \frac{e^{iaz}}{2iz}dz = \int_{-\infty}^{\infty} \frac{e^{iaz}}{2iz}dz + \int_{C\epsilon \ Cw} \frac{e^{iaz}}{2iz}dz = 0$$

Or

$$\oint_C \frac{e^{iaz}}{2iz}dz = \int_{-\infty}^{\infty} \frac{e^{iaz}}{2iz}dz + \frac{-i\pi}{2i} = 0,$$

using theorem 4.3.1b. Or

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{2iz}dz = \frac{\pi}{2}$$

The second term is

$$J_2 = -\int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz}dz$$

If this is integrated over the same contour

$$\oint_C \frac{e^{-iaz}}{2iz}dz = \int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz}dz + \int_{C\epsilon \ Cw} \frac{e^{iaz}}{2iz}dz + \int_{CR} \to \infty$$

The CR integral is infinity because for $a > 0$ the Jordan lemma holds for the upper half of the complex plane. So we take the bottom contour and include the pole.

$$\oint_C \frac{e^{-iaz}}{2iz}dz = \int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz}dz + \int_{C\epsilon \ Cw} \frac{e^{iaz}}{2iz}dz = -2\pi i Res(F(z=0))$$

where the negative on the residue is because the direction of the contour is clockwise. Or

$$\oint_C \frac{e^{-iaz}}{2iz}dz = \int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz}dz + \frac{-\pi}{2} = (-1)2\pi i Res(F(z=0))$$

Or

$$\int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz}dz = +\frac{\pi}{2} - \pi$$

Thus

$$-\int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz}dz = +\frac{\pi}{2}$$

So

$$\int_{-\infty}^{\infty} \frac{e^{iaz} - e^{-iaz}}{2iz}dz = \pi = \int_{-\infty}^{\infty} \frac{sin(ax)}{x}dx$$

14.3 Method 4

At times, a point z seems like a singularity, but when the limits are computed, then it is not a singularity.

Note: *If an integral has to go through a singularity then there is no choice but to go through it using an ϵ contour and see if this is an integrable singularity by taking ϵ to zero. There is no avoiding it by deforming the contour. Avoiding the 'singularity' by starting with an indented contour can be done only when the seeming 'singularity' is not a singularity, i.e., the function is analytic in the domain. With a singularity it is not.*

However, in this case, there is no singularity in $sin(ax)/x$ to begin with at $x = 0$. So when we break it into

$$\int_{-\infty}^{\infty} \frac{sin(ax)}{x} dx = \int_{-\infty}^{\infty} \frac{e^{iax} - e^{-iax}}{2ix} dx \to \int_{-\infty}^{\infty} \frac{e^{iaz} - e^{-iaz}}{2iz} dz$$

the complex integral is continuous at $z = 0$ and analytic all the way along the real axis. First let us observe that around the origin

$$\lim_{\epsilon \to 0} \oint_{\epsilon} \frac{e^{iaz} - e^{-iaz}}{2iz} dz, \quad with z = \epsilon^{i\theta}, \quad dz = i\epsilon^{i\theta} d\theta$$

we get

$$\lim_{\epsilon \to 0} \oint_{\epsilon} \frac{e^{ia\epsilon(cos\theta + isin\theta)} - e^{-ia\epsilon(cos\theta + isin\theta)}}{2i\epsilon^{i\theta}} i\epsilon^{i\theta} d\theta = 0.$$

If f is continuous in a domain D and integrals round closed contours are zero, then the function is analytic in D. Thus, the function is not singular and has no integral contributions from $z = 0$. So nothing new can come integrating through the $z = 0$ point. Hence, we choose an indented contour to begin with as shown in figure below (figure 6). The value of the integral must be the same passing through the origin or on the indented contour.

lecture 11 ends here.

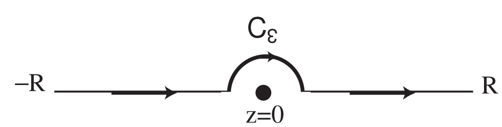


Figure 6: Contour for problem 3 method 4.

First Term:

$$J_1 = \int_{-\infty}^{\infty} \frac{e^{iaz}}{2iz} dz$$

We have two terms in the integral above. We take the first term. We now choose to deform the already indented path. If we continuously deform the path now to $+\infty$ this function is analytic throughout and follows Jordan's Lemma. Given $a > 0$ and $1/z$ goes to zero uniformly as z goes to infinity. Thus,

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{2iz} dz = 0.$$

Second Term:

$$J_2 = -\int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz} dz$$

For now we keep the negative sign out and just remember to bring it back at the end. This term because of the $-iaz = -ia(x+iy) = -iax + ay$ goes to infinity as y becomes large positive in the upper half plane. This function is non-analytic at $z = \infty$ in the upper half. (If the value of a function becomes infinity then it has a singularity at that point). So we deform it the contour downward as shown in the figure 7. So for the second integral, we

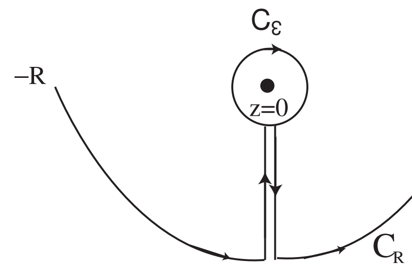


Figure 7: Contour for problem 3 method 4a.

have the downward contour and the integral is

$$\int_{-\infty}^{\infty} \frac{e^{-iaz}}{2iz} dz.$$

If this is integrated over the new downward contour

$$\oint_C \frac{e^{-iaz}}{2iz} dz = \int_{CR} + \int_{C\epsilon\ \ Cw} \frac{e^{-iaz}}{2iz} dz + \left(\int_{-\infty\ to\ 0} + \int_{0\ to\ -\infty} \right) \frac{e^{-iaz}}{2iz} dz$$

We will hold onto the end points, i.e., $+\infty$ and $-\infty$, and continuously deform the contour downward, into CR at $-\infty$. The value of this C_R integral by Jordan's Lemma is zero. The two upward and downward integrals sum to zero, since the function is analytic and uniquely defined. The integral limits are opposite. Thus, we are left with the integral round the singularity in the clockwise direction. This gives a residue

$$\int_{C\epsilon\ \ Cw} \frac{e^{-iaz}}{2iz} dz = -2\pi i \frac{1}{2i} = -\pi$$

Thus remembering that there is a negative sign

$$\int_{-\infty}^{\infty} \frac{e^{iaz} - e^{-iaz}}{2iz} dz = \pi$$

as before.

15 Problem 4

Consider

$$I = \int_0^{\infty} \frac{xdx}{x^3+1} dx$$

We write it as

$$J = \oint_C \frac{zdz}{z^3+1} = \int_0^{\infty} \frac{xdx}{x^3+1} + \int_{C_R} \frac{zdz}{z^3+1} + \int_C \frac{zdz}{z^3+1}$$

on the contour shown in figure 8. For finding the singularities of the integrand we have

$$\begin{aligned} z^3+1 &= 0 \Rightarrow \\ z^3 &= e^{(i\pi+i2\pi k)} \quad (18)\\ z &= e^{(i\pi/3+i2\pi k/3)}, k=0,1,2 \quad (19)\\ z &= e^{(i\pi/3)}, e^{(i\pi)}, e^{(i5\pi/3)} \quad (20)\end{aligned}$$

These are at 60^o, 180^o and 300^o. In the video lectures I have made a mistake and written these as 60^o, 120^o and 300^o (**this is incorrect**). Below we take a closed contour which encloses only the first pole. One can see that the C_R integral goes to zero. Thus,

$$J = \int_0^\infty \frac{xdx}{x^3+1} + \int_C \frac{zdz}{z^3+1} = 2\pi i\ Res(e^{(i\pi/3)})$$

A word about residues: If $f(z)$ is a ratio of two analytic functions, $p(z)$ and $q(z)$ and $f(z)$ has a simple pole at z_0, then the residue at $z=z_o$ **is also** found as

$$Res_{z=z_o}\frac{p(z)}{q(z)} = \frac{p(z_o)}{q'(z_o)}$$

provided $q'(z_o)\neq 0$. Here, the residue is

$$Res(e^{(i\pi/3)}) = \frac{p}{q'}|_{z=e^{(i\pi/3)}} = \frac{e^{(i\pi/3)}}{3e^{(2i\pi/3)}}$$

Thus

$$J = \int_0^\infty \frac{xdx}{x^3+1} + \int_C \frac{zdz}{z^3+1} = \frac{2\pi i}{3}e^{(-i\pi/3)}$$

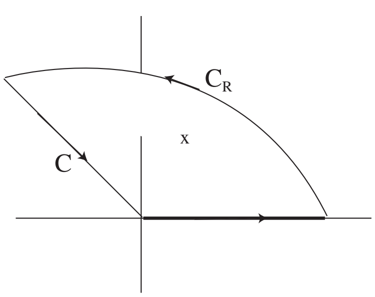


Figure 8: Contour for problem 4.

