Partitioning instantaneous degrees-of-freedom and its application to three-degrees-of-freedom parallel manipulators

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Abstract—*In general, a rigid body moving in space can possess three translational and three rotational degrees-of-freedom. In many situations, the motion of the rigid body is constrained, and therefore has less than six-degrees-of-freedom. In such cases, it is often important to understand how the degrees-of-freedom are distributed between pure translation and pure rotation. In this paper, we present a general approach towards the* partitioning *of the available instantaneous degrees-of-freedom for the constrained motion of rigid-bodies. The approach is based on computing the eigenvalues and eigenvectors of certain matrices associated with the instantaneous motion of a rigid body. The eigenvalue problem involves the solution of* at most *a cubic polynomial, and hence the eigenvalues can be obtained in closed form in all the cases. The approach is applied to several well known three-degrees-of-freedom spatial parallel manipulators. It is well-known that parallel manipulator can gain one or more degrees-of-freedom at a gain-type singularity. The general approach is also applied to determine if the gained degree(s)-of-freedom are in translational or rotational in nature.*

Keywords: Partitioning, instantaneous degrees-of-freedom, eigen-problem, gain singularity

I. Introduction

It is well known that the rigid body displacements form a Lie group, known as the *Special Euclidean Group* denoted by $SE(3)$ (see, for example, [1]). The linear and angular velocities associated with rigid body motions are real 3-vectors each, but can be suitably composed into vectors in six dimensions, known as the *twists* in kinematic literature. The twists lie in the Lie algebra associated with $SE(3)$, which is denoted by $se(3)$. The twists can also be described by a linear combinations of *screws*, which are elements of $\mathbb{P}^5$ and can be thought of simply as twists of unit magnitude [2]. This description of the twists allows one to obtain the the resultant twist of a n-degrees-of-freedom rigid body motion in terms of the n independent input screws (known as a n-*screw system*). Furthermore, it also leads naturally to the notion of a set of *principal screws*, which form a *canon-*

ical basis of a system of screws. The knowledge of the principal screws can reveal important properties of a rigid body motion, and therefore it has received a lot of attention in the past century. Most of the important results of this type of analysis can be found in the seminal works of Ball [3] and Hunt [4].

In reference [5], [6], the authors take a slightly different viewpoint and analyse the space of twists directly instead of the underlying space of screws. One of the major advantages of such analysis is that it leads to an understanding of the *nature* of the degrees-of-freedom of a rigid body motion, i.e., in a general n-degrees-of-freedom motion, it can clearly obtain the number of degrees-of-freedom involved in *pure* translational motion (equivalently, number of screws with infinite pitch) and the remaining involved in rotational as well translational motion in a general screw mode (equivalently, the number of screws with finite pitch). For instance, in a 3-degrees-of-freedom motion, we can have either 3, 2, 1 or 0 linearly independent angular velocities, and 0, 1, 2 or 3 linearly independent translational velocities, respectively. The first combination, namely 3 angular and 0 translational velocities, is a pure rotational motion, and the last combination, namely 0 angular and 3 translational velocities, is pure translational motion. These two combinations in a three-degrees-of-freedom motion are relatively intuitive to ascertain. However, screw theory does not answer this question in a direct way. In this paper, we use the approach and concepts presented in [5], [6] to present a systematic approach to obtain the exact division of linear and angular velocities, and extend the results presented in [7]. The approach for three-degrees-of-freedom motion is discussed in detail in section II. For general n-degrees-of-freedom motion the reader is referred to [6], [7].

The identification of the twists associated with angular and purely linear velocities allows us to determine a partitioning of possible degrees-of-freedom of a rigid body motion. We present an analytical approach which can be directly used for serial, parallel and hybrid manipulators. The concept of partitioning degrees-of-freedom is illustrated with the help of several well known three-degree-of-freedom parallel manipulators in section III.

A parallel manipulator is known to gain one or more

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degrees-of-freedom of freedom at a *gain* singularity [8]. However, it is not easy to determine if the gain of the degree(s)-of-freedom is in terms of angular or linear velocity. Using the analytical approach to determine the partitioning of the degrees-of-freedom, we can determine if the gained degree-of-freedom is in linear or angular velocity. This specific advantage of the approach developed in this paper is shown for several three-degrees-of-freedom spatial parallel manipulators, at their respective gain-singular configurations, in section IV.

The paper is organised as follows: in section II, we briefly discuss the formulations for the distribution of linear and angular velocities, and derive the closed-form expression for the principal twists in the ω basis. We also present the concept of partitioning of available degrees-of-freedom and present a classification for three-degrees-of-freedom motion. In section II-D, we present the formulation of the relevant matrices in terms of the constraint equations in parallel manipulators in non-singular and gain singularity configurations. In section III, we present the formulation for a 3-UPU wrist manipulator, a 3-UPU manipulator capable of pure translational motion, a 3-RPS manipulator, a cylindrical manipulator and the Delta robot with parallelogram linkage in a leg. We present results showing the partitioning of degrees-of-freedom in a typical non-singular configuration for these parallel manipulators. In section IV, we present the singularity analysis of the chosen three degrees-of-freedom manipulators and determine the gained degrees-of-freedom In section V, we present the conclusions.

II. Mathematical formulation

In this section, a brief description of the mathematical formulation is presented. The theoretical developments presented in this paper follow [5], [6] closely, and the readers are directed to the same for further details. This section also briefly presents the formulation of matrices relevant for analysis of parallel manipulators.

