1

(1) $$f(z) = \frac{(z^2-1) - \sin(z-1)}{(z-1)} \quad \text{at } z=1.$$

$$\lim_{z\to 1} f(z) = \lim_{z\to 1} (z+1) - \frac{\sin(z-1)}{(z-1)}$$

$$= 2 - 1 = 1$$

As $z=1$ is not a singular point for $f(z)$, hence the residue of $f(z)$ at $z=1$ is 0 (zero).

(2) $$J = \oint_C \frac{e^z}{z^2-(1/4)}\, dz \quad \text{wher } C: |z| = 1$$

$$f(z) = \frac{e^z}{z^2-\frac{1}{4}} = \frac{p(z)}{q(z)}$$

poles are at $z = \pm 1/2$

Residue at $z = 1/2$ equals to $\text{Res}(\frac{1}{2}) = \lim_{z\to\frac{1}{2}} (z-\frac{1}{2}) \frac{e^z}{(z^2-1/4)}$

$$= e^{1/2}$$

Similarly

$$\text{Res}(-1/2) = \lim_{z\to -1/2} (z+\tfrac{1}{2}) \frac{e^z}{(z^2-1/4)} = -e^{-1/2}$$

Using the cauchy residue theorom

$$J = \oint_C f(z)\, dz = 2\pi i\left[\text{Res}(1/2) + \text{Res}(-1/2)\right]$$

$$= 2\pi i\left[e^{1/2} - e^{-1/2}\right]$$

$$= 2\pi i \; 2\sinh(1/2)$$

$$= 4\pi i \sinh(1/2)$$

③ $f(z) = \dfrac{\cos(z)}{z^5}$

Singular point $z = 0$

order of singularity $= 5$

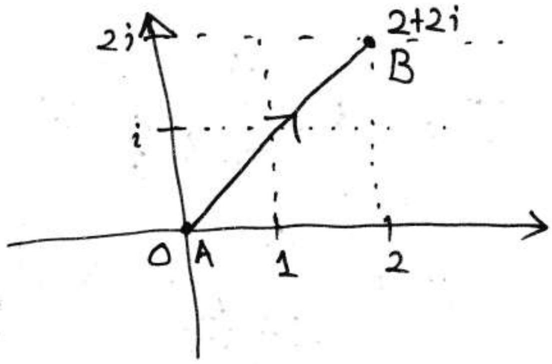
Residue $= \lim_{z\to 0} \dfrac{1}{(5-1)!} \dfrac{d^4}{dz^4}\left(z^5 \dfrac{\cos(z)}{z^5}\right)$

$= \dfrac{1}{4!}$

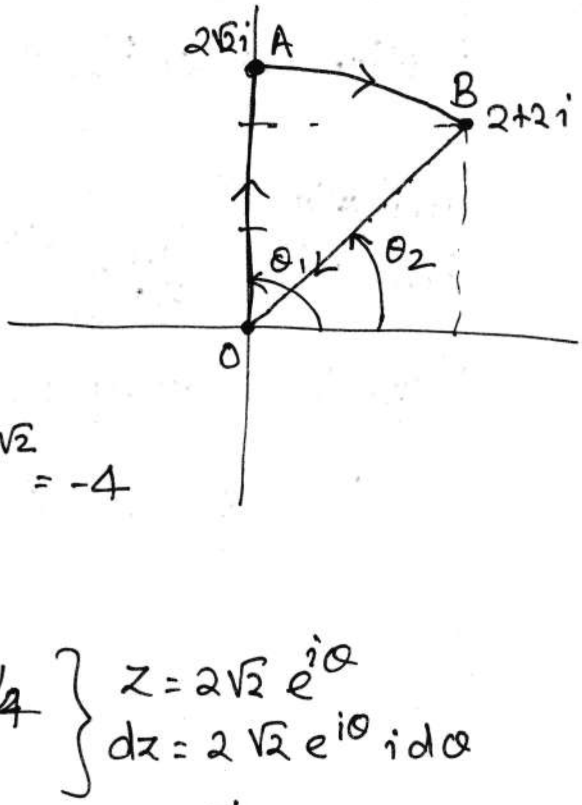
④ $\displaystyle\int_0^{2+2i} z\,dz = \int_0^2 (t + it)\,dt\,(1+i)$

$\displaystyle = 2i \frac{t^2}{2}\Big|_0^2 = 4i$

$$\boxed{\int_0^{2+2i} z\,dz = 4i}$$



⑤ $\displaystyle\int_0^{2\sqrt{2}i} z\,dz + \int_{2\sqrt{2}i}^{2+2i} z\,dz$



Along $\overline{OA}$

$z = it \Rightarrow dz = i\,dt$

$\displaystyle\int_0^{2\sqrt{2}i} z\,dz = \int_0^{2\sqrt{2}} it\,i\,dt = -\frac{t^2}{2}\Big|_0^{2\sqrt{2}} = -4$