Let us look at the integral along the straight line contour C.

$$\int_C \frac{zdz}{z^3+1}$$

Set

$$\begin{aligned} z &= re^{i\theta}, dz = dre^{i\theta} \\ z &= 0, r=0 \quad (21)\\ z &= \infty, r=\infty \quad (22)\end{aligned}$$

$$\int_C \frac{zdz}{z^3+1} = \int_\infty^0 \frac{re^{i\theta}dre^{i\theta}}{r^3e^{3i\theta}+1} = -\int_0^\infty \frac{rdre^{2i\theta}}{r^3+1} = -Ie^{i4\pi/3}$$

since $\theta = 2\pi/3$. So **(Lecture 12 ends here.)**

$$I(1-e^{i4\pi/3}) = \frac{2\pi i}{3}e^{(-i\pi/3)}$$

$$I = \frac{2\pi}{3\sqrt{3}}$$

16 Problem 4a

Let us do the same problem using the method of contour deformation. We consider the original integral I as an integral of a complex integrand.

$$I = \int_0^\infty \frac{zdz}{z^3+1}$$

where the integral is still on the real axis. Now we deform this straight line 'contour' as shown in the figure 9. Remember to fix the end points when deforming the contour. Since

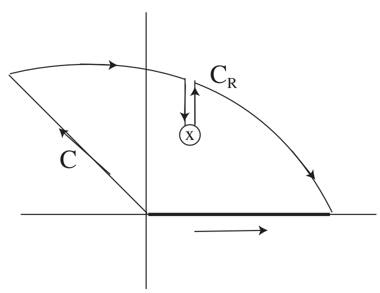


Figure 9: Contour for problem 4a.

we cannot cut across a pole, the contour encircles the pole with two opposite going segments. The function is well defined on these segments. Thus we get

$$I = I_c + I_{C_R} + I_{two-opposites} + I_{round-the-pole}$$

I_{C_R} and $I_{two-opposites}$ are zero so we are left with

$$I = I_c + I_{round-the-pole}$$

I_c from the previous method is

$$Ie^{i4\pi/3}$$

(mind the sign). Thus,

$$I(1-e^{i4\pi/3}) = I_{round-the-pole} = 2\pi i \; Res(e^{(i\pi/3)})$$

This gives the same answer as before.

Appendix: Branch Cut

The study of branch cuts of multi-valued complex functions will be carried through the specific function $f(z) = \left(z^2 - z_0^2\right)^{\frac{1}{2}}$.

Factorizing into linear factors we get $f(z) = \left(z^2 - z_0^2\right)^{\frac{1}{2}} = (z - z_0)^{\frac{1}{2}}\,(z + z_0)^{\frac{1}{2}}$. Consider the function $(z - z_0)^{\frac{1}{2}}$ in a small neighbourhood around z_0. It is given by $|z - z_0|\, e^{\frac{i\theta}{2}}$, where θ is the argument of $(z - z_0)$. Because of the ambiguity in phase, if we make the convention $\beta < \theta < 2\pi + \beta$, then traversing a complete circle around z_0 from β to $2\pi + \beta$ we go continuously from $|z - z_0|\, e^{\frac{i\beta}{2}}$ to $|z - z_0|\, e^{i\left(\pi + \frac{\beta}{2}\right)} = -\,|z - z_0|^{\frac{1}{2}}\, e^{\frac{i\beta}{2}}$.

Thus, though the value of the function changes continuously on moving around the circle, on completion of the circle a *jump* is encountered between the starting and ending point. This is a typical example of a *branch point* of a multi-valued complex function. Branch point of a multi-valued function is defined as a point in the complex plane at which by traversing a closed circle around it of any radius, the value of the function at the starting point and the ending point are unequal. It was shown by the previous argument that the function $(z - z_0)^{\frac{1}{2}}$ has a branch point at z_0. Similarly the function $(z + z_0)^{\frac{1}{2}}$ has a branch point at $-z_0$. Thus, the function $\left(z^2 - z_0^2\right)^{\frac{1}{2}}$ has branch points at z_0 and $-z_0$. It may be proved that these are the only branch points of the above function , *i.e.,* no other branch points exist.

A branch cut is a curve joining branch points. By definition, points across the branch cut have a jump in the function value. The function is undefined along the branch, however by specifying the branch cut we make the function analytic and single-valued over the remaining part of the complex plane. Obviously, there are many curves which may join the branch points and hence there are different branch cuts possible for the multi-valued function. Each branch cut specifies a different branch of the multi-valued function. Needless to say, in applications, we will generally choose the easiest. Here, we illustrate a few choices for branch cuts for the function $\left(z^2 - z_0^2\right)^{\frac{1}{2}}$ and illustrate how in each case it specifies the single-valued continuous branch.

1. This choice of branch cut is shown graphically in figure 10. The thick line denotes the branch cut. The function value jumps between two sides of the branch. To get the jump on the right of $-z_0$ we select

   $$(z + z_0)^{\frac{1}{2}} = |z + z_0|^{\frac{1}{2}}\, e^{i\frac{\phi}{2}}, \quad \text{where } 0 < \phi = Arg(z + z_0) < 2\pi.$$

   Similarly, to get the jump on the left of z_0 we select

   $$(z - z_0)^{\frac{1}{2}} = |z - z_0|^{\frac{1}{2}}\, e^{i\frac{\theta}{2}}, \quad \text{where } -\pi < \theta = Arg(z - z_0) < \pi.$$

   Thus,

   $$\left(z^2 - z_0^2\right)^{\frac{1}{2}} = |z - z_0|^{\frac{1}{2}}\, |z + z_0|^{\frac{1}{2}}\, e^{i\frac{\theta + \phi}{2}},$$

   where $-\pi < \theta = Arg(z - z_0) < \pi, \;\; 0 < \phi = Arg(z + z_0) < 2\pi$.

2. This choice of branch cut is shown graphically in figure 11. The thick line denotes the branch cut. The function value jumps between two sides of the branch. To get the jump on the left of $-z_0$ we select

   $$(z + z_0)^{\frac{1}{2}} = |z + z_0|^{\frac{1}{2}}\, e^{i\frac{\phi}{2}} \quad \text{where } -\pi < \phi = Arg(z + z_0) < \pi$$

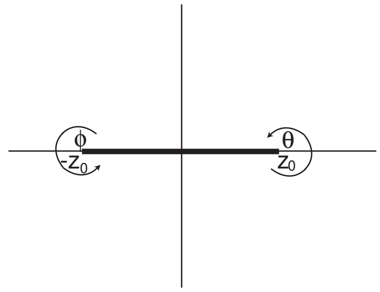


Figure 10: Branch cut joining branch points $-z_0$ and z_0

Similarly, to get the jump on the right of z_0 we select

$$(z-z_0)^{\frac{1}{2}} = |z-z_0|^{\frac{1}{2}}\, e^{i\frac{\theta}{2}}, \quad \text{where } 0 < \theta = Arg(z-z_0) < 2\pi.$$

Thus,

$$\left(z^2 - z_0^2\right)^{\frac{1}{2}} = |z-z_0|^{\frac{1}{2}}\, |z+z_0|^{\frac{1}{2}}\, e^{i\frac{\theta+\phi}{2}},$$

where $0 < \theta = Arg(z-z_0) < 2\pi \quad -\pi < \phi = Arg(z+z_0) < \pi$. (Lecture 13 ends here)

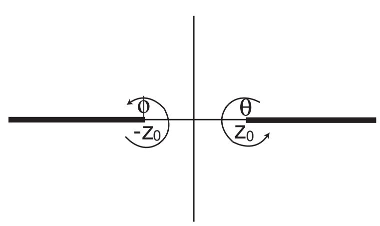


Figure 11: Branch cut going to from $-z_0$ going to ∞ in the left side and from z_0 to ∞ in the right side. Branch Cut joins the branch points through infinity.

Lecture 14 describes the branch cut in detail
Lecture 15 describes the finite branch cut in the beginning and continues with the following problem.

17 Problem 5

Consider

$$I = \int_0^\infty \frac{dx}{\sqrt{x}(1+x^2)}$$

on a key hole contour given below

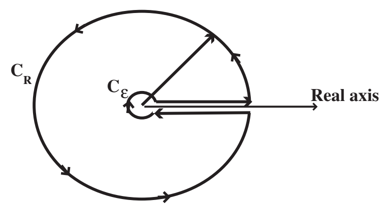


Figure 12: Contour for problem 5.

We replace the integral by

$$J = \oint_C \frac{dz}{\sqrt{z}(1+z^2)}$$

The square root function has a branch cut along the positive x axis. So we get

$$J = \int_\epsilon^\infty + \int_{C_R} + \int_\infty^\epsilon + \int_{C_\epsilon}$$

The C_R integral is zero by the reason of polynomial degrees. For the C_ϵ we have

$$\begin{aligned} z &= \epsilon e^{i\theta} \\ dz &= \epsilon i e^{i\theta} d\theta \end{aligned} \tag{23}$$

$$\int_{C_\epsilon} = \lim_{\epsilon \to 0} \int_{2\pi}^0 \frac{\epsilon e^{i\theta} i d\theta}{\sqrt{\epsilon} e^{i\theta/2}(1+\epsilon^2 ...)} = 0 \tag{24}$$

Thus, we are left with

$$J = \int_\epsilon^\infty + \int_\infty^\epsilon = upper + Lower\ Branch$$

$$\begin{aligned} z &= re^{i\theta} \\ dz &= dre^{i\theta} \end{aligned} \tag{25}$$

$$J = \lim_{r\to\infty} \left(\int_0^r \frac{dre^{i\theta}}{\sqrt{r}e^{i\theta/2}(1+r^2e^{2i\theta})}|_{\theta=0} + \int_r^0 \frac{dre^{i\theta}}{\sqrt{r}e^{i\theta/2}(1+r^2e^{2i\theta})}|_{\theta=2\pi} \right) \tag{26}$$

$$J = \lim_{r\to\infty} \left(\int_0^r \frac{dr}{\sqrt{r}(1+r^2)}|_{\theta=0} + \int_r^0 \frac{dr}{\sqrt{r}e^{i\pi}(1+r^2)} \right)$$

$$J = \lim_{r\to\infty} 2\int_0^r \frac{dr}{\sqrt{r}(1+r^2)} = 2\pi i Res(i,-i)$$

The residues are at $i, -i$.

$$\frac{1}{\sqrt{z}(z+i)}|_{z=i} + \frac{1}{\sqrt{z}(z-i)}|_{z=-i}$$

$$\frac{1}{\sqrt{z}(z+i)}|_{z=e^{i\pi/2}} + \frac{1}{\sqrt{z}(z-i)}|_{z=e^{i3\pi/2}}$$

$$\frac{1}{e^{i\pi/4}2i} + \frac{1}{e^{i3\pi/4}(-2i)}$$

$$2I = \frac{1}{2i}\left(\frac{1}{e^{i\pi/4}} + \frac{1}{e^{-i\pi/4}}\right)2\pi i$$

$$2I = \frac{1}{2i}2cos(\pi/4)2\pi i$$

$$I = cos(\pi/4)\pi = \frac{\pi}{\sqrt{2}}$$

Lecture 15 ends here.