A. *The partitioning of degrees-of-freedom*

The angular velocity of a chosen output link (or end-effector), ω, and the linear velocity of the chosen point on the link, v can be written as

$$\omega = J_\omega \dot{\theta}, \quad v = J_v \dot{\theta} \tag{1}$$

where $\dot{\theta}$ denotes the vector of input joint rates and J_ω and J_v denote appropriate Jacobian matrices. One can consider a *dual* linear combination of these matrices, to arrive at a *dual Jacobian matrix* [5]:

$$\hat{J} = J_\omega + \epsilon J_v \tag{2}$$

where ϵ is the *dual unit* with the property $\epsilon^2 = 0, \epsilon \neq 0$. One can further define the matrix:

$$\hat{g} = \hat{J}^T \hat{J} \tag{3}$$
$$= g + \epsilon g_0, \text{ where } g = J_\omega^T J_\omega, \ g_0 = J_\omega^T J_v + J_v^T J_\omega$$

Following [5], one can study the eigenproblem of the matrix $\hat{g}$, and find the *principal twists* associated with the motion of the manipulator. Some of the main results of the study are given below.

1. Since there can be at most three components of ω, irrespective of the number of degrees-of-freedom, the $\text{rank}(g) \leq 3$.

2. If n denotes the degrees-of-freedom of the motion, then for $n > 3$, there are m vanishing eigenvalues of the matrix g, where $m \geq (n-3)$. Correspondingly, there are m eigenvectors in the null-space of g (or equivalently, in the null-space of J_ω). These eigenvectors, denoted by $\dot{\theta}_i^n$, $i = 1, \ldots, m$, can be obtained as the non-trivial solution to the equation:

$$g\dot{\theta}_i^n = 0, \ i = 1, \ldots, m \tag{4}$$

The eigenvectors $\dot{\theta}_i^n$ can be obtained by any standard linear algebra technique. The corresponding principal twists are obtained as:

$$\hat{\mathcal{V}}_i^n = J_\omega \dot{\theta}_i^n + \epsilon J_v \dot{\theta}_i^n$$
$$= 0 + \epsilon \mathbf{V}_i, \quad i = 1, \cdots, m \tag{5}$$

The significance of the zero-valued real part is that these twists are *not* associated with any angular velocity and they represent *pure translational motions*.

3. The number of pure linear velocity components, V_i, can at the most be three. To find the actual number of independent linear velocities, one can construct the matrix J_V such that V_i constitutes the ith column of J_V, and further define:

$$g_V = J_V^T J_V \tag{6}$$

The number of non-zero eigenvalues of g_V gives the number of linearly independent linear velocity components associated with the rigid-body motion, and the column-space of g_V gives the corresponding directions [5].

It is clear from the above discussion, that at a given instant, the degrees-of-freedom of a rigid-body can be considered in terms of its angular and linear velocities, and expressed as the following formula:

$$\text{degrees-of-freedom} = \text{rank}(g) + \text{rank}(g_V) \tag{7}$$

Equation (7) formalizes the concept of the *partitioning of degrees-of-freedom* introduced in [5]. This concept of *instantaneous* definition of degree-of-freedom helps in the following:

• Identification of the nature of independent degrees-of-freedom in the case of multi-degree-of-freedom constrained rigid-body motion. In our formulation, this division comes out *naturally*, and examples are provided in the paper to illustrate this point.

• Identification of the nature of singularity: It is well-known in literature, that at a singular configuration, the degree-of-freedom of a manipulator changes from its regular value. However, it is not easy to identify if the gained or lost degree-of-freedom is translational or rotational. As we show later in this paper, the effects of gain singularity in a number of parallel manipulators are captured by our formulation.

B. Classification of three-degrees-of-freedom non-singular motions

In reference [6], [7], a detailed classification of non-singular spatial motions based on equation (7) is presented. In this section, we present, for sake of completeness, the special case of possible partitions of three-degrees-of-freedom in a chosen end-effector. We also present the analytical solutions for the eigenvalue problems associated with each of the classes. These equations are used for the examples given in the paper.

A chosen end-effector in a three-degree-of-freedom parallel manipulator can exhibit a)three rotations and zero translation, b) two rotations and one translation, c) one rotation and two translations, and d) three translations and zero rotations. These are termed as class χ_{30}, χ_{21}, χ_{12} and χ_{03}, respectively (see [6], [7] for details). For 3-degree-of-freedom motion, the characteristic equation of g is given by [5]:

$$\lambda^3 - 3\lambda^2 + \frac{1}{2}(3 - A_1)\lambda + \frac{1}{2}(1 + A_1 - 4A_2) = 0 \quad (8)$$

where $A_1 = \cos 2\phi_{12} + \cos 2\phi_{23} + \cos 2\phi_{31}$, $A_2 = c_{12}c_{23}c_{31}$. Here, c_{ij} indicates $\cos \phi_{ij}$, where ϕ_{ij} is the angle between the ith and the jth input screws.

Based on the values of A_1, A_2, and consequently the values of the coefficients of the cubic equation, one can recognize the following subclasses within the three-screw system.

1. Class χ_{30}: The input screws belong to this category when all the roots of equation (8) are non-zero.

2. Class χ_{21}: In this case one of the roots of equation (8) vanish. The conditions can be written as

$$
\begin{aligned}
1 + A_1 - 4A_2 &= 0 \\
3 - A_1 &\neq 0
\end{aligned}
\quad (9)
$$

It may be verified easily, that the geometric implication of equation (9) is that the three input screws axes are all parallel to a single plane, i.e. the angular velocity vector has zero component in the direction perpendicular to the plane.

The residual quadratic equation is:

$$\lambda^2 - 3\lambda + \frac{1}{2}(3 - A_1) = 0 \quad (10)$$

This equation can be solved for the two non-zero eigenvalues, and the corresponding eigenvectors can be mapped to the respective principal twists. The third eigenvector in the null-space of g maps to the single pure translational velocity in this case, and we require this velocity is non-zero for the screw system to belong to this category.

3. Class χ_{12}: This class is complementary to the last one, and it requires that two of the roots of equation (8) vanish. The conditions can be written as

$$
\begin{aligned}
1 + A_1 - 4A_2 &= 0 \quad (11) \\
3 - A_1 &= 0 \quad (12)
\end{aligned}
$$

In this case there is only one independent rotational motion possible. The two eigenvectors in the null-space of g give rise to two pure translational velocities. For the motion to be non-singular, i.e. of 3-degree-of-freedom, rank(g_V) needs to equal two in this case.