Along the arc $\widehat{AB}$

$\left.\begin{array}{l}\tan\theta_2 = \frac{2}{2} \Rightarrow \theta_2 = \tan^{-1}(1) = \pi/4 \\ \tan\theta_1 = \infty \Rightarrow \theta_1 = \pi/2\end{array}\right\}\begin{array}{l} z = 2\sqrt{2}\,e^{i\theta} \\ dz = 2\sqrt{2}\,e^{i\theta}\,i\,d\theta\end{array}$

$\displaystyle\int_{2\sqrt{2}i}^{2+2i} z\,dz = \int_{\pi/2}^{\pi/4} 2\sqrt{2}e^{i\theta}\,2\sqrt{2}e^{i\theta}\,i\,d\theta = 8\int_{\pi/2}^{\pi/4} e^{2i\theta}\,i\,d\theta = 4(i+1)$

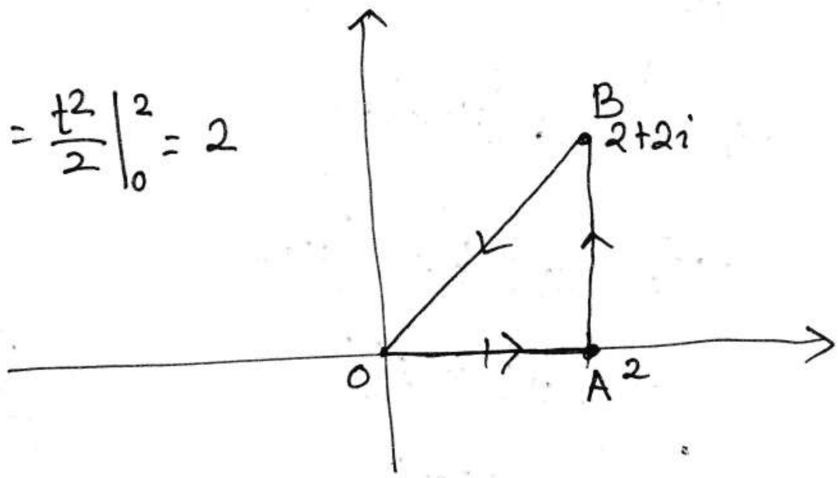
Hence

$$\int_0^{2\sqrt{2}i} z\,dz + \int_{2\sqrt{2}i}^{2+2i} z\,dz = -4+4(i+1) = 4i$$

Along $\overline{BO}$ Similarly $\int_{2+2i}^{0} z\,dz = -\int_0^{2+2i} z\,dz = -4i$. Hence $\oint_C z\,dz = 4i-4i = 0$

(6) Along $\overline{OA}$

$$\int_0^2 z\,dz = \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2$$



Along $\overline{AB}$

$$z = 2+it$$
$$\Rightarrow dz = i\,dt$$

$$\int_2^{2+2i} z\,dz = \int_0^2 (2+it)\,i\,dt = \int_0^2 2i\,dt - t\,dt = 4i-2$$

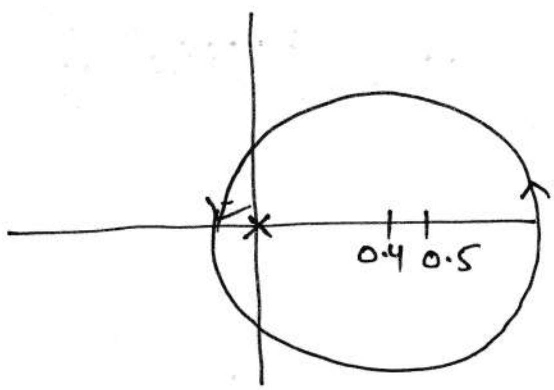
Along $\overline{BO}$ (see Question NO 4)