18 Problem 6

I will briefly deviate here from the branch cut idea and solve one more traditional problem using two methods.
Consider

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}dx}{1+e^x}, \quad a < 1$$

$$J = \oint_C \frac{e^{az}dz}{1+e^z} = I + [extra\ terms]$$

$$J = I + \int_{R+2i\pi}^{-R+i2\pi} + \int_R^{R+i2\pi} + \int_{-R}^{-R+i2\pi}$$

$$J = I + term2 + term3 + term4$$

See the contour in figure 13.
The poles are at

$$z = e^{-i\pi+i2\pi k}, k = 0, 1, 2..$$

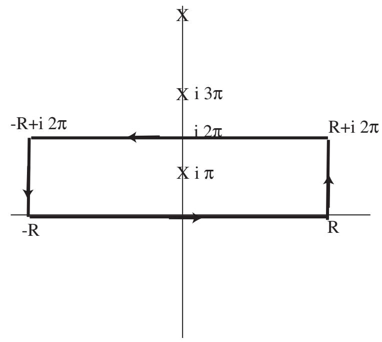


Figure 13: Contour for problem 6.

18.1 term 2

$$\lim_{R\to\infty}\int_{R+2i\pi}^{-R+i2\pi}\frac{e^{az}dz}{1+e^z}$$

$$\begin{aligned} z &= t+i2\pi \\ dz &= dt && (27) \\ z &= R+i2\pi \Rightarrow t=R && (28) \\ z &= -R+i2\pi \Rightarrow t=-R && (29) \end{aligned}$$

The integral becomes

$$\lim_{R\to\infty}\int_{R+2i\pi}^{-R+i2\pi}\frac{e^{az}dz}{1+e^z} = \lim_{R\to\infty}\int_{R}^{-R}\frac{e^{ai2\pi}e^{at}dt}{1+e^{t+i2\pi}} = \lim_{R\to\infty}\int_{R}^{-R}\frac{e^{ai2\pi}e^{at}dt}{1+e^{t}} = -Ie^{ia2\pi}$$

18.2 term 3

$$\lim_{R\to\infty}\int_{R}^{R+i2\pi}\frac{e^{az}dz}{1+e^z}$$

$$\begin{aligned} z &= R+ip \\ dz &= idp && (30) \\ z &= R \Rightarrow p=0 && (31) \\ z &= R+i2\pi \Rightarrow p=2\pi && (32) \end{aligned}$$

The integral becomes

$$\lim_{R\to\infty}\int_{0}^{2\pi}\frac{e^{aR}e^{aip}idp}{1+e^Re^{ip}} = \lim_{R\to\infty}\int_{0}^{2\pi}e^{(a-1)R}\frac{e^{aip}idp}{e^{-R}+e^{ip}} = 0$$

since $a<1$. Similarly term 4 is also zero. We have the residue at $z=i\pi$ left. The residue is

$$\frac{e^{ia\pi}}{e^{i\pi}}$$

Finally considering I and term 2 along with the residue we get

$$I(1-e^{ia2\pi}) = -2i\pi e^{ia\pi}$$

$$I(-2i)sin(a\pi)e^{ia\pi} = -2i\pi e^{ia\pi}$$

Or

$$I = \frac{\pi}{sin(a\pi)}$$

19 Problem 6a

We will do the same problem by the method of deformation of contours. See figure 14. Consider

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}dx}{1+e^x} = \int_C \frac{e^{az}dz}{1+e^z}$$

Now it is an integral on the complex domain. We keep the end points at infinity fixed. If we continuously deform (or stretch) the contour the value of the integral remains the same as long as we do not cut across a pole. Thus, we continuously deform the contour upward and as we hit a pole we deform the contour round it. The length of the contour keeps stretching. There will be vertical parts of the contour going in opposite directions and the poles get encircled in the anti-clockwise direction. We go till infinity upwards and in R. Thus the C_R integral goes to 0.

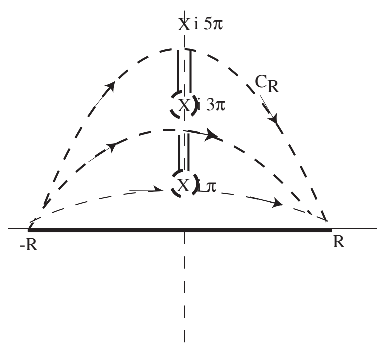


Figure 14: Contour for problem 6a.

Thus, we are left with integrations around the poles, i.e., residues only

$$J = 2\pi i \left(\frac{e^{ia\pi}}{e^{i\pi}} + \frac{e^{3ia\pi}}{e^{i\pi}} + \frac{e^{5ia\pi}}{e^{i\pi}} + ... \right)$$

Or

$$J = -2\pi i \left(e^{ia\pi} + e^{3ia\pi} + e^{5ia\pi} + ... \right)$$

Using the geometric series formula with $r = e^{ia2\pi}$

$$J = -2\pi i e^{ia\pi} \left(1 + e^{2ia\pi} + e^{4ia\pi} + ... \right)$$

Or

$$J = -2\pi i \frac{e^{ia\pi}}{1 - e^{2ia\pi}}$$

$$J = -2\pi i \frac{e^{ia\pi}}{e^{ia\pi}(e^{-ia\pi} - e^{ia\pi})}$$

$$J = \frac{\pi}{sin(a\pi)}$$

Lecture 16 ends here.

20 Problem 7

Consider the integral (from the book by Ablowitz and Fokas)

$$I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx$$

We replace this with the contour integral

$$J = \oint_C \frac{\sqrt{z^2-1}}{1+z^2} dz$$

Using polar coordinates

$$(z^2-1)^{1/2} = \sqrt{\rho_1 \rho_2} e^{i(\phi_1+\phi_2)/2}, \quad 0 \le \phi_1, \phi_2 < 2\pi$$

where

$$\begin{aligned} (z-1)^{1/2} &= \sqrt{\rho_1} e^{i\phi_1/2}, \rho_1 = |z-1| \\ (z+1)^{1/2} &= \sqrt{\rho_2} e^{i\phi_2/2}, \rho_2 = |z+1| \end{aligned} \qquad (33)$$

with this choice of the branch we find

$$(z^2-1)^{1/2} = \begin{cases} \sqrt{x^2-1} & 1 < x < \infty \\ -\sqrt{x^2-1} & -\infty < x < -1 \\ i\sqrt{1-x^2} & -1 < x < 1, y \to 0^+ \\ -i\sqrt{1-x^2} & -1 < x < 1, y \to 0^- \end{cases}$$

Using the expressions in the contour integral J, it follows that

$$\begin{aligned} J = \int_{-1+\epsilon_1}^{1-\epsilon_2} \frac{i\sqrt{1-x^2}}{1+x^2} dx \quad &+ \quad \int_{1-\epsilon_2}^{-1+\epsilon_1} \frac{-i\sqrt{1-x^2}}{1+x^2} dx + \left(\int_{C_{\epsilon 1}} + \int_{C_{\epsilon 2}} + \int_{C_R} \right) \frac{(z^2-1)^{1/2}}{1+z^2} dz \\ &= \quad i2\pi \left[\left(\frac{(z^2-1)^{1/2}}{2z} \right) |_{e^{i\pi/2}} + \left(\frac{(z^2-1)^{1/2}}{2z} \right) |_{e^{i3\pi/2}} \right] \end{aligned} \qquad (34)$$

We note that the crosscut integrals vanish

$$\left(\int_{L_0} + \int_{L_i} \right) \frac{(z^2-1)^{1/2}}{1+z^2} dz = 0$$

because L_0 and L_i are chosen in a region where $(z^2-1)^{1/2}$ is continuous and single valued, and L_0 and L_i are arbitrarily close to each other. Theorem 4.3.1a shows that $\int_{C_{\epsilon i}} \to 0$ as $\epsilon \to 0$, i.e.,

$$\begin{aligned} |\int_{C_{\epsilon_i}} \frac{(z^2-1)^{1/2}}{1+z^2} dz| \quad &\le \quad \int_0^{2\pi} \frac{(|\epsilon_i^2 e^{2i\theta} - 1|^{1/2})}{1+\epsilon_i^2 e^{2i\theta}} \epsilon_i d\theta \;\; \text{using the inequalities at the end} \\ &< \quad \int_0^{2\pi} \frac{\sqrt{\epsilon_i^2+1}}{1-\epsilon_i^2} \epsilon_i d\theta = \frac{\sqrt{\epsilon_i^2+1}}{1-\epsilon_i^2} \epsilon_i \int_0^{2\pi} d\theta \quad \to 0 \; as \; \epsilon_i \to 0 \end{aligned} \qquad (35)$$

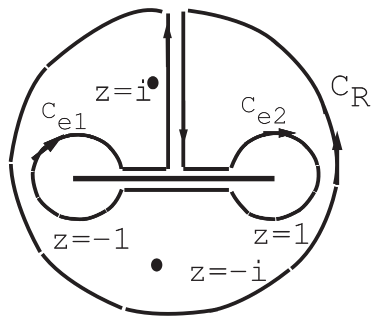


Figure 15: Contour for problem 7.