4. Class χ_{03}: As in the case of class χ_{02}, this requires that J_ω is a null matrix, and we analyse the matrix g_V to obtain the distribution of the pure translational velocities. We assume here that this form is non-degenerate, i.e. all the eigenvalues of g_V are non-zero. An obvious class of manipulators generating this kind of motion is the serial 3-P translational manipulators.

C. Singularities and transitions between various classes of rigid-body motion

The end-effector of a spatial manipulator can have motions belonging to different classes at different subsets of its workspace. More importantly, some change in the architecture parameters of a manipulator can cause its motion to shift from one class to the other, without changing the total degrees-of-freedom (see the examples of a 3-UPU manipulator as a wrist and as a purely translational device). Further, a manipulator can *lose* or *gain* degree(s)-of-freedom at a singularity [8], [9]. All the above can be generalised as a *transition* from one motion class to another. Using the corresponding analytical formula one can analyse all of these phenomena in a unified setting and provide a vital tool for the design of parallel manipulators. In the illustrative examples in this paper, the nature of the gained singularity is computed.

D. Equivalent Jacobian matrices and condition for gain singularity in parallel manipulators

The theory developed above is applicable to serial and parallel manipulators alike. However, there are some additional considerations for its application to parallel manipulators. For instance, in a parallel manipulator, the choice of the output link (or the *end-effector*) is not so obvious in

all cases - in this paper, we study only *platform-type* manipulators, and the end-effector is taken to be the platform in each case. Further, the motion of the end-effector is affected not only by n active joints $\boldsymbol{\theta}$, but also m *passive* (i.e., non-actuated) joints $\boldsymbol{\phi}$. Therefore, the first-order kinematic equations (1) can be rewritten for this case as:

$$\boldsymbol{\omega} = \boldsymbol{J}_{\omega}^{eq}\dot{\boldsymbol{q}} = \boldsymbol{J}_{\omega\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{\omega\phi}\dot{\boldsymbol{\phi}}$$
$$\boldsymbol{v} = \boldsymbol{J}_{v}^{eq}\dot{\boldsymbol{q}} = \boldsymbol{J}_{v\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{v\phi}\dot{\boldsymbol{\phi}} \quad (13)$$

where $\boldsymbol{q} = (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T)^T \in \mathbb{R}^{n+m}$. In this paper, the point of interest, $\boldsymbol{p}$, is taken to be the origin of the reference frame $\{P\}$, attached rigidly to the centroid of the moving platform, which is taken to be symmetric in all the cases. The angular velocity of the moving platform can be obtained from its rotation matrix, ${}_{P}^{0}\boldsymbol{R}$, with respect to the inertial reference $\{0\}$ as follows:

$$ {}^{0}\boldsymbol{\omega}_P = \frac{d}{dt}({}_{P}^{0}\boldsymbol{R})\, {}_{P}^{0}\boldsymbol{R}^T = \boldsymbol{J}_{\omega}^{eq}\dot{\mathbf{q}} \quad (14)$$

where ${}_{P}^{0}\boldsymbol{R}$ is the rotation matrix of moving platform $\{P\}$ with respect to the fixed platform $\{0\}$. Note that ${}_{P}^{0}\boldsymbol{R}$ is a function of both $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$, and hence equation (14) can be cast in the form of equation (13). The linear velocity of the moving platform can be obtained as derivative of the position vector $\boldsymbol{p}$, which is given by:

$$\boldsymbol{p} = \frac{1}{3}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)$$
$$\Rightarrow \boldsymbol{v}_{\boldsymbol{p}} = \frac{d\boldsymbol{p}}{dt} = (\dot{\boldsymbol{p}}_1 + \dot{\boldsymbol{p}}_2 + \dot{\boldsymbol{p}}_3) = \boldsymbol{J}_{v}^{eq}\dot{\boldsymbol{q}} \quad (15)$$

where $\boldsymbol{p}_i, \ i = 1, 2, 3$ denote the position vectors of the vertices of the triangular platform in frame $\{0\}$.

The remaining task is to convert equations (14, 15) to forms analogous to equation (1). For doing this, $\dot{\boldsymbol{\phi}}$ must be described in terms of $\dot{\boldsymbol{\theta}}$, as done in the following.

The m passive variables can be determined from a set of m *constraint equations*, which essentially ensure the closure of the loops in the mechanism. These can be written as:

$$\boldsymbol{\eta}(\boldsymbol{q}) = \boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \mathbf{0}, \ \boldsymbol{\eta} \in \mathbb{R}^m \quad (16)$$

Differentiating equation (16) with respect to time t, and rearranging, we get:

$$\boldsymbol{J}_{\eta\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{\eta\phi}\dot{\boldsymbol{\phi}} = \mathbf{0}, \quad \boldsymbol{J}_{\eta\theta} = \frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\theta}}, \ \boldsymbol{J}_{\eta\phi} = \frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{\phi}} \quad (17)$$

where the ith column of the $m \times n$ matrix $\boldsymbol{J}_{\eta\theta}$ consists of the partial derivatives of $\boldsymbol{\eta}(\boldsymbol{q})$ with respect to the actuated variables $\theta_i, \ i = 1, \ldots, n$ and the ith column of $m \times m$ matrix $\boldsymbol{J}_{\eta\phi}$ consists of the partial derivatives of $\boldsymbol{\eta}(\mathbf{q})$ with respect to the passive variables $\phi_i, \ i = 1, \ldots, m$.