$$\int_{2+2i}^{0} z\,dz = -\int_0^{2+2i} z\,dz = -4i$$

Hence

$$\int_0^2 z\,dz + \int_2^{2+2i} z\,dz + \int_{2+2i}^{0} z\,dz = 2+4i-2-4i = 0$$

(7) $$J = \oint_C \frac{\cos(z)}{z^5}\,dz$$



$$= 2\pi i\ \mathrm{Res}(z=0)$$

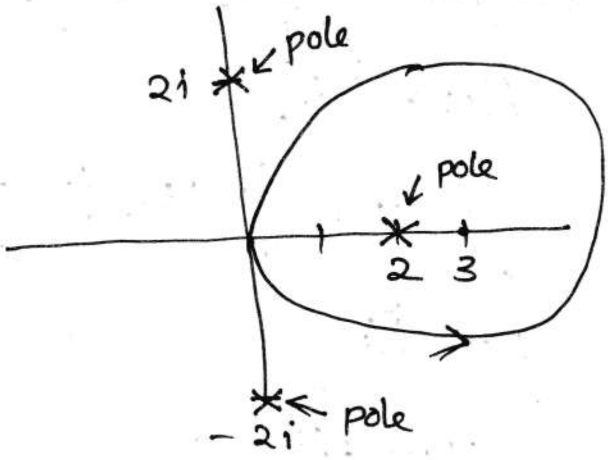
$$= 2\pi i \times \frac{1}{4!}$$

$$\boxed{J = \frac{\pi i}{12}}$$

(8) $f(z) = \dfrac{5z^2+17}{z^3-2z^2+4z-8} = \dfrac{5z^2+17}{(z-2i)(z+2i)(z-2)}$

(a) $J = \oint_C \cancel{8}\, f(z)\, dz$

where $C: |z-3| = 3$

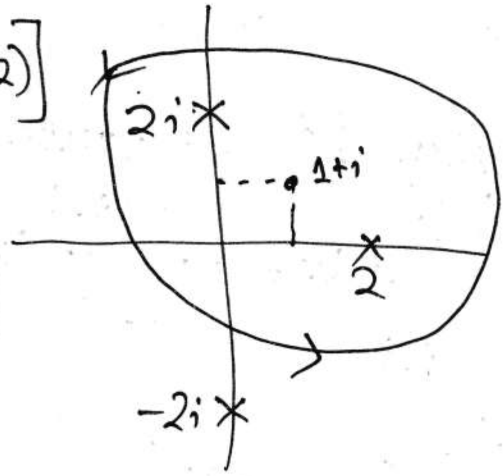


$$J = 2\pi i \, \text{Res}(z=2)$$

$$= 2\pi i \times \frac{37}{8}$$

$$= \frac{37\pi i}{4}$$

(b) $J = 2\pi i \left[\text{Res}(z=2i) + \text{Res}(z=2) \right]$



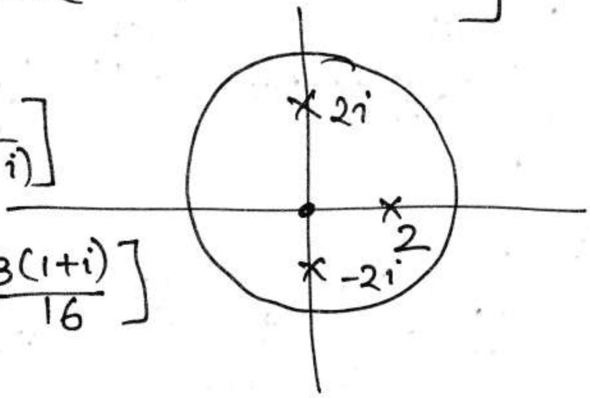
$$= 2\pi i \left[\frac{37}{8} + \frac{(-20+17)}{4i(2i-2)} \right]$$

$$= 2\pi i \left[\frac{37}{8} + \frac{3(1-i)}{16} \right]$$

$$J = \frac{(77i+3)}{8}\pi$$

(c)

$$J = 2\pi i\left[\operatorname{Res}(z=2) + \operatorname{Res}(z=2i) + \operatorname{Res}(z=-2i)\right]$$



$$= 2\pi i\left[\frac{37}{8} + \frac{3}{(1+i)} + \frac{3}{(1-i)}\right]$$

$$= 2\pi i\left[\frac{37}{8} + \frac{3(1-i)}{16} + \frac{3(1+i)}{16}\right]$$

$$= \frac{2\pi i}{8} \times 40$$

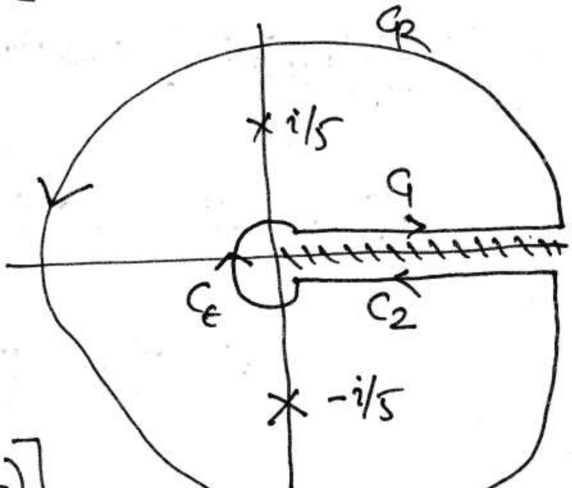
$$= 10\pi i$$

(9) (a)

$$J = \int_0^\infty \frac{1}{\sqrt{5x}\,(1+25x^2)}\,dx \quad , \; f(x) = \frac{1}{\sqrt{5x}\,(1+25x^2)}$$

As mentioned in the lectures, for $0 < \theta < 2\pi$ $\sqrt{z}$ introduces a branch cut from $z=0$ to ∞.