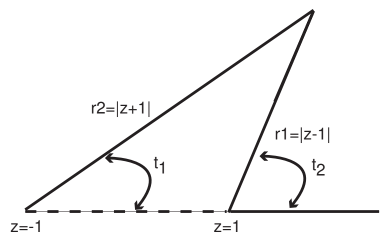


Figure 16: Square root function of problem 7.

Lecture 17 ends here.
The contribution from C_R is calculated as follows

$$\int_{C_R} \frac{(z^2-1)^{1/2}}{1+z^2} dz = \int_0^{2\pi} \frac{(R^2e^{2i\theta}-1)^{1/2}}{1+R^2e^{2i\theta}} iRe^{i\theta} d\theta$$

We note that $(R^2e^{2i\theta}-1)^{1/2} \approx Re^{i\theta}$ as $R \to \infty$ because the chosen branch implies $lim_{z\to\infty}(z^2-1)^{1/2} = z$. Hence,

$$\lim_{R\to\infty} \int_{C_R} \frac{(z^2-1)^{1/2}}{1+z^2} dz = 2\pi i$$

Calculations of the residues requires computing the correct branch of $(z^2-1)^{1/2}$. Note that the poles are being encircled in the counter clockwise direction.

$$\begin{aligned} \frac{\sqrt{z^2-1}}{2z}\Big|_{z=e^{i\pi/2}} &= \frac{\sqrt{2}e^{i(3\pi/4+\pi/4)/2}}{2i} = \frac{1}{\sqrt{2}} \\ \frac{\sqrt{z^2-1}}{2z}\Big|_{z=e^{i3\pi/2}} &= \frac{\sqrt{2}e^{i(5\pi/4+7\pi/4)/2}}{-2i} = \frac{1}{\sqrt{2}} \end{aligned} \tag{36}$$

Taking $\epsilon_i \to 0$, $R \to \infty$ and substituting the above results in the expression for J, we find

$$2i \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx = 2i\pi(\sqrt{2}-1)$$

Thus

$$I = \pi(\sqrt{2}-1)$$

21 Problem 7a

Now, the same integral will be computed using the method of path deformation. So,

$$I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx$$

As before, this is replaced with

$$J = \oint_C \frac{\sqrt{z^2-1}}{1+z^2} dz$$

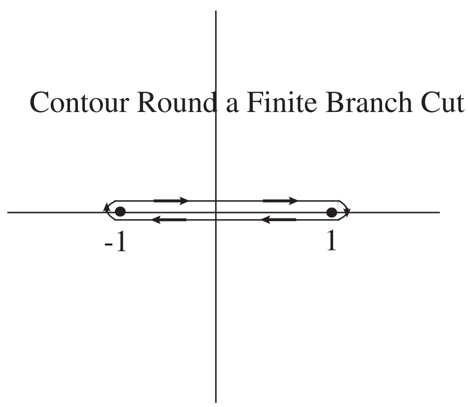
All definitions are the same as before, i.e.,

$$(z^2-1)^{1/2} = \sqrt{\rho_1 \rho_2} e^{i(\phi_1+\phi_2)/2}, \quad 0 \le \phi_1, \phi_2 < 2\pi$$

where

$$\begin{aligned} (z-1)^{1/2} &= \sqrt{\rho_1} e^{i\phi_1/2}, \rho_1 = |z-1| \\ (z+1)^{1/2} &= \sqrt{\rho_2} e^{i\phi_2/2}, \rho_2 = |z+1| \end{aligned} \tag{37}$$

The chosen contour is shown in figure 17. Notice the two isolated poles. Thus,



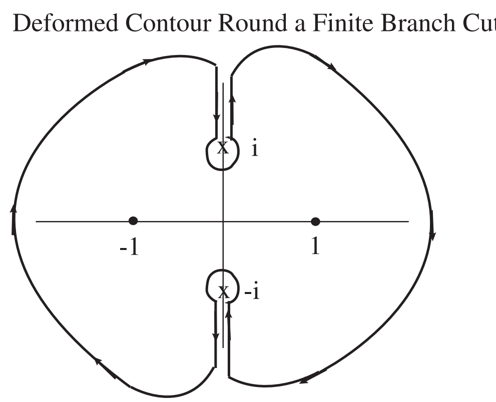


Figure 17: Left side contour is around the finite branch cut. Right side contour is the deformed contour with radius going to infinity. The contour cannot cross the poles at $\pm i$.

$$J = \int_{-1\ upper}^{1} f(z)dz + \int_{C_{\epsilon 1}} f(z)dz + \int_{1\ lower}^{-1} f(z)dz + \int_{C_{\epsilon 2}} f(z)dz$$

The integrals around $C_{\epsilon 1}$ and $C_{\epsilon 2}$ go to zero since z is represented as

$$z = \epsilon e^{i\theta}, \quad \text{and} \quad dz = \epsilon\ i\ e^{i\theta} d\theta$$

on the two contours. Thus the integrand becomes

$$\frac{\sqrt{z^2-1}}{1+z^2} dz = \frac{\sqrt{\epsilon^2 e^{2i\theta} - 1}}{1+\epsilon^2 e^{2i\theta}} \epsilon\ i\ e^{i\theta} d\theta.$$

And the radius of the contour goes to zero. Therefore

$$Lim\ \ \epsilon \to 0 \frac{\sqrt{\epsilon^2 e^{2i\theta} - 1}}{1+\epsilon^2 e^{2i\theta}} \epsilon\ i\ e^{i\theta} d\theta = \frac{\sqrt{-1}}{1} \epsilon\ i\ e^{i\theta} d\theta = 0,$$

since ϵ survives only in the numerator and goes to zero. The upper and the lower integrals as before become

$$J = 2i \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx$$

Now we do not have a way of computing this. Hence, we deform the contour as shown in figure 17b. Using this contour, the C_R integral is in the cloclwise direction. The up and down integrals together go to zero. We are left with two counter-clockwise contours round the two singularities which will give us residues. Therefore

$$J = 2iI = \oint_{CR\ clockwise} f(z)dz + 2\pi i \sum Residues(f(z), z = i, -i)$$

The C_R integral gave us $2\pi i$ last time, when we went in the counter clockwise direction. This time, it is in the clockwise direction and so we get $-2\pi i$. The counter clockwise integrations round the singularities give $2\pi i\sqrt{2}$. Thus,

$$2iI = -2\pi i + 2\pi i\sqrt{2}$$

$$\text{Or} \quad I = \pi(\sqrt{2} - 1),$$

as before. **Lecture 18 ends here**.

22 Problem 8

Consider the integral

$$I = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$$

We replace with the integral

$$J = \oint_C \frac{dz}{\sqrt{z(z-1)}}$$

over the contour in figure 18. This is an asymmetric finite branch cut.

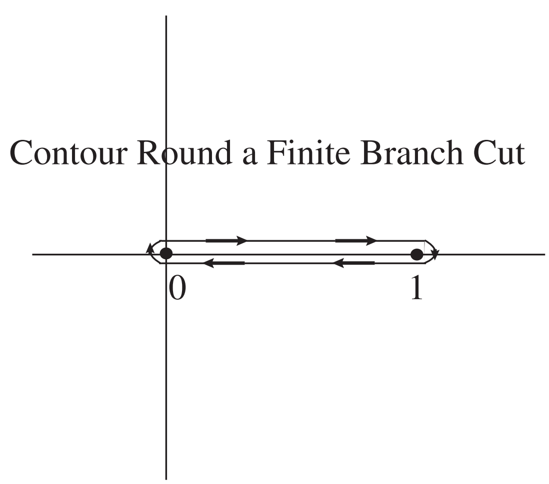


Figure 18: An asymmetric finite branch cut. The branch points are at 0 and 1.