If $\det(\boldsymbol{J}_{\eta\phi}) \neq 0$, i.e., the matrix $\boldsymbol{J}_{\eta\phi}$ is not singular, we can solve for $\dot{\boldsymbol{\phi}}$ from equation (17) and write

$$\dot{\boldsymbol{\phi}} = -\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta}\dot{\boldsymbol{\theta}} = \boldsymbol{J}_{\phi\theta}\dot{\boldsymbol{\theta}}, \quad \boldsymbol{J}_{\phi\theta} = -\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta} \quad (18)$$

We can now substitute $\dot{\boldsymbol{\phi}}$ from equation (18) in equation (13) and get

$$\boldsymbol{\omega} = \boldsymbol{J}_{\omega\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{\omega\phi}\boldsymbol{J}_{\phi\theta}\dot{\boldsymbol{\theta}}$$
$$\boldsymbol{v} = \boldsymbol{J}_{v\theta}\dot{\boldsymbol{\theta}} + \boldsymbol{J}_{v\phi}\boldsymbol{J}_{\phi\theta}\dot{\boldsymbol{\theta}} \quad (19)$$

By comparison with equation (1), one can obtain the equivalent Jacobian matrices as:

$$\boldsymbol{J}_{\omega}^{eq} \triangleq \boldsymbol{J}_{\omega\theta} + \boldsymbol{J}_{\omega\phi}\boldsymbol{J}_{\phi\theta} = \boldsymbol{J}_{\omega\theta} - \boldsymbol{J}_{\omega\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta} \quad (20)$$

$$\boldsymbol{J}_{v}^{eq} \triangleq \boldsymbol{J}_{v\theta} + \boldsymbol{J}_{v\phi}\boldsymbol{J}_{\phi\theta} = \boldsymbol{J}_{v\theta} - \boldsymbol{J}_{v\phi}\boldsymbol{J}_{\eta\phi}^{-1}\boldsymbol{J}_{\eta\theta} \quad (21)$$

It is clear that the above formulation of the Jacobian matrices hold under the condition $\det(\boldsymbol{J}_{\eta\phi}) \neq 0$. When this is violated, i.e., the manipulator is at a constraint/gain-type singularity, it is customary to investigate the motion arising out of the *gained* degree(s)-of-freedom alone (see, e.g., [5], [8]). In order to do so, the gained passive velocity is found out first, while the actuators are held fixed, i.e., $\dot{\boldsymbol{\theta}} = \mathbf{0}$. From equation (17), we get:

$$\boldsymbol{J}_{\eta\phi}\dot{\boldsymbol{\phi}}_i^n = \mathbf{0}, \ i = 1, \ldots, \text{nullity}(\boldsymbol{J}_{\eta\phi}) \quad (22)$$

The gained angular and linear velocities can be obtained as:

$$\boldsymbol{\omega} = \boldsymbol{J}_{\omega\phi}\dot{\boldsymbol{\phi}}^n$$
$$\boldsymbol{v} = \boldsymbol{J}_{v\phi}\dot{\boldsymbol{\phi}}^n \quad (23)$$

Equations (23) imply that even with the actuator locked the end-effector of the parallel manipulator can *instantaneously* have non-zero linear and/or angular velocities and the manipulator *gains* one or more degree(s)-of-freedom.

Similar to the analysis with $\boldsymbol{g}$, we can solve the eigenvalue problem arising out of $\boldsymbol{\omega} = \boldsymbol{J}_{\omega\phi}\dot{\boldsymbol{\phi}}^n$. We can furthermore classify the gained degrees-of-freedom as follows:
• For the non-singular motion of class χ_{30}, if (with actuators locked) the eigenvalue(s) of $\boldsymbol{J}_{\omega\phi}{}^T\boldsymbol{J}_{\omega\phi}$ is(are) zero, then the gained motion(s) is(are) translational.
• For class χ_{21} or χ_{12}, if (with actuators locked) the eigenvalue(s) of $\boldsymbol{J}_{\omega\phi}{}^T\boldsymbol{J}_{\omega\phi}$ is(are) zero, then the gained motion(s) is(are) purely translational, rotational or both.
• For class χ_{03}, if (with actuators locked) the manipulator gains one or more degrees-of-freedom, then the gained motion(s) is(are) purely rotational.
It may be noted that in the above classification, we *assume* that the gained degree-of-freedom does not lead to redundant degrees-of-freedom. In the next section, we use the concept of equivalent Jacobian to partition degrees-of-freedom in several three-degrees-of-freedom parallel manipulators. We also study the gained degree-of-freedom at a gain singularity.

III. Degree-of-freedom partitioning for three-degrees-of-freedom parallel manipulators

In this section, we illustrate the theory developed with several examples of three-degrees-of-freedom parallel manipulators. The manipulators chosen are well-known in literature, such as: a) a 3-UPU manipulator as a wrist and a purely translational manipulator, b) a 3-RPS parallel manipulator, c) a "cylindrical" robot, and d) the well-known Clavel's "Delta" robot.

A. The 3-UPU manipulator as a wrist

The 3-UPU manipulator, as a parallel wrist, was first proposed by Di Gregorio [10]. The 3-UPU manipulator, as the name signifies, consist of three 'legs' each of which can be modeled as R-R-P-R-R chain. With reference to the figure 1, the points P_i, B_i $i = 1, 2, 3$, are the centres of the universal joints in the moving platform and the fixed base, respectively. For the 3-UPU manipulator to function as a wrist, in each leg the axes of first revolute joints making up the U joint in the fixed base and the axis of the last revolute joints making up the U joint in the moving platform meet at a point P. Moreover, the point P is chosen as the origin of the fixed base, i.e, O is same as P. Figure 1 shows a typical leg of the 3-UPU wrist. In this figure, a_i and b_i are the constant lengths of the segments P_iP and B_iP respectively; l_i is the variable length of the segment B_iP_i and denotes the joint variable of the actuated prismatic joint. The point P is fixed in the stationary platform and is chosen as the origin of a reference system attached to the same. Assigning co-ordinate axis according to the standard convention [15], the the Denavit-Hartenberg (DH) parameters [15]) for a leg can be obtained. These are given in table I. In the table, the
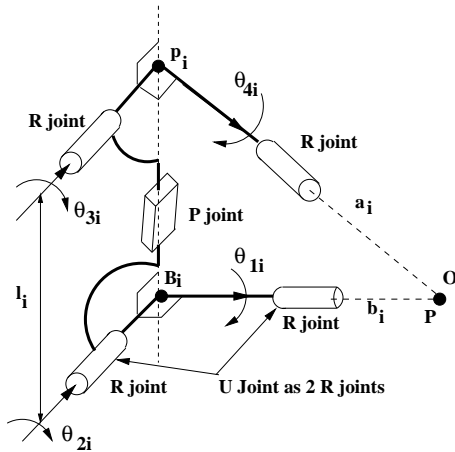

