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Instantaneous Properties of Multi-Degrees-of-Freedom Motions—Point Trajectories

A general framework is presented for the study of the properties of trajectories generated by points embedded in rigid bodies undergoing multi-degrees-of-freedom motions. Quantities are developed to characterize point trajectories generated by different mechanisms and to distinguish between different positions along the same trajectory. Point trajectories are classified into three types according to whether the number of degrees of freedom is less than, equal to, or greater than the dimension of the space in which the motion takes place. Local and global motion properties are developed for each of these three cases. A new way of using the redundant degrees of freedom in (redundant) mechanisms is presented. These analysis techniques are applied to two- and three-degrees-of-freedom mechanisms containing rotary and prismatic joints.

1 Introduction

Over the past 20 years, there has been rapidly growing interest in mechanical manipulators, mechanical hands, walking machines and many other so-called robotic devices and smart products. The motion of the links of such machines are governed by more than one independent variable, in contrast to conventional machines which have only one controllable degree of freedom.

One way to study the motion of links, which are assumed to be rigid bodies, of a multi-degree-of-freedom mechanism, is to consider the motion of points embedded in the moving links. For a one-degree-of-freedom motion, a point's path in three-space is a twisted curve. For a two-degrees-of-freedom motion a point's path is a surface. Properties of curves and surfaces are well known [1-5]. However, little is known about the properties of the trajectories of points embedded in a moving rigid body undergoing more than two-degrees-of-freedom motion. Similarly, the effects of the structural parameters of the multi-degrees-of-freedom mechanism (e.g., the dimensions of the links) on the properties of the trajectories traced out by the points of the moving body have not been studied.

The majority of the previous works in connection with global properties of point trajectories fall under two categories: (1) Global theorems on curves and surfaces [5, 6]; (2) Workspaces of mechanical manipulators [7-9]. In the second category, the workspace boundary has been of primary interest.

In this paper, a general framework for the study of the properties of point trajectories is presented. Point trajectories are classified into three classes according to whether the number of degrees of freedom is less than, equal to, or greater than the dimension of the space of the point trajectory. First- and second-order local properties and a few global properties of the point trajectories are developed. Quantities are developed to characterize point trajectories generated by different mechanisms and to distinguish between different positions along the same trajectory. These quantities are obtained by using concepts from differential geometry. Two of these quantities are based on a transmission efficiency defined in terms of "areas" and "volumes." We present a novel way to utilize the extra freedoms in a system having motion redundancy. The analysis techniques and the new results are illustrated with examples of two- and three-degrees-of-freedom open-loop chains containing rotary and sliding joints.

Contributed by the Mechanisms Committee and presented at the Design Engineering Technical Conference, Columbus, Ohio, October 5-8, 1986, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received at ASME Headquarters, July 1, 1986. Paper No. 86-DET-19.

2.1 Mathematical Formulation

In the most general terms, the path generated by a point moving under an m-degrees-of-freedom motion can be represented by a set of equations giving the coordinates of the point, in a fixed reference frame, as functions of m independent motion parameters. In the case of manipulators or multi-degrees-of-freedom mechanisms the m independent parameters are typically the rotations or translations at the joints. The actual equations depend on the mechanism's structural parameters, e.g., link lengths, offsets and twist angles. We can symbolically write these equations as

$$\Psi:(\theta_1, \ldots, \theta_m) \rightarrow (x,y,z) \tag{1}$$

Here, $(\theta_1, \ldots, \theta_m)$ are the m independent motion parameters, (x, y, z) are the coordinates of a point embedded in the moving rigid body (measured in a reference three-space, $\mathbf{R}^3$) and Ψ may be thought of as a mapping which takes points in the motion parameter space to points in the space of the motion. The functions Ψ depends on the point chosen in the moving rigid body and the mechanism's structure. As mentioned before, for a one-degree-of-freedom motion ($m = 1$), the point's path in three-space is a twisted curve and, for a two-degrees-of-freedom motion ($m = 2$), a point's path is a surface. Under a three-degrees-of-freedom motion ($m = 3$), a point's path is a "solid" region (with a boundary) in a three-space. For four- or higher-degrees-of-freedom motion ($m \geq 4$), the point's path is still a solid (with a boundary) but now we may have infinitely many values of the independent motion parameters for a given (x, y, z). When this situation arises, we have a so-called redundant motion.