Using polar coordinates

$$(z(z-1))^{1/2} = \sqrt{\rho_1 \rho_2} e^{i(\phi_1+\phi_2)/2}, \quad 0 \le \phi_1, \phi_2 < 2\pi$$

where

$$\begin{aligned} (z-1)^{1/2} &= \sqrt{\rho_1} e^{i\phi_1/2}, \rho_1 = |z-1| \\ (z)^{1/2} &= \sqrt{\rho_2} e^{i\phi_2/2}, \rho_2 = |z| \end{aligned} \tag{38}$$

with this choice of the branch we find

$$(z(z-1))^{1/2} = \begin{cases} \sqrt{x(x-1)} & 1 < x < \infty \\ -\sqrt{x(x-1)} & -\infty < x < -1 \\ i\sqrt{x-x^2} & -1 < x < 1, y \to 0^+ \\ -i\sqrt{x-x^2} & -1 < x < 1, y \to 0^- \end{cases}$$

Using the expressions in the contour integral J, it follows that

$$\begin{aligned} J = Lim_{\epsilon_1,\epsilon_2 \to 0} \int_{-1+\epsilon_1}^{1-\epsilon_2} \frac{dx}{i\sqrt{x-x^2}} &+ Lim_{\epsilon_1,\epsilon_2 \to 0} \int_{1-\epsilon_2}^{-1+\epsilon_1} \frac{dx}{-i\sqrt{x-x^2}} + \left(\int_{C_{\epsilon 1}} + \int_{C_{\epsilon 2}} \right) \frac{dz}{\sqrt{z(z-1)}} \\ &= 0 \end{aligned} \tag{39}$$

for there is no isolated singularity inside the contour. The $C_{\epsilon 1}$ and $C_{\epsilon 2}$ contour integrations are zero. This is because on the C_ϵ contour,

$$z = \epsilon e^{i\theta}, \quad \text{and} \quad dz = \epsilon e^{i\theta} i d\theta$$

Thus,

$$\frac{dz}{\sqrt{z(z-1)}} = Lim_{\epsilon \to 0} \frac{\epsilon e^{i\theta} i d\theta}{\sqrt{\epsilon e^{i\theta}(\epsilon e^{i\theta}-1)}} = Lim_{\epsilon \to 0} \sqrt{\epsilon} e^{i\theta/2} i d\theta = 0$$

The above is true for $C_{\epsilon 1}$ and $C_{\epsilon 2}$. Thus,

$$J = \oint_C \frac{dz}{\sqrt{z(z-1)}} = \frac{2I}{i}$$

As before, we do not know how to calculate the value of I. The Cauchy Residue Theorem does not help us here. The Cauchy Residue Theorem talks about **isolated** singularities and not branch cuts. Although, a branch cut is a region where the function is singular, it does not have unique values, but it is not an isolated singularity. And we do not have a theorem to compute the integral above. The only other thing we can do is to deform the contour to infinity. Since, there are no other isolated singularities of the function in the domain till infinity. The contour can be deformed to a circular contour C_R of infinite radius. This contour runs in the clockwise direction, just like the original contour round the branch cut. A similar deformed contour was used in the earlier problem. Here there are no isolated singularities.

The contribution from C_R is calculated as follows

$$\lim_{R \to \infty} \int_{C_R} \frac{dz}{\sqrt{z(z-1)}} dz = \lim_{R \to \infty} \int_0^{2\pi} \frac{iRe^{i\theta} d\theta}{\sqrt{R^2 e^{2i\theta}}} = \int_0^{2\pi} i d\theta = -2i\pi$$

The negative sign because it is a clockwise contour.

Thus

$$\frac{2I}{i} = -2i \int_{-1}^{1} \frac{dx}{\sqrt{x(1-x)}} = -2i\pi$$

Thus

$$I = \pi$$

23 Problem 9: Pole on a branch cut

Given the integral

$$\int_0^\infty \frac{x^{-p}dx}{1-x}, \quad 0 < p < 1$$

As before we replace this with

$$\oint_C \frac{z^{-p}dz}{1-z}, \quad 0 < p < 1, \quad z = Re^{i\theta}, ; \quad 0 \le \theta \le 2\pi$$

Consider the contour in figure 19.

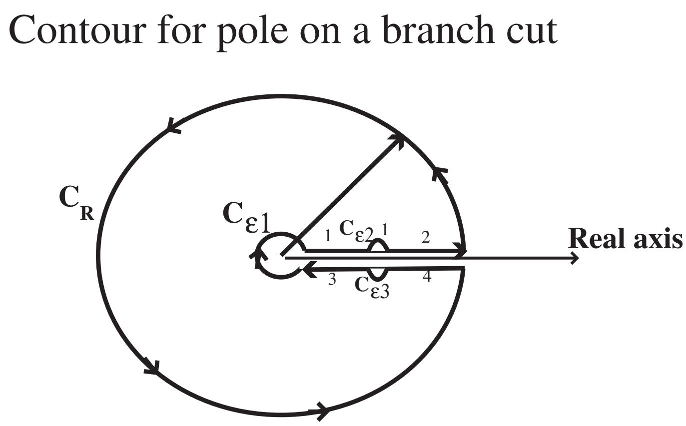


Figure 19: Contour for a pole on the branch cut. Pole is at $z = 1$

For this definition of θ, if we start very close to the x axis, θ is very close to 0, then z is equal to R over here. If I go full circle and let $\theta = 2\pi$ then $z = Re^{i2\pi} = R$. Hence, z is continuous. But $z^{-p} = R^{-p}$ at $\theta = 0$ and $z^{-p} = R^{-p}e^{-i2\pi p}$ at $\theta = 2\pi$. For $p = 1/2$, I get $R^{-1/2}$ and $-R^{-1/2}$ at $\theta = 0, 2\pi$. So, as z goes around, the function does not come back to the same value. Similarly, if I choose $p = 1/3$, then also they will be a jump. At $\theta = 0$, I will get $R^{-1/3}$ and at $\theta = 2\pi$ I will get $R^{-1/3}e^{i2\pi/3}$. Again, the function jumps values. So, it is a branch cut. It starts at z equal to 0 and extends to infinity.

So, now, I will choose a contour as shown in the figure. Now, there is a pole at $z = 1$ on the branch cut. I need an epsilon contour both above the cut and below the cut. The contour encloses no poles. Thus,

$$J = \oint_C \frac{z^{-p}dz}{1-z} = \int_1 [] + \int_2 [] + \int_3 [] + \int_4 [] + \int_{CR} [] + \int_{C_{\epsilon_1}} [] + \int_{C_{\epsilon_2}} [] + \int_{C_{\epsilon_3}} [] = 0$$

<u>CR integral</u>

On CR,

$$Z = Re^{i\theta}, \quad \text{and,} \quad dZ = Rie^{i\theta}d\theta$$

and on CR, the integrand becomes

$$\lim_{R\to\infty} \frac{R^{-p}e^{-i\theta p}Rie^{i\theta}d\theta}{1 - Re^{i\theta}} = O(\frac{1}{R^p}) = 0$$

and so the CR integral goes to zero.

Portions 1 and 2 are given by

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int_{\epsilon_1}^{1-\epsilon_2}\frac{z^{-p}dz}{1-z}+\lim_{\epsilon_2\to 0}\int_{1+\epsilon_2}^{\infty}\frac{z^{-p}dz}{1-z}=$$

Substituting

$$Z=Re^{i\theta},\quad \text{and,}\quad dZ=Rie^{i\theta}d\theta,\ \ \theta=0$$

portions 1 and 2 become

$$\int_0^1\frac{R^{-p}dR}{1-R}+\int_1^{\infty}\frac{R^{-p}dR}{1-R}=\int_0^{\infty}\frac{R^{-p}dR}{1-R}=I,$$

the integral we want from the beginning in the Cauchy Principal Value sense.
Portions 4 and 3 are given by

$$\lim_{\epsilon_2\to 0}\int_{\infty}^{1+\epsilon_2}\frac{z^{-p}dz}{1-z}+\lim_{\epsilon_1,\epsilon_2\to 0}\int_{1-\epsilon_2}^{\epsilon_1}\frac{z^{-p}dz}{1-z}=$$

Substituting

$$Z=Re^{i\theta},\quad \text{and,}\quad dZ=Rie^{i\theta}d\theta,\ \ \theta=2\pi$$

portions 1 and 2 become

$$\int_{\infty}^1\frac{R^{-p}e^{-i2\pi p}dRe^{i2\pi}}{1-Re^{i2\pi}}+\int_1^0\frac{R^{-p}e^{-i2\pi p}dRe^{i2\pi}}{1-Re^{i2\pi}}=\int_{\infty}^0\frac{R^{-p}dRe^{-i2\pi p}}{1-R}=Ie^{-i2\pi p},$$

where an extra $e^{-i2\pi p}$ factore has come in. The sum of portions 1,2,3,4 become

$$I(1-e^{-i2\pi p})$$

Lecture 19 ends here.

The C_{ϵ_1} integral
The integral C_{ϵ_1} becomes

$$\lim_{\epsilon_1\to 0}\int_{C\epsilon_1}\frac{z^{-p}dz}{1-z},\quad z=\epsilon_1e^{i\theta},\quad dz=\epsilon_1ie^{i\theta}d\theta$$

Or

$$\lim_{\epsilon_1\to 0}\int_{2\pi}^0\frac{\epsilon_1^{-p}e^{-i\theta p}\epsilon_1e^{i\theta}d\theta}{1-\epsilon_1e^{i\theta}}=\lim_{\epsilon_1\to 0}\int_{2\pi}^0\epsilon_1^{1-p}g(\theta,p)d\theta=0,$$

since $p<1$.
The C_{ϵ_2} integral
The integral C_{ϵ_2} becomes

$$\lim_{\epsilon_2\to 0}\int_{C\epsilon_2}\frac{z^{-p}dz}{1-z},\quad z=e^{i0}+\epsilon_2e^{i\phi},\quad dz=\epsilon_2ie^{i\phi}d\phi$$

Or

$$\lim_{\epsilon_2\to 0}\int_{\pi}^0\frac{(1+\epsilon_2e^{i\phi})^{-p}\epsilon_2ie^{i\phi}d\phi}{1-(1+\epsilon_2e^{i\phi})}=i\pi.$$

The C_{ϵ_3} integral
The integral C_{ϵ_3} becomes

$$\lim_{\epsilon_3 \to 0} \int_{C\epsilon_3} \frac{z^{-p} dz}{1-z}, \quad z = e^{i2\pi} + \epsilon_3 e^{i\phi}, \quad dz = \epsilon_3 i e^{i\phi} d\phi$$

Or

$$\lim_{\epsilon_3 \to 0} \int_{2\pi}^{\pi} \frac{(e^{i2\pi} + \epsilon_3 e^{i\phi})^{-p} \epsilon_3 i e^{i\phi} d\phi}{1 - (e^{i2\pi} + \epsilon_3 e^{i\phi})} = \int_{2\pi}^{\pi} \frac{(e^{i2\pi})^{-p} i \, d\phi}{-1} = i\pi e^{-i2\pi p}.$$

The total J becomes

$$J = I(1 - e^{-i2\pi p}) + i\pi(1 - e^{-i2\pi p}) = 0$$

Or

$$I(1 - e^{-i2\pi p}) + i\pi(1 + e^{-i2\pi p}) = 0$$

Or

$$I = -i\pi \frac{(1 + e^{-i2\pi p})}{(1 - e^{-i2\pi p})} = -\pi cot(p\pi)$$

Lecture 20 continues with the L-shaped branch cut.