Fig. 1. A typical leg for the 3-UPU wrist

variables l_i are active joint variables and θs are the passive joint variables. Once the DH parameters are known, the link transformation matrices $^{i-1}_iT$ can be computed easily (see, e.g., [15]). The product $^0_1T^1_2T...^4_5T$ gives the posi-

Link	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	b	θ_{1i}
2	$\pi/2$	0	0	θ_{2i}
3	$\pi/2$	0	l_i	π
4	$\pi/2$	0	0	θ_{3i}
5	$\pi/2$	0	a	θ_{4i}

TABLE I. DH parameters of a leg of a 3-UPU wrist

tion vector of the point P which would be a function of l_i and the four θs. We can also obtain the position vector of the points B_i.

From the DH table, we can observe that there are 4 θ's for each leg and, potentially, the kinematics of the 3-UPU wrist can involve all the 12 passive variables. However, due to the special choice of P, only 9 passive variables are of interest and we have to deal with 9 loop-closure equations. The first 6 constraint equations are formed by equating the X, Y and Z coordinate of the end point P from each leg:

$$\begin{aligned} x_i - x_j &= 0 \\ y_i - y_j &= 0 \\ z_i - z_j &= 0 \end{aligned} \qquad (24)$$

where $i, j = 1, 2, 3$ and $i \neq j$. The other three equations are formed by equating the point P to the origin of the base:

$$P_x = 0, \quad P_y = 0 \quad P_z = 0 \qquad (25)$$

The 9 equations above are functions of nine θ variables and the three actuated variables. At a typical non-singular configuration and for given values of the actuated variables, these 9 non-linear equations can be solved for the passive variables. Once all the passive variables are known, following the formulation shown in section II, we obtain the equivalent Jacobian matrices, J^{eq}_ω and J^{eq}_v. The matrix $g = J^{eqT}_\omega J^{eq}_\omega$ is constructed next and its eigenproblem is solved, leading to the classification of the manipulator[1].

For the 3-UPU wrist, we have used $b_i = a_i = 1$, ($i = 1, 2, 3$) and computed several sets of numerical results. Two representative results are as follows:

- For active variables values 1.5240, 0.9980, 1.2430, the 9 passive variables values are 1.2873, 0.7044, 0.7044, 1.0705, -1.0484, 5.2348, 1.1328, -0.9001 and 5.3830. The eigenvalues of g are 0.296×10^{20}, 69.4340 and 0.3197×10^{19}.
- For active variables values 3/2, 1, 3/2, the 9 passive variables values are 1.0, 0.7227, 0.7227, 0.7167, 1.0472, 1.0472, 1.0, 0.7227 and 0.7227. The eigenvalues of g are 0.255×10^{18}, 32.0 and 15.4012.

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[1]In this work, for all examples, except for the numerical solution of the passive variables, all other steps use symbolic expressions and results are obtained in closed-form. Further, all lengths are in metres, and all angles in radians.

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It can be observed that the eigenvalues of the matrix $\boldsymbol{g}$ are non-zero and hence the moving platform in the 3-UPU manipulator has *purely rotational* degrees-of-freedom. As expected, the 3-UPU wrist belongs to the class χ_{30}.

B. *The 3-UPU manipulator as purely translation device*

The 3-UPU mechanism discussed in example 1 can also be configured as a purely translation device as presented in Tsai and Joshi [11]. In this configuration, for the each R-R-P-R-R chain in a leg, the first and fifth rotary joint axis and the second and fourth rotary joint axis are constrained to be parallel. One leg of the 3-UPU manipulator with this geometry is shown in figure 2. Again, the direct kinematics of the
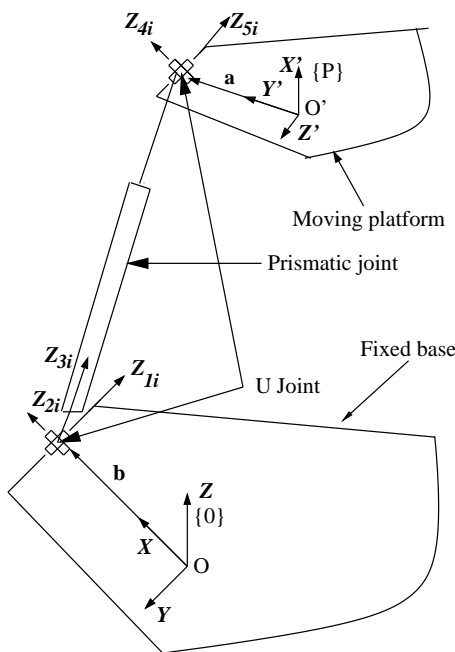