Consider $I = \oint_c \frac{1}{\sqrt{5x}\,(1+25z^2)}\,dz$



$$I = \oint_c \frac{1}{\sqrt{5z}\,(1+25z^2)}\,dz$$

$$= \left[\int_{C_1} + \int_{C_R} + \int_{C_3} + \int_{C_\epsilon}\right] f(z)\,dz$$

$$= 2\pi i\left[\operatorname{Res}(z=i/5) + \operatorname{Res}(z=-i/5)\right]$$

$$\int_{C_R} \frac{1}{\sqrt{5z}\,(1+25z^2)}\,dz = 0 \quad \left\{\begin{array}{l}\text{As the power of the numerator} \\ \text{is smaller than the denominator}\end{array}\right\}$$

Next along C_ϵ, $z = \epsilon e^{i\theta}$
$\Rightarrow dz = \epsilon e^{i\theta}\, i\, d\theta$

$$\int_{C_\epsilon} \frac{dz}{\sqrt{5z}\,(1+25z^2)} = \lim_{\epsilon\to 0} \int \frac{i\epsilon e^{i\theta} d\theta}{\sqrt{5\epsilon e^{i\theta}}\,(1+25\epsilon^2 e^{2i\theta})} = 0$$

Hence,

$$I = \left[\int_{C_1} + \int_{C_3}\right] f(z)\, dz$$

$$= 2\pi i \left[\mathrm{Res}(i/5) + \mathrm{Res}(-i/5)\right]$$

Along C_1, $\theta = 0$ and along C_2, $\theta = 2\pi$

Hence along C_1 and C_2, $z = re^{i\theta} \Rightarrow dz = dr\, e^{i\theta}$

$$\int_{C_1} \frac{e^{i\theta} dr}{\sqrt{5re^{i\theta}}\,(1+25r^2 e^{2i\theta})}\Bigg|_{\theta=0} = \int_0^\infty \frac{dr}{\sqrt{5r}\,(1+25r^2)} = J$$

$$\int_{C_2} \frac{e^{i\theta} dr}{\sqrt{5re^{i\theta}}\,(1+25r^2 e^{2i\theta})}\Bigg|_{\theta=2\pi} = \int_\infty^0 \frac{-dr}{\sqrt{5r}\,(1+25r^2)} = J$$

$$\therefore I = 2J = 2\pi i \left[\mathrm{Res}(i/5) + \mathrm{Res}(-i/5)\right]$$

$$\Rightarrow J = \pi i \left[\frac{1}{10 i e^{i\pi/4}} - \frac{1}{10 i e^{3\pi/4}}\right]$$

$$= \frac{\pi}{10}\, 2\cos(\pi/4)$$

$$\boxed{J = \frac{\pi}{5\sqrt{2}}}$$

(9b) $J = \int_0^{\infty} \frac{1}{\sqrt[3]{x}\,(1+x^2)}\, dx$

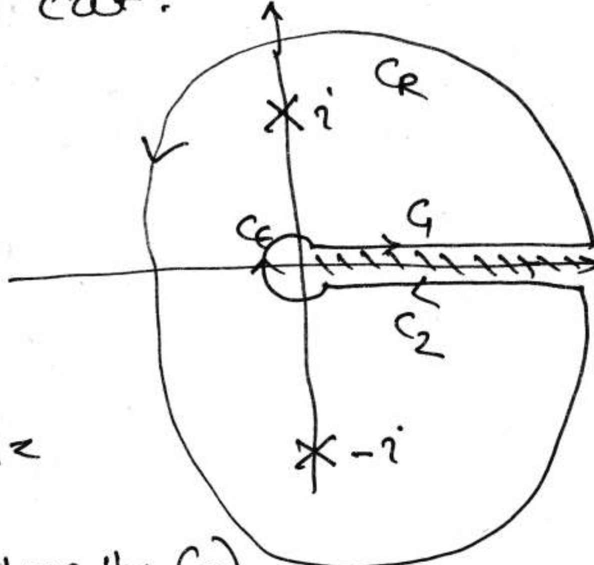
Consider, $I = \oint_C \frac{1}{z^{1/3}(1+z^2)}\, dz$, $f(z) = \frac{1}{z^{1/3}(1+z^2)}$

Consider, $z = re^{i\theta} \Rightarrow z^{1/3} = e^{i\theta/3}\, r^{1/3}$ where $0 \leq \theta \leq 2\pi$

at $\theta = 0$, $z^{1/3} = r^{1/3}$

$\theta = 2\pi$, $z^{1/3} = r^{1/3} e^{i2\pi/3} = r^{1/3}\left(\frac{-1+\sqrt{3}i}{2}\right)$

Hence, on the +ve x axis there is jump in $z^{1/3}$.
Hence, it becomes the branch cut.