24 L shaped branch cut

This portion is extra material. This portion will not be part of the syllabus for those who are taking the nptel certification exam.
There are other branch cuts that one must take under specific circumstances. Here I will show how an L-shaped branch cut can arise. Consider the following integral:

$$I^{mnpq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[1 - (-1)^m \cos \lambda a] \, [1 - (-1)^n \cos \mu b] \, \left(k^2 - \lambda^2 - \mu^2\right)^{\frac{1}{2}}}{(\lambda^2 - k_m^2) \left(\lambda^2 - k_p^2\right) (\mu^2 - k_n^2) \left(\mu^2 - k_q^2\right)} \, \mathrm{d}\lambda \, \mathrm{d}\mu. \tag{40}$$

In the equation above, we denote the integral over λ as

$$I_1^{mp}(\mu) = \int_{-\infty}^{\infty} \frac{[1 - (-1)^m \cos \lambda a] \, \left(k^2 - \lambda^2 - \mu^2\right)^{\frac{1}{2}}}{(\lambda^2 - k_m^2) \left(\lambda^2 - k_p^2\right)} \, \mathrm{d}\lambda, \tag{41}$$

and use the fact that for an even function $f(\lambda)$

$$\int_{-\infty}^{\infty} f(\lambda) \cos \lambda a \, \mathrm{d}\lambda = \int_{-\infty}^{\infty} f(\lambda) \, \mathrm{e}^{\mathrm{i}\lambda a} \, \mathrm{d}\lambda.$$

Thus,

$$I_1^{mp}(\mu) = \int_{-\infty}^{\infty} \frac{\left[1 - (-1)^m \, \mathrm{e}^{\mathrm{i}\lambda a}\right] \, \left(k^2 - \lambda^2 - \mu^2\right)^{\frac{1}{2}}}{(\lambda^2 - k_m^2) \left(\lambda^2 - k_p^2\right)} \, \mathrm{d}\lambda. \tag{42}$$

24.1 Branch points and branch cuts

The integrand of $I_1^{mp}(\mu)$ has square root branch points at

$$\lambda_{1,2} = \pm \left(k^2 - \mu^2\right)^{\frac{1}{2}}.$$

Depending on the value of μ and hence the location of the branch points $\lambda_{1,2}$, $I_1^{mp}(\mu)$ has to be evaluated differently - **Case 1**: when $|\mu| < k$; $\lambda_{1,2} = \pm\sqrt{k^2 - \mu^2}$, i.e., the branch points lie on the positive and the negative real axis and **Case 2**: when $|\mu| > k$; $\lambda_{1,2} = \pm \mathrm{i}\sqrt{\mu^2 - k^2}$, i.e., the branch points lie on the positive and the negative imaginary axis.
Consider the first case in which $\lambda_{1,2}$ lie on the real axis, i.e.,

$$\lambda_1 = (k^2 - \mu^2)^{\frac{1}{2}} \qquad \text{and} \qquad \lambda_2 = -\lambda_1.$$

For $z > 0$, the radiated pressure wave has the form $\mathrm{e}^{\mathrm{i}\xi z - \mathrm{i}\omega t}$ with $\xi = (k^2 - \lambda^2 - \mu^2)^{\frac{1}{2}} = (\lambda_1^2 - \lambda^2)^{\frac{1}{2}}$. A growing wave is physically inadmissible and hence the imaginary part of ξ must be positive. For example, for a purely real λ, such that $\lambda > \lambda_1$, we must have

$$\xi = \mathrm{i}(\lambda^2 - \lambda_1^2)^{\frac{1}{2}}.$$

Thus, it is necessary that we choose a feasible definition for ξ so that a growing wave solution never occurs. We will now select an appropriate branch cut and definition for the function $(k^2 - \lambda^2 - \mu^2)^{\frac{1}{2}}$ by looking at it as a product of square roots, i.e.,

$$\left(\lambda_1^2 - \lambda^2\right)^{1/2} = (\lambda_1 - \lambda)^{1/2} (\lambda_1 + \lambda)^{1/2} = |\lambda_1 - \lambda|^{1/2} \mathrm{e}^{\mathrm{i}\gamma/2} |\lambda_1 + \lambda|^{1/2} \mathrm{e}^{\mathrm{i}\theta/2}. \tag{43}$$

The complex functions $(\lambda_1 - \lambda)$ and $(\lambda_1 + \lambda)$ are shown in Figs. 20(a) and 20(b), respectively. From Fig. 20(a), as γ varies from 0 to 2π, the resulting branch cut of

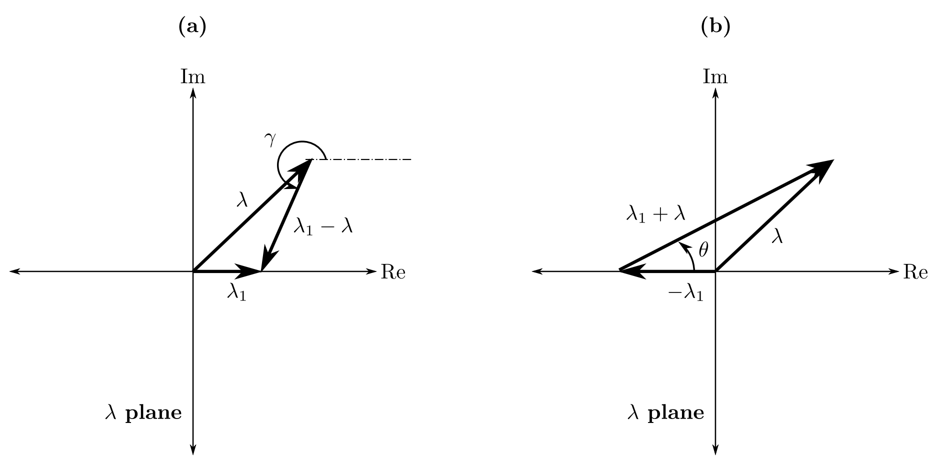


Figure 20: Vectors of $\lambda_1 - \lambda$ and $\lambda_1 + \lambda$ (case 1) in the complex λ plane.

$(\lambda_1 - \lambda)^{1/2}$ runs along the real axis from λ_1 to $-\infty$ (see Fig. 21(a)). We may now select the following function definition for $(\lambda_1 - \lambda)^{1/2}$ so that the branch cut modifies to an

'L' shaped one as shown Fig. 21(b):

$$(\lambda_1-\lambda)^{1/2}=\begin{cases} |\lambda_1-\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}\gamma/2} & for Re(\lambda)>0 \\ -\,|\lambda_1-\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}\gamma/2} & for Re(\lambda)<0 and Im(\lambda)>0 \\ |\lambda_1-\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}\gamma/2} & for Re(\lambda)<0 and Im(\lambda)<0 \end{cases} \tag{44}$$

It will be described later how the above modification of branch cut (and the one which will be explained next) prevent the function $(\lambda_1^2-\lambda^2)^{1/2}$ from assuming any negative imaginary values.

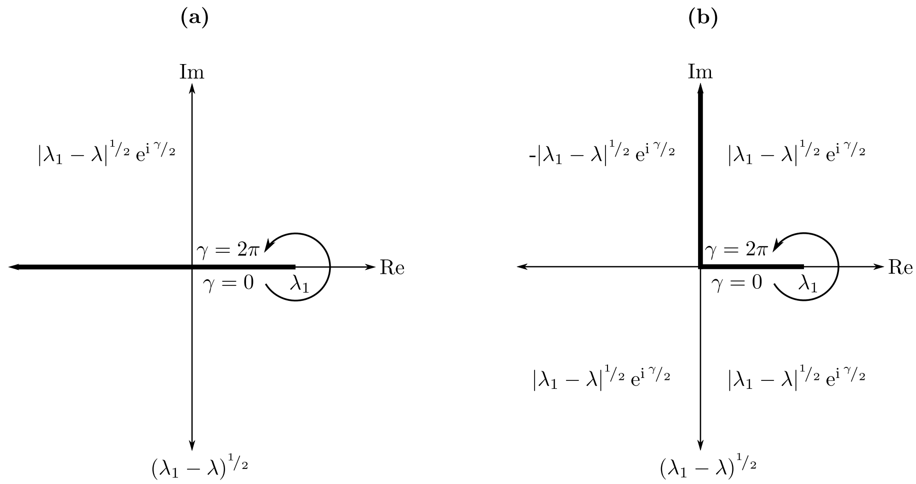


Figure 21: (a) Initial and (b) modified branch cuts of $\sqrt{\lambda_1-\lambda}$ (case 1) in the complex λ plane.