Fig. 2. A typical leg in the 3-UPU translational robot

3-UPU manipulator involve 12 passive variables and 3 actuated variables. To solve for the passive variables we form the constraint equations as follows: 1) 6 equations are obtained by equating the position vectors of the centre of the moving platform obtained by following each leg from the fixed origin, 2) three more equations are obtained by using the fact that the points on the top platform a triangle whose lengths are constants, and 3) three equations are obtained by using the fact that the first and last rotary joint axes in each of the UPU legs are parallel. The first 6 equations are similar to equations (24). The second set of three equations are given by:

$$(\boldsymbol{p}_i - \boldsymbol{p}_j) \cdot (\boldsymbol{p}_i - \boldsymbol{p}_j) - 3a^2 = 0, \quad i,j = 1,2,3, i \neq j \quad (26)$$

where $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ are the position vectors of the vertices of the top platform whose sides are $\sqrt{3}a$. Finally, the remain-

ing three equations are given by:

$$(\boldsymbol{Z}_{1i} \cdot \boldsymbol{Z}_{5i}) - 1 = 0 \quad (27)$$

Nine of these 12 equations are functions of 9 variables and the remaining three equations are functions only four passive joint variables. Following the theory developed in section II, for a given set of three actuated variables, the passive variables are to be solved numerically. Then the eigenproblem for $\boldsymbol{g}$ is solved and based on the eigenvalues, the mechanism can be characterised. In this example, it turns out the equivalent angular velocity Jacobian matrix $\boldsymbol{J}_\omega^{eq}$ is *trivially null*. This implies that the 3-UPU with first and fifth and second and fourth joint axes parallel is a *purely translational device*, i.e., it belongs to the class χ_{03}.

C. *The 3-RPS manipulator*

The 3-RPS parallel manipulator was introduced in 1988 by Lee and Shah [12] and has since been studied extensively by several researchers. The manipulator, as shown in figure 3, consists of a fixed and a moving platform. The fixed platform is connected to the moving platform by means of three legs, each of which has a rotary, a prismatic and a spherical joint. The prismatic joints are actuated and all other joints are passive. This gives rise to three-degrees-of-freedom for the moving platform.
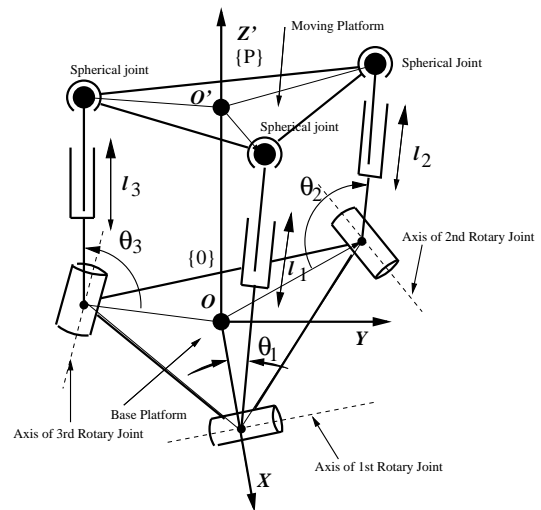

Fig. 3. The 3-RPS robot

For the sake of convenience, we have chosen the fixed base and moving top platforms as equilateral triangles and the rotary joint axes to lie in the plane of the fixed platform. The Denavit-Hartenberg parameters for each R-P-S chain can be easily derived and we can observe that there are three actuated (l_i, $i = 1,2,3$ and three passive variables ϕ_i, $i = 1,2,3$ which need to be solved for [15]. The three constraint equations to solve for the passive variables are

given by

$$
\begin{aligned}
(\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_1) - 3a^2 &= 0 \\
(\boldsymbol{p}_3 - \boldsymbol{p}_2) \cdot (\boldsymbol{p}_3 - \boldsymbol{p}_2) - 3a^2 &= 0 \\
(\boldsymbol{p}_1 - \boldsymbol{p}_3) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_3) - 3a^2 &= 0
\end{aligned}
$$

where $\boldsymbol{p}_i$, $i = 1, 2, 3$ are the position vectors of the three points on the moving platform where the legs are connected and $\sqrt{3}a$ is the distance between any two of them. These three equations are numerically solved for given values of actuated variables and as in the previous examples the eigenvalues of the matrix $\boldsymbol{g}$ are calculated. At two typical non-singular configuration and for $a = 0.5$ and $b = 1$, we get the following results:
• For active variables values 0.5, 1 and 2, the 3 passive variables are given by 0.4, 0.7535, 0.2402 and the eigenvalues of $\boldsymbol{g}$ are 19.6269, -0.250×10^{-15}, and 1.1675.
• For active variables values 1, 1, and 1, the 3 passive variables are given by 1.0472, 1.0472, 1.0472 and the eigenvalues of $\boldsymbol{g}$ are 3.5556, 0.14×10^{-38}, and 3.5556.
It is seen that two eigenvalues are non-zeros and one eigenvalue is zero. This implies that the moving platform has two rotational degrees-of-freedom and one pure translational degree-of-freedom. The eigenvectors of the equivalent rotational Jacobian matrix can be used to obtain the angular velocity vector. This is given as:

$$
{}^0\boldsymbol{\omega}_P = \begin{pmatrix} 0.1830 \\ 0.1830 \\ 0.5773 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} 0.7320 \\ 0.7320 \\ 0.5773 \end{pmatrix} \dot{\theta}_2 + \begin{pmatrix} 0 \\ 0 \\ 0.5773 \end{pmatrix} \dot{\theta}_3
$$

The 3-RPS manipulator was originally proposed as a parallel wrist in reference [12]. It can be seen from the above analysis that this is not possible – the manipulator belongs to the class $\boldsymbol{\chi}_{21}$ since there are only two rotational degrees-of-freedom.