Poles

$1+z^2 = 0 \Rightarrow z = \pm i$

$I = \oint_C f(z)\, dz$

$= \left[\int_{C_1} + \int_{C_R} + \int_{C_2} + \int_{C_\epsilon}\right] f(z)\, dz$

Consider $z = Re^{i\theta} \Rightarrow dz = Re^{i\theta} i\, d\theta$ (along the C_R)

$$\int_{C_R} f(z)\, dz = \lim_{R\to\infty} \int_0^{2\pi} \frac{iRe^{i\theta}\, d\theta}{R^{1/3} e^{i\theta/3}(R^2 e^{2i\theta}+1)} = 0$$

Similarly $z = \epsilon e^{i\theta}$ (along C_ϵ)

$$\int_{C_\epsilon} f(z)\, dz = \lim_{\epsilon\to 0} \int_{2\pi}^{0} \frac{i\epsilon\, e^{i\theta}\, d\theta}{\epsilon^{1/3} e^{i\theta/3}(1+\epsilon^2 e^{2i\theta})} = 0$$

So, $I = \left[\int_{C_1} + \int_{C_2}\right] f(z)\, dz = 2\pi i\left[\text{Res}(i) + \text{Res}(-i)\right]$

Next,

$$\int_{C_1} \frac{1}{z^{1/3}(1+z^2)}\,dz \qquad \left\{ \begin{array}{c} \text{Consider } z = re^{i\theta} \\ dz = dr\, e^{i\theta} \\ \text{at } \theta = 0 \end{array} \right\}$$

$$= \int_0^{\infty} \frac{e^{i\theta}\, dr}{r^{1/3}(1+r^2e^{2i\theta})\, e^{i\theta/3}} \Bigg|_{\theta=0}$$

$$= \int_0^{\infty} \frac{dr}{r^{1/3}(1+r^2)} = J$$

Next,

$$\int_{C_2} f(z)\,dz = \int_{\infty}^{0} \frac{dr\, e^{i\theta}}{r^{1/3}\, e^{i\theta/3}(1+r^2e^{2i\theta})} \Bigg|_{\theta=2\pi}$$

$$= \int_{\infty}^{0} \frac{dr\; e^{2\pi i}}{r^{1/3}\; e^{2\pi/3\, i}(1+r^2e^{i4\pi})} \times \frac{e^{-\pi i}}{e^{-\pi i}}$$

$$= \int_{\infty}^{0} \frac{-dr}{r^{1/3}(1+r^2)} \times \frac{1}{e^{-\pi i/3}}$$

$$= J e^{\pi i/3}$$

Hence,

$$I = J + Je^{\pi i/3} = 2\pi i\,[\text{Res}(i) + \text{Res}(-i)]$$

$$\Rightarrow J = \frac{2\pi i}{(1+e^{\pi i/3})} \left[\frac{1}{z^{1/3}\cdot 2z}\Bigg|_{z=i} + \frac{1}{z^{1/3}\cdot 2z}\Bigg|_{z=-i} \right]$$

$$\Rightarrow \boxed{J = \frac{\pi}{\sqrt{3}}}$$

(9c)

$$I = \int_{-1/a}^{1/a} \frac{\sqrt{1-a^2x^2}}{1+a^2x^2}\,dx$$

consider $J = \oint_C \frac{\sqrt{a^2z^2-1}}{a^2z^2+1}\,dz$

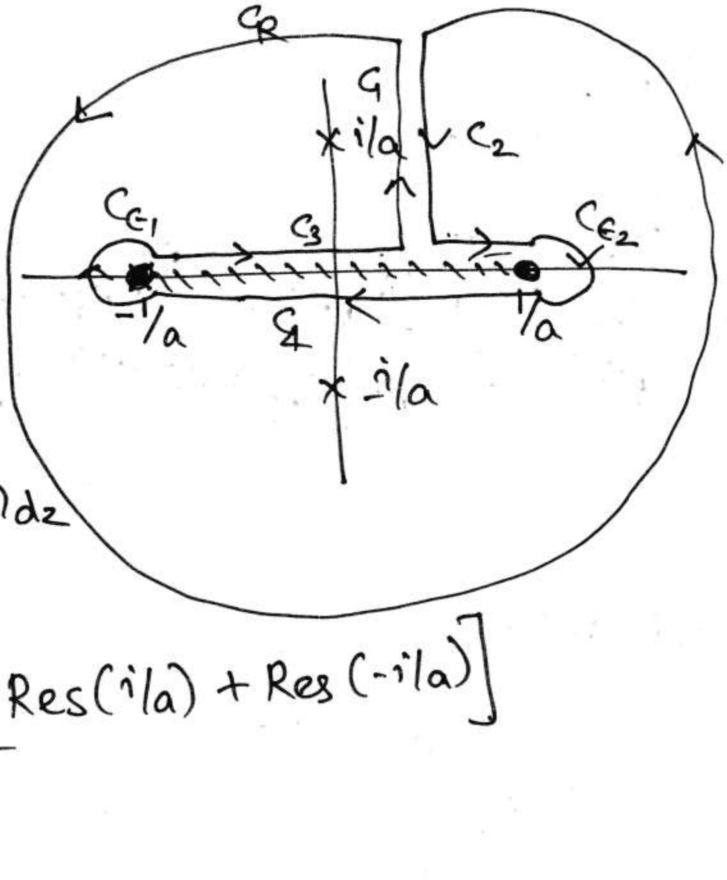
where C is given in the figure.