Next, assume that θ, the argument of $(\lambda_1+\lambda)^{1/2}$, varies from 0 to 2π. The resulting branch cut of $(\lambda_1+\lambda)^{1/2}$ extends from $-\lambda_1$ to ∞ along the real axis, as shown in Fig. 22(a). It is then modified to an 'L' shaped one by choosing the following function definition for $(\lambda_1+\lambda)^{1/2}$:

$$(\lambda_1+\lambda)^{1/2}=\begin{cases} |\lambda_1+\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}\theta/2} & for Re(\lambda)<0 \\ -\,|\lambda_1+\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}\theta/2} & for Re(\lambda)>0 and Im(\lambda)<0 \\ |\lambda_1+\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}\theta/2} & for Re(\lambda)>0 and Im(\lambda)>0 \end{cases} \tag{45}$$

The modified branch cut is shown in Fig. 22(b).
Combining the definitions of $(\lambda_1-\lambda)^{\frac{1}{2}}$ (Eq. (45)) and $(\lambda_1+\lambda)^{\frac{1}{2}}$ (Eq. (45)), $(\lambda_1^2-\lambda^2)^{\frac{1}{2}}$ can be defined as

$$\left(\lambda_1^2-\lambda^2\right)^{1/2}=\begin{cases} |\lambda_1-\lambda|^{1/2}\,|\lambda_1+\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}(\gamma+\theta)/2} & for Re(\lambda)>0 and Im(\lambda)>0 \\ -\,|\lambda_1-\lambda|^{1/2}\,|\lambda_1+\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}(\gamma+\theta)/2} & for Re(\lambda)<0 and Im(\lambda)>0 \\ |\lambda_1-\lambda|^{1/2}\,|\lambda_1+\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}(\gamma+\theta)/2} & for Re(\lambda)<0 and Im(\lambda)<0 \\ -\,|\lambda_1-\lambda|^{1/2}\,|\lambda_1+\lambda|^{1/2}\,\mathrm{e}^{\mathrm{i}(\gamma+\theta)/2} & for Re(\lambda)>0 and Im(\lambda)<0 \end{cases} \tag{46}$$

The arguments γ and θ vary from 0 to 2π. The resulting branch cut of $(\lambda_1^2-\lambda^2)^{\frac{1}{2}}$ is depicted in Fig. 23.
For example, Fig. 24 depicts the values of arguments γ and θ along the real axis when $|Re(\lambda)|>\lambda_1$. It can be seen that for all the four cases, as shown in the figure,

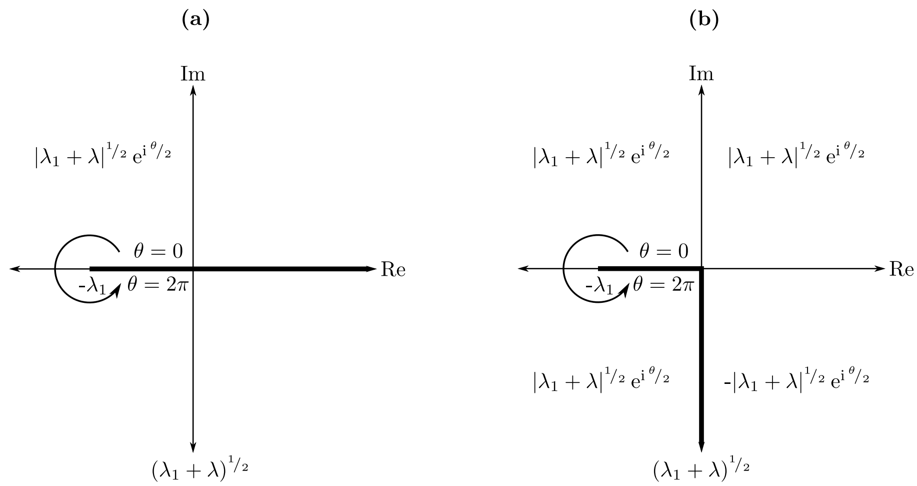


Figure 22: (a) Initial and (b) modified branch cuts of $(\lambda_1+\lambda)^{\frac{1}{2}}$ (case 1) in the complex λ plane.

$(\lambda_1^2-\lambda^2)^{\frac{1}{2}} = \mathrm{i}\,|\lambda_1^2-\lambda^2|^{\frac{1}{2}}$. Although we do not show all possibilities, the selected definition of $(\lambda_1^2-\lambda^2)^{\frac{1}{2}}$ (Eq. 46) and the associated branch cut prevent a growing wave in the z direction. Finally, the contour that is equivalent to the J contour in the textbook problems looks as in figure 25 for the L-shaped cut.

L shaped cut ends in lecture 21.

25 Laplace Transform: lectures 22 and 23

We frequently use Laplace transforms in solving odes and pdes. Since our systems are excited at time $t=0$, we can use a one-sided Laplace Transform which is defined as

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

where s is a complex variable. Note that this integral exists provided e^{-st} can over power the increasing value of $f(t)$ at infinity. Mostly this is possible because real systems give out finite valued signals which even decay with time.

25.1 Functions of exponential order

A function f(t) is said to be of exponential order if there exist real constants M and γ such that

$$|f(t)| \leq Me^{\gamma t}$$

for all t greater than some value t_0. Find out which of the following functions are of exponential order.

$$a)3t+2, b)t^3 - sint, c)e^{2t+1}+5t, d)e^{t^2}, e)t^t.$$

Answer: Since polynomials tend to infinity slower than $e^{\gamma t}$ for any positive value of γ, (a) and (b) are of exponential order. The function (c) is of exponential order with $\gamma \geq 2$. e^{t^2} and t^t grow faster than any $e^{\gamma t}$ thus they are not of exponential order.

Suppose

$$f(t) = Me^{\gamma t},$$

with γ real valued and positive. then

$$L\{Me^{\gamma t}\} = M\int_0^\infty e^{-(s-\gamma)t}dt = -M\frac{e^{-(s-\gamma)t}}{s-\gamma}|_0^\infty$$

At $t = \infty$, the expression above goes to zero only if $Re(s-\gamma) > 0$, or we require that the complex variable s be restricted to the half plane $Re(s) > \gamma$. Then we get

$$L\{Me^{\gamma t}\} = \frac{M}{s-\gamma}$$

So if a function is exponential order then it has a Laplace Transform. Which ones are exponential order? a) $4e^{-4t} + 3t$ b) $t^{-t} + t^2$ c) $sinht$, d) $3e^{5t}$, e) e^{logt} f) e^{t^4} Let f(t) be continuous and of exponential order, i.e.,

$$|F(t)| \leq Me^{\gamma t}$$

for all $t > 0$. Then the Laplace transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

exists for $Re(s) > \gamma$ and

$$|F(s)| = |\int_0^\infty f(t)e^{-st}dt| < \int_0^\infty |f(t)e^{-st}|dt < \int_0^\infty |Me^{\gamma t - st}|dt$$

Or

$$|F(s)| < \frac{M'}{|s|},$$

for some M' and some s.

25.2 The inverse Laplace transform

Using the Cauchy Integral Formula we write

$$F(s) = \frac{1}{2\pi i}\oint_{cc}\frac{F(z)dz}{z-s}$$

Remember that $|F(s)|$ is bounded and is analytic for $Re(s) > \gamma$. For the integral above, let us use the following contour. The contour entirely is to the right of γ. It has a vertical straight line portion and a semi-circular arc of radius R which is large as shown in the figure 26. Upon reversing the direction we have

$$F(s) = \frac{1}{2\pi i}\oint_{cw}\frac{F(z)dz}{s-z}$$

Now $F(z)$ which is the Laplace transform decays as $M'/|z|$ and hence as $R \to \infty$, the integrand decays as follows

$$\frac{F(z)}{s-z}dz \to \frac{M}{Re^{i\theta}}\frac{1}{Re^{i\theta}}Rie^{i\theta}d\theta$$

Or

$$|\frac{F(z)}{s-z}dz| \to |\frac{M}{R}\frac{1}{R}Rd\theta| \to 0$$

as $R \to \infty$ on the curved portion of the contour. We are left with the straight line part of the integral

$$F(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(z)dz}{s-z}$$

Let us now apply the inverse Laplace operator on s

$$\begin{aligned}
f(t) = L^{-1}\{F(s)\} &= \frac{1}{2\pi i}L^{-1}\int_{c-i\infty}^{c+i\infty}\frac{F(z)dz}{s-z} \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(z)L^{-1}\{\frac{1}{s-z}\}dz \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(z)e^{zt}dz \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds,
\end{aligned} \tag{47}$$

which is the Laplace inverse.