D. The cylindrical manipulator

This parallel robot was proposed by Wang and Liu [13]. As shown in figure 4, it consists of a fixed base and a moving platform linked together by three legs. Two of the three legs have identical geometry, each consisting of a two-degrees-of-freedom universal joint and two rotary joints. The third leg consists of a planar four-bar parallelogram mechanism and three rotary joints. One of the rotary joints in each leg is actuated. In this example, the top moving platform and the fixed base are assumed to be isosceles triangles. All the revolute joint axes in all the legs are parallel to each other. The revolute joints attached to the base are actuated and all others are passive. The DH parameters of the first and the second legs are identical, whereas the DH parameters for the third leg are different because of the use of a parallel four-bar mechanism. The DH parameters of the first and second legs are given in table II below. The DH parameters for the third leg are given in table III. It may noted
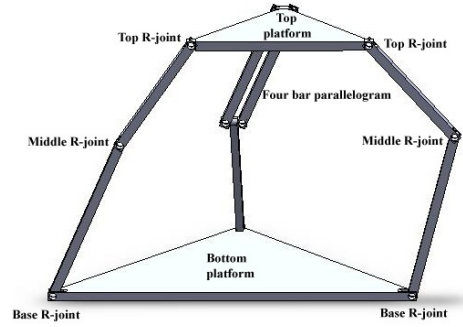

Fig. 4. A three-degree-of-freedom cylindrical robot

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_{1k}
2	0	l_1	0	ϕ_{1j}
3	0	l_2	0	ϕ_{2j}
4	$\pi/2$	0	0	ϕ_{3j}

TABLE II. DH parameters of first and second legs of the cylindrical manipulator

that the due to the four-bar linkage the link length a_{i-1} and offset d_i are not constant and depend on the joint rotation.

There are altogether twelve joint variables – three are actuated and remaining nine are passive variables. To solve for the nine passive variables, we need 9 constraint equations. Six of the constraint equations are obtained by equating the coordinates of the position vectors of a point on the top moving platform[2] and these equations are similar to equation (24). The remaining three equations are similar to the ones use for the 3-RPS manipulator and are given as:

$$
\begin{aligned}
(\boldsymbol{p}_1 - \boldsymbol{p}_2) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2) - 4a^2 &= 0 \\
(\boldsymbol{p}_2 - \boldsymbol{p}_3) \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_3) - 2a^2 &= 0 \\
(\boldsymbol{p}_3 - \boldsymbol{p}_1) \cdot (\boldsymbol{p}_3 - \boldsymbol{p}_1) - 2a^2 &= 0 \quad (28)
\end{aligned}
$$

The first 6 equations are function of nine passive variables and the last three equations are function of four passive variables. In the non-singular configuration the nine passive variables can be obtained for given values of actuated variables by numerically by solving the 9 non-linear equations. As explained in section II, the angular velocity and the equivalent Jacobian matrix, $\boldsymbol{J}_{\boldsymbol{\omega}}^{eq}$, are computed from the

[2]For this manipulator, it is more useful to choose the point on the moving platform on one side of the isosceles triangle and not the centroid as in other examples – this choice yields the simplest constraint equations.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	l_1	0	ϕ_{j1}
3	$\pi/2$	$l_2 \sin(\phi_{j2})$	$l_2 \cos(\phi_{j2})$	ϕ_{j3}

TABLE III. DH parameters for the third leg of the cylindrical robot

joint values and the eigenvalues of the matrix g are computed to understand the partitioning of the threedegrees-of-freedom. For $b = 1$, $a = 0.6$ and all link lengths equal to 1.2, we present two sample results.

• For active variables values -0.4000, -2.0944 and -1.9500, the 9 passive variables are given by 2.4454, 2.60483, 1.5742, 1.7938, -1.2789, 1.5674, 1.0667, 0.5939 and 1.0589 and the eigenvalues of g are 0, 6.159, and 0.

• For active variables values -0.4854, -0.4854, -0.4854, the 9 passive variables are given by 0.6103, 1.3783, -1.5968, 0.6274, -1.7756, 1.5448, -0.5690, 1.5781 and 0.8960 and the eigenvalues of g are 0, 12.335, 0.

It can be seen that the moving platform of the cylindrical robot has one rotational degree-of-freedom, and the other two degrees-of-freedom are translational. Therefore, the manipulator belongs to the class χ_{12}, which is complementary to the class of the 3-RPS manipulator.

E. Clavel's Delta manipulator

In the early 80's, Clavel presented the concept of using parallelograms to build a parallel robot. In this example, we use the approach developed in section II to partition the degrees-of-freedom of the Clavel's Delta manipulator, as described in Zsombor-Murray [14]. In the Delta manip-
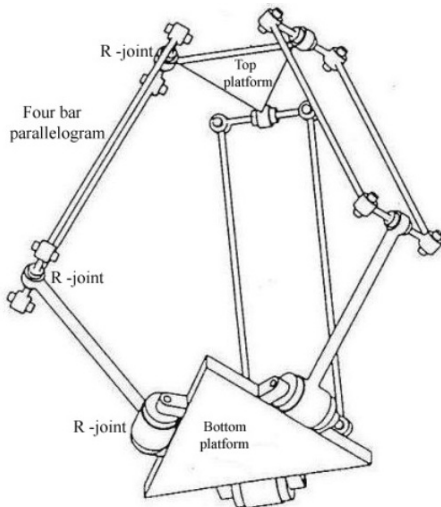

Fig. 5. Clavel's Delta robot (from [14])

ulator, as shown in figure 5, the fixed frame supports three actuated revolute (R) joints. These rotary joint axes form an equilateral planar triangle. The other end of each link, connected to the base revolute, supports another R joint whose axis is parallel to the one at the base. The moving platform also supports three R joints whose axes form another triangle which is similar to and maintains the same orientation as the one on the base platform. The base triangle R axes are held parallel to those on top platform because the moving platform and the intermediate revolute joints are

connected by a parallelogram four-bar linkage whose rotary joint axes are all perpendicular to the rotary joint axes to which it is connected. The Denavit-Hartenberg parameters for each leg of the Delta robot are given in table IV. In this table, the θ_{i1}s are the active joint variables and ϕs

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_{i1}
2	0	l_1	0	ϕ_{j1}
3	0	$l_2 \sin(\phi_{j2})$	$l_2 \cos(\phi_{j2})$	0
4	0	0	0	ϕ_{j3}