Branch cut.

$$\sqrt{a^2z^2-1} = \sqrt{(az+1)(az-1)} = a\sqrt{(z-1/a)(z+1/a)}$$

Hence, branch cut is from $\left(\frac{-1}{a} \text{ to } \frac{1}{a}\right)$.

Poles $a^2z^2+1=0 \Rightarrow z = \pm i/a$



$$J = \oint_C f(z)\,dz$$
$$= 2\pi i\left[\mathrm{Res}(i/a) + \mathrm{Res}(-i/a)\right]$$

$$\Rightarrow \left[\int_{C_1} + \int_{C_R} + \int_{C_2} + \int_{C_{\epsilon_2}} + \int_{C_4} + \int_{C_{\epsilon_1}} + \int_{C_3}\right] f(z)\,dz$$
$$= 2\pi i\left[\mathrm{Res}(i/a) + \mathrm{Res}(-i/a)\right]$$

Next,

$$\int_{C_R} f(z)\,dz \quad \left\{ \text{Along } C_R,\ z = Re^{i\theta} \Rightarrow dz = Re^{i\theta} i\,d\theta \right\}$$

$$= \lim_{R\to\infty} \int_0^{2\pi} \frac{\sqrt{a^2R^2e^{2i\theta} - 1}}{1 + R^2e^{2i\theta}a^2} Re^{i\theta} i\,d\theta$$

$$= \lim_{R\to\infty} \int_0^{2\pi} \frac{\cancel{a}\cancel{Re^{i\theta}}\cancel{Re^{i\theta}}\, i\,d\theta}{\cancel{R^2}\cancel{e^{2i\theta}}\cancel{a^2}} \quad \left\{\text{As } R\to\infty\right\}$$

$$= \frac{2\pi i}{a}$$

Next,

$$\int_{C_{\epsilon_1}} f(z)\,dz \quad \left\{ z = -\frac{1}{a} + \epsilon_1 e^{i\theta} \Rightarrow dz = \epsilon_1 e^{i\theta} i\,d\theta \text{ and } \epsilon_1 \to 0 \right\}$$

$$= \lim_{\epsilon_1\to 0} \int_0^{2\pi} \frac{a\left(1/a^2 + \epsilon_1^2 e^{2i\theta} - \frac{2}{a}\epsilon_1 e^{i\theta}\right)^{1/2} i\,\epsilon_1 e^{i\theta}\,d\theta}{a^2\left(1/a^2 + \epsilon_1^2 e^{2i\theta} - \frac{2}{a}\epsilon_1 e^{i\theta}\right) + 1}$$

$$= 0$$

Similarly,

$$\int_{C_{\epsilon_2}} f(z)\,dz = 0$$

Next,

$$\int_{C_1} f(z)\,dz = -\int_{C_2} f(z)\,dz \Rightarrow \left[\int_{C_1} + \int_{C_2}\right] f(z)\,dz = 0$$

Hence,

$$J = \oint_C f(z)dz = \left[\int_{C_R} + \int_{C_3} + \int_{C_4}\right] f(z)\,dz$$

$$= 2\pi i\left[\mathrm{Res}(i/a) + \mathrm{Res}(-i/a)\right]$$

Next,

$$\int_{C_3} f(z)\,dz = \int_{C_3} \frac{i\sqrt{1-x^2 d^2}}{1+x^2 d^2} \quad \text{for } 0 \leq \theta \leq 2\pi$$

$$\int_{C_4} f(z)\,dz = \int_{C_4} \frac{-i\sqrt{1-x^2 d^2}}{1+x^2 d^2} \quad \text{for } 0 \leq \phi \leq 2\pi$$