25.3 Examples

Let us consider n inverse Laplace problem.

$$\begin{aligned}
F(s) &= \frac{1}{(s-2)^2} \qquad (48) \\
f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{e^{st}ds}{(s-2)^2}
\end{aligned}$$

The vertical line segment at $s = C$ is to the right of $s = 2$, the singularity of $F(s)$. Let us again use a closed contour as shown in the figure 27. The contour has a vertical part and a circular arc part with a radius R. Let R be such that the region encloses all the singularities, $s = 2$ in this case. If the integral over the circular arc goes to zero then we need only to find the residues of $F(s)$.

$$f(t) = \frac{1}{2\pi i}\oint\frac{e^{st}ds}{(s-2)^2} = \frac{1}{2\pi i}\left(\int_{c-i\infty}^{c+\infty}\frac{e^{st}ds}{(s-2)^2} + \int_{arc}\right) = 2\pi i Res(F(s)) \tag{49}$$

Now

$$|e^{ts}| = |e^{tRe(s)}e^{tIm(s)}| = e^{tRe(s)}.$$

Here $t > 0$ and $Re(s) < C$ and on most of the arc $Re(s) < 0$. Thus, for a fixed value of t

$$|\frac{1}{2\pi i}\int_{arc}\frac{e^{st}ds}{(s-2)^2}| < \frac{1}{2\pi}\int_{arc}|\frac{e^{st}ds}{(s-2)^2}| = \frac{1}{2\pi}\int_{arc}|\frac{e^{Re(s)t}R}{R^2}| = 0 \tag{50}$$

as $R \to \infty$. Thus

$$f(t) = \frac{1}{2\pi i}\oint\frac{e^{st}ds}{(s-2)^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\frac{e^{st}ds}{(s-2)^2} = 2\pi i Res(F(s))/(2\pi i) \tag{51}$$

After the residues, the integral value is

$$\frac{d}{ds}e^{st}|_{s=2} = te^{st}|_{s=2} = te^{2t}$$

where the $2\pi i$ is cancelled.

26 Some useful inequalities

If z_1 and z_2 are any two complex numbers

$$\begin{aligned} |z_1 + z_2| &\leq |z_1| + |z_2| \\ |z_1 + z_2| &\geq ||z_1| - |z_2|| \\ |z_1 - z_2| &\leq |z_1| + |z_2| \\ |z_1 - z_2| &\geq ||z_1| - |z_2|| \end{aligned} \tag{52}$$

27 Finding residues at singularities

If we have a function that is a ratio of two polynomials, for example,

$$\frac{z}{(z-1)(z-2)}, \quad \textit{singular at } z = 1, \textit{ and } z = 2$$

- then the residue at $z = 1$ is found as follows:

$$Res(z=1) = \frac{z}{(z-1)(z-2)}(z-1)|_{z=1} = \frac{z}{(z-2)}|_{z=1} = -1$$

- the residue at $z = 2$ is found as follows:

$$Res(z=2) = \frac{z}{(z-1)(z-2)}(z-2)|_{z=2} = \frac{z}{(z-1)}|_{z=2} = 2$$

- If $f(z)$ is a ratio of two polynomials $p(z)$ and $q(z)$ then the residue at $z = z_o$ is also found as

$$Res_{z=z_o}\frac{p(z)}{q(z)} = \frac{p(z_o)}{q'(z_o)}$$

  provided $q'(z_o) \neq 0$. If $q(z_o) = 0$, then this is the only way.
- If

$$f(z) = \frac{\phi(z)}{(z-z_o)^m}$$

  If $m = 1$

$$Res_{z=z_o}f(z) = \phi(z = z_o)$$

  and if $m \geq 2$ then

$$Res_{z=z_o}f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}\phi(z=z_o)}{dz^{m-1}}$$

- Let two functions p and 1 be analytic at a point z_o, and suppose that $p(z_0) \neq 0$. If q has zeros of order m at z_0 then the quotient $p(z)/q(z)$ has a pole of order m there.
- if F has a simple pole at $z = Z_0$, then

$$Res(F, z_0) = \lim_{z \to z_0} (z - z_0) f(z_0)$$

- if F has a simple of order k at $z = Z_0$, then

$$Res(F, z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z_0)$$

- A branch point is not a simple pole.

28 References

"Complex variables and applications", J.W. Brown and R.V.Churchill. Mcgraw Hill publication.
"Complex Variables: Introduction and Applications", M. J. Ablowitz and A. S. Fokas. Cambridge University press.

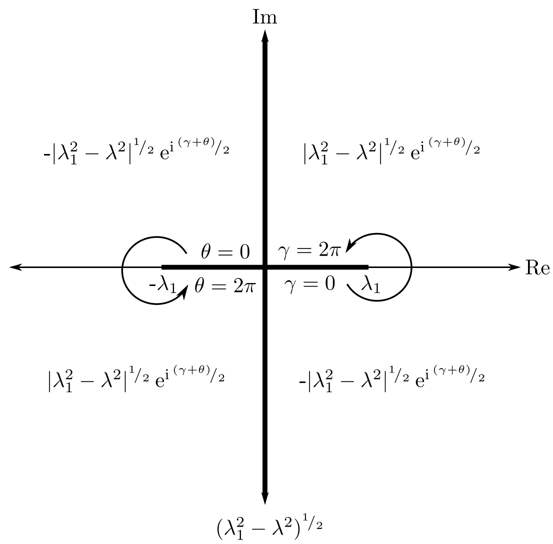


Figure 23: Branch cut of $(\lambda_1^2 - \lambda^2)^{1/2}$ (case 1) in the complex λ plane.

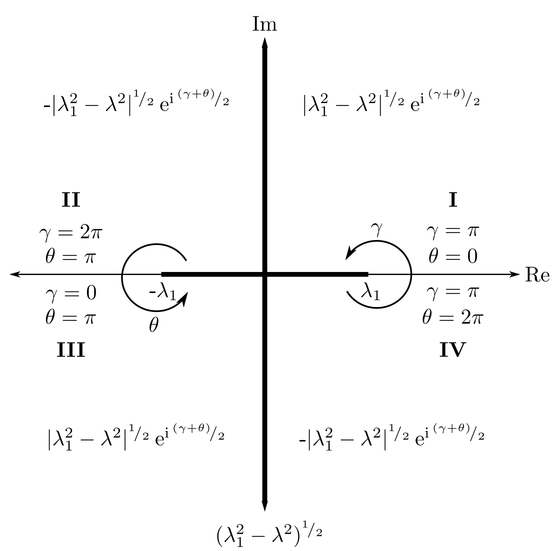


Figure 24: Argument values of $(\lambda_1^2 - \lambda^2)^{\frac{1}{2}}$ along the real axis when $|Re(\lambda)| > \lambda_1$ (case 1).

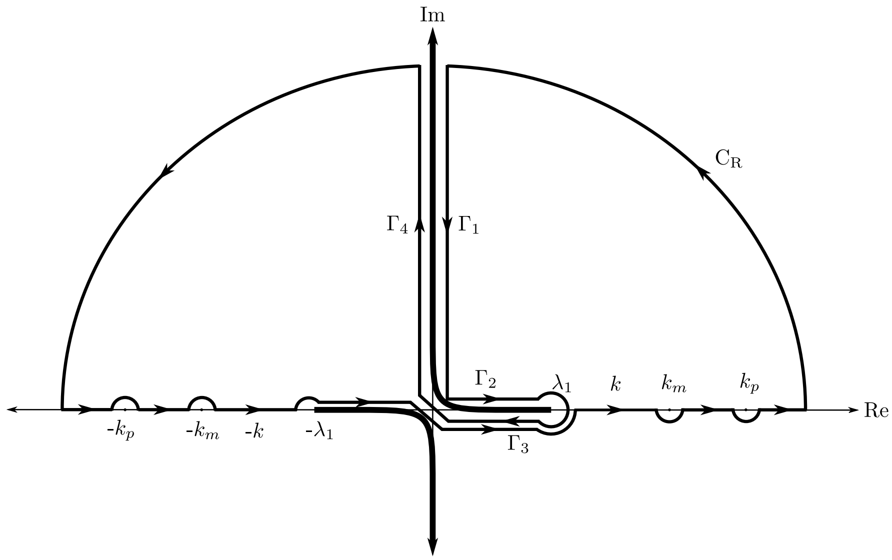


Figure 25: Integration contour of $I_1^{mp}(\mu)$ for case 1 ($|\mu| < k$) when $k_m, k_p > k$ and $k_m \neq k_p$.

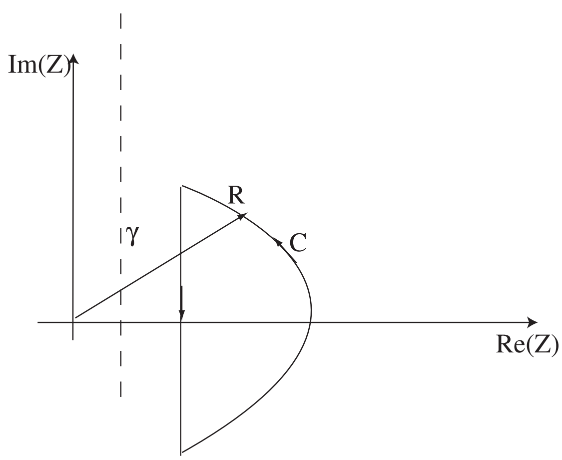


Figure 26: Figure for Laplace inverse.

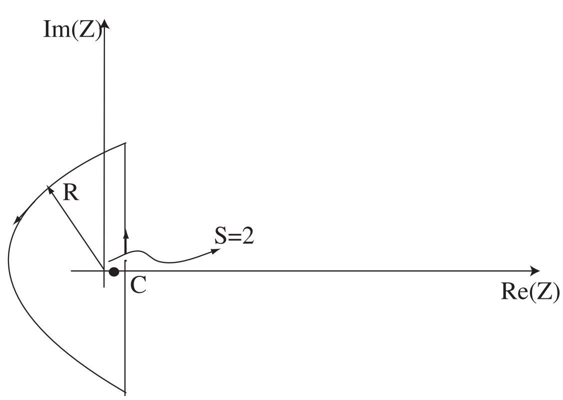


Figure 27: Example for Laplace inverse.