TABLE IV. DH parameters of one leg of the Delta robot

are the passive joint variables. Hence, there are 9 passive joint variables which need to be solved for. The required 9 constraint equations are obtained as follows: 6 equations are formulated by equating the point coordinates of the origin of the coordinate system fixed to top moving platform. These are similar to equations (24). The three remaining equations are obtained from the fact that the distance between the any two of three points, P_i, $i = 1, 2, 3$, on the top platform are constant and known. These three equations are given below:

$$(\boldsymbol{p}_i - \boldsymbol{p}_j) \cdot (\boldsymbol{p}_i - \boldsymbol{p}_j) - 3/4a^2 = 0, i \neq j, \ i, j = 1, 2, 3, \quad (29)$$

where P_i and P_j are the vertices of the top platform. Six of these 9 equations are functions of all the 9 passive variables and the remaining three equations are functions of only four passive joint variables. These equations are non-linear and need to be solved numerically. Once the passive variables are known the equivalent angular velocity Jacobian can be obtained and the eigenproblem for g can be solved. In this example, it turns out that the matrix $\boldsymbol{J}_\omega^{eq}$ is trivially null. Hence, the motion of the top platform is purely translational, as reported by Zsombor-Murray [14]. Therefore the manipulator belongs to the class χ_{03}, i.e., the complementary class of the 3-UPU parallel wrist.

IV. Gain singularity and gained motion

The results of the study of partitioning of degrees-of-freedom shown in section III are for the manipulators at non-singular configurations. As discussed in section II-D, a parallel manipulators can gain one or more degrees-of-freedom where $\det(\boldsymbol{J}_{\eta\phi})$ vanishes. We now use the approach developed in section II to determine the nature of the gained degree-of-freedom, namely, whether it is translational or rotational. We present results for each of the five three-degrees-of-freedom parallel manipulators discussed in section III.

The 3-UPU Wrist
In order to find a gain-singular configuration, we need to solve for the equation $\det(\boldsymbol{J}_{\eta\phi}) = 0$ in addition to the 9 constraint equations shown in the first example of section III. The 10 equations are solved numerically and it

is found that the values 0.7567, 1.5713, 2.238 for actuated variables and 0.0673, 1.1828, 1.1828, 0.2737, 0.06671, 0.6671, -0.7969, -1.1828, -1.1828 for passive variables satisfy the 10 equations. Two eigenvalues of $J_{\eta\phi}^T J_{\eta\phi}$ is found to be zero. In the non-singular configuration, the 3-UPU wrist has three rotational degrees-of-freedom and one possible inference is that the gained degrees-of-freedom are *translational*.

3-UPU Translational robot
For the 3-UPU translational robot, it was found that the actuated joint values of $(1, 1, 1)$ and the set -0.0006, -1.5708, 1.5708, -3.1410, -0.0006, -1.5708, 1.5708, -3.1410, -3.1422, 1.5708, -1.5708, 6.2838 for the passive variables satisfy all the constraints and condition for gain singularity. One of the eigenvalue of $J_{\eta\phi}^T J_{\eta\phi}$ is zero. Since the non-singular 3-UPU manipulator in this case is of class χ_{03}, we can conclude that the gained degree-of-freedom is *rotary*.

Clavel's Delta robot
A gain-singular configuration of the Delta robot is given by the following set of joint variables – -3.2087, 0.2094, 0.2992 for actuated variables and 2.2366, 1.6119, 1.7721, 0.7606, -1.9890, 1.2078, 3.8179, 1.1072, -1.9394 for passive variables. One eigenvalue of $J_{\eta\phi}^T J_{\eta\phi}$ is zero implying that the manipulator has gained one degree-of-freedom. The eigenvalues of $J_{\omega\phi}^T J_{\omega\phi}$ are found to be 0, 0 and 1. The Delta robot in a non-singular configuration belongs to the class χ_{03}. Hence, we can infer that the gained degree-of-freedom is *rotational*.

The 3-RPS manipulator
For the 3-RPS parallel manipulator, the gain singularity occurs at the set of values given by 0.5300, 0.4800, 2.0015 for actuated joints and 0.1304, 0.1455, 0.0553 for the passive joints. One eigenvalue of $J_{\eta\phi}^T J_{\eta\phi}$ is found to be zero for this set of joint variables. One eigenvalue of $J_{\omega\phi}^T J_{\omega\phi}$ is found to be very close to zero (0.8×10^{-7}) and two are non-zero (0.6719, 7.342) at this configuration. The non-singular 3-RPS belongs to the class χ_{21} implying that it has two rotational degrees-of-freedom and one translational degree-of-freedom. This partitioning of degrees-of-freedom are same in the singular configuration and hence we can infer that the gained degree-of-freedom is *translational*.

The Cylindrical robot
In the cylindrical robot, the constraints and condition of gain singularity are satisfied by actuated variable values of -1.5430, -1.9800, -2.8526 and 2.4626, -1.1807, 1.5710, 3.1752, -3.2030, 1.5708, 1.4856, 1.9090, 3.8477 for passive variables. One eigenvalue of $J_{\eta\phi}^T J_{\eta\phi}$ was found to be zero at this configuration implying that there is a gain of one degree of freedom. The eigenvalues of $J_{\omega\phi}^T J_{\omega\phi}$ are found to be 0.9316, 2.1214, and 0, i.e., one is zero. The non-singular cylindrical robot belongs to the class χ_{12} and hence the gained degree-of-freedom is *rotational*.

V. Conclusions

In this paper, we have presented a formulation for partitioning the instantaneous degree-of-freedom of a rigid body moving in space. The general theory involves solution of an eigenvalue problem which in turn involves solution of at most a cubic equation. Hence, the approach presented in this paper is semi-analytical. The developed theory has been applied to five different parallel manipulators each having three degrees of freedom. In each case, the developed approach allows us to determine whether the available degrees of freedom of the output link are translational or rotational. The developed approach also allows us to determine the kind of gained motion at a gain singularity. For future work, we plan to extend the developed approach to four and five degrees of freedom parallel manipulators.

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