Hence,

$$J = \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \left[\int_{-1/a+\epsilon_1}^{1/a-\epsilon_2} \frac{i\sqrt{1-a^2x^2}}{1+a^2x^2}\,dx + \int_{1/a-\epsilon_2}^{1/a+\epsilon_1} \frac{-i\sqrt{1-a^2x^2}}{1+a^2x^2}\,dx\right] + \frac{2\pi i}{a}$$

$$= 2\pi i\left[\mathrm{Res}(i/a) + \mathrm{Res}(-i/a)\right]$$

$$\Rightarrow \int_{-1/a}^{1/a} \frac{i\sqrt{1-a^2x^2}}{1+x^2a^2}\,dx + \int_{1/a}^{-1/a} \frac{-i\sqrt{1-x^2a^2}}{1+x^2a^2}\,dx + \frac{2\pi i}{a}$$

$$= 2\pi i\left[\mathrm{Res}(i/a) + \mathrm{Res}(-i/a)\right]$$

$$\Rightarrow 2Ii + \frac{2\pi i}{a} = 2\pi i\left[\mathrm{Res}(i/a) + \mathrm{Res}(-i/a)\right]$$

$$\mathrm{Res}(i/a) = \left.\frac{\sqrt{a^2z^2-1}}{2a^2 z}\right|_{z=i/a} = \frac{\sqrt{-1}\,\sqrt{2}}{2i} = \frac{1}{\sqrt{2}\,a}$$

for $y>0$, $\sqrt{-1}=i$

for $y<0$, $\sqrt{-1}=-i$

~~Here~~
Similarly,

$$\text{Res}(-i/a) = \left.\frac{\sqrt{a^2z^2-1}}{2a^2z}\right|_{z=-i/a} = \frac{\sqrt{-1}\sqrt{2}}{2a(-i)} = \frac{1}{\sqrt{2}\,a}$$

Hence,

$$2I_i + \frac{2\pi i}{a} = 2\pi i\left[\frac{1}{\sqrt{2}a} + \frac{1}{\sqrt{2}a}\right]$$

$$\Rightarrow I + \frac{\pi}{a} = \frac{\pi}{a}\sqrt{2}$$

$$\Rightarrow I = \frac{\pi}{a}(\sqrt{2}-1)$$

9d. $J = \int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx$

Consider $x = 2y \Rightarrow dx = 2dy$

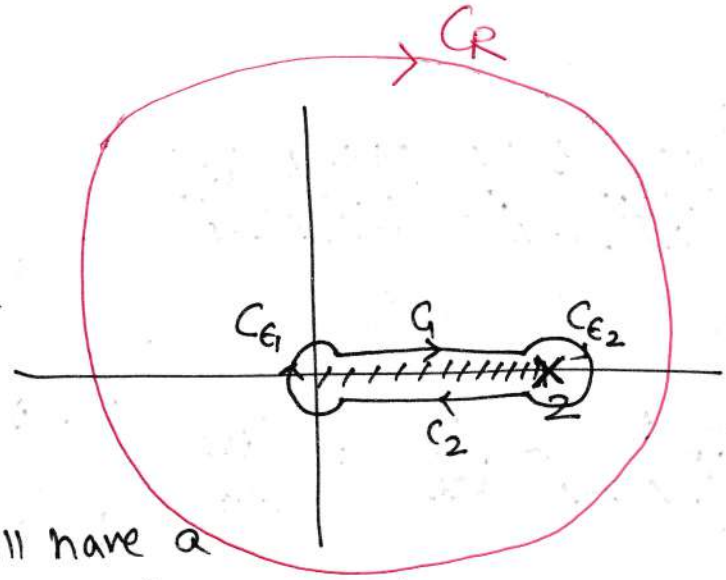
$$J = \int_0^1 \frac{2\,dy}{\sqrt{2y(2-2y)}} = \int_0^1 \frac{dy}{\sqrt{y(1-y)}} = \pi$$

[Detailed derivation was done in the class]
(See the next page)

(9d)

$$J = \int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx$$

$$I = \int_0^2 \frac{1}{\sqrt{z(z-2)}}\,dz$$



$\sqrt{(z-0)(z-2)} \to$ It will have a Branch cut from $z=0$ to 2

$$\left[\int_{C_{\epsilon_1}} + \int_{C_1} + \int_{C_2} + \int_{C_{\epsilon_2}}\right] \frac{1}{\sqrt{z(z-2)}}\,dz = \lim_{R\to\infty}\int_{C_R} \frac{1}{\sqrt{z(z-2)}}\,dz \; (-1)$$

$(\because$ Clockwis

Now consider $z = \epsilon_1 e^{i\theta} \Rightarrow dz = \epsilon_1 e^{i\theta}\, i\,d\theta$

$$\lim_{\epsilon_1\to 0}\int_{C_{\epsilon_1}} \frac{1}{\sqrt{z(z-2)}}\,dz = \lim_{\epsilon_1\to 0} \frac{\epsilon_1 e^{i\theta} i\,d\theta}{\sqrt{\epsilon_1 e^{i\theta}(\epsilon_1 e^{i\theta}-2)}} = 0$$