2.2 Local Properties

In general, the equations represented by (1) are highly nonlinear, even for simple mechanisms, and hence it is very difficult to determine properties of the point trajectory which depend on the entire motion (so-called global properties). It is much easier to obtain the differential properties of the motion, that are valid in the neighborhood of the position under consideration. These are the so-called first-, second-, and other higher-order properties at a position, and are obtained by finding the first, second and higher derivatives at that position. Although properties obtained from these derivatives are local, they often give us valuable insight into the global properties of the point trajectory.

If the motion is a function of one independent variable, we can talk of the tangent to the curve, the curvature, and the torsion. For a curve the slope of the straight line (tangent to the curve) is a first-order property, the curvature is a second-order property, and the torsion, which describes the twisting of the curve, is a third-order property. These properties totally describe the curve at a point to, respectively, the first, second and third orders. These properties are *geometric* in nature. If we take time as the independent variable, the first-, second-, and third-order properties are the velocity, acceleration and jerk of a point's motion along the path. These time-based quantities are our primary concern in this paper. They are referred to as, respectively, the first-, second-, and third-order *motion* properties.

The first-order properties of a point trajectory due to multi-degrees-of-freedom motions are obtained from the first partial derivatives of the function Ψ with respect to θ_i, $i = 1, \ldots, m$. The matrix of the first partial derivatives, whose columns are $\partial\boldsymbol{\Psi}/\partial\theta_i$, is called the Jacobian matrix, $J(\boldsymbol{\Psi})$, [5]. Denoting the dimension of the space in which the motion is taking place by n, we can see that $J(\boldsymbol{\Psi})$ is a $n \times m$ matrix. The velocity, $\mathbf{v}$, of a point (x, y, z) located in the fixed space by the position vector $\mathbf{p}$, is obtained by evaluating the elements of $J(\boldsymbol{\Psi})$ at $\mathbf{p}$; denoting this as $J(\boldsymbol{\Psi})_\mathbf{p}$ we have

$$\mathbf{v} = J(\boldsymbol{\Psi})_\mathbf{p}\dot{\boldsymbol{\Theta}} \tag{2}$$

where, $\dot{\boldsymbol{\Theta}}$ is an m-dimensional vector of the rate of change of the parameters (i.e., $\dot{\boldsymbol{\Theta}} = (\dot{\theta}_1, \ldots, \dot{\theta}_m)^T$). By using a different $\dot{\boldsymbol{\Theta}}$, we obtain a different velocity for the moving point. By varying $\dot{\boldsymbol{\Theta}}$, we can get a *distribution of velocities* of the moving point. This first-order property is of primary importance and will be developed in detail for point trajectories due to multi-degrees-of-freedom motions.

If there is no constraint on the $\dot{\theta}_i$'s, the magnitude of the velocity vector can be arbitrary and not much can be said about the magnitude of $\mathbf{v}$. It is much more instructive to use a normalizing constraint on the $\dot{\theta}_i$'s. There are two natural ways to put a constraint on $\dot{\boldsymbol{\Theta}}$:

(1) One could use a linear equation relating the $\dot{\theta}_i$'s. This equation can be of the form, $\dot{\theta}_k$ equals a constant or $\dot{\theta}_k$ equals a linear combination of all (or some) of the $\dot{\theta}_i$'s. However, the use of such relations effectively reduces the number of independent parameters and hence is not of much use in studying m-degrees-of-freedom motions.

(2) One could bound the magnitude of