Similarly, $\lim_{\epsilon_2\to 0}\int_{C_{\epsilon_2}} \frac{1}{\sqrt{z(z-2)}}\,dz = 0$

Now

$$\left[\int_{C_1} + \int_{C_2}\right] \frac{1}{\sqrt{z(z-2)}}\,dz = \lim_{R\to\infty}\int_{C_R} \frac{1}{\sqrt{z(z-2)}}\,dz \; (-1)$$

Now by definition.

along C_1 $\sqrt{z(z-2)} = i\sqrt{x(2-x)}$

along C_2 $\sqrt{z(z-2)} = -i\sqrt{x(2-x)}$

Hence $\cancel{\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left[\int_{0+\epsilon_1}^{2-\epsilon_2} + \int_{2-\epsilon_2}^{0+\epsilon_1}\right]}$

Hence,

$$\lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}\left[\int_{0+\epsilon_1}^{2-\epsilon_2}\frac{1\,dx}{i\sqrt{x}\sqrt{2-x}}+\int_{2-\epsilon_2}^{0+\epsilon_1}\frac{1\,dx}{-i\sqrt{x(2-x)}}\right] = \lim_{R\to\infty}\int_{C_R}\frac{-1}{\sqrt{z(z-2)}}\,dz$$

$$\Rightarrow \frac{2}{i}\int_0^2\frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{R\to\infty}\int_{C_R}\frac{-1}{\sqrt{z(z-2)}}\,dz$$

$$\Rightarrow \frac{2J}{i} = \lim_{R\to\infty}\int_{C_R}\frac{-1}{\sqrt{z(z-2)}}\,dz \quad \text{———} \ (*)$$

consider $z = Re^{i\theta}$ wher $R\to\infty$

$\Rightarrow dz = Re^{i\theta}d\theta\, i$

Eq (*) becomes

$$\Rightarrow \frac{2J}{i} = \lim_{R\to\infty}\int_0^{2\pi}\frac{-1\quad Re^{i\theta}\,i\,d\theta}{\sqrt{Re^{i\theta}(Re^{i\theta}-2)}}$$

$$\Rightarrow \frac{2J}{i} = -2\pi i$$

$$\Rightarrow \boxed{J = \pi}$$

(10) (a) $$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\, e^{st}\, ds = \mathrm{Res}(s=3)$$

$$\Rightarrow f(t) = \frac{1}{(s-3)}(s-3)\, e^{st}\Big|_{s=3} = e^{3t}$$

(b) $$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{3}{s^2+9}\, e^{st}\, ds = \left[\mathrm{Res}(3i) + \mathrm{Res}(-3i)\right]$$

$$\mathrm{Res}(3i) = \frac{3e^{st}}{(s-3i)(s+3i)}(s-3i)\Big|_{s=3i} = \frac{3e^{3it}}{6i}$$

$$\mathrm{Res}(-3i) = \frac{3e^{st}}{(s-3i)(s+3i)}(s+3i)\Big|_{s=-3i} = \frac{-3e^{-3it}}{6i}$$

$$f(t) = \frac{3}{6i}\left[e^{3it} - e^{-3it}\right]$$

$$= \sin(3t).$$

(c) $$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{s}{s^2+4}\, e^{st}\, ds = \mathrm{Res}(2i) + \mathrm{Res}(-2i)$$

$$\mathrm{Res}(2i) = \frac{se^{st}}{s+2i}\Big|_{s=2i} = \frac{e^{2it}}{2}$$

$$\mathrm{Res}(-2i) = \frac{e^{-2it}}{2}$$

$$f(t) = \frac{e^{2it} + e^{-2it}}{2} = \cos(2t)$$

(d) $$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{2}{s^3}\, e^{st}\, ds$$

$$= \operatorname{Res}(s=0)$$

$$= \frac{1}{(3-1)!}\, \frac{d^2}{ds^2}\left(2e^{st}\right)\Big|_{s=0} = t^2$$

(e) $$f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{5+s}{s^2+1}\, e^{st}\, ds = \left[\operatorname{Res}(i) + \operatorname{Res}(-i)\right]$$

$$= \frac{(5+s)\, e^{st}}{s-i}\Big|_{s=-i} + \frac{(5+s)\, e^{st}}{s+i}\Big|_{s=i}$$

$$= \frac{1}{2i}\left[(5+i)\, e^{it} - (5-i)\, e^{-it}\right]$$

$$= \frac{1}{2i}\left[10i\, \sin t + 2i\cos t\right]$$

$$= 5\sin t + \cos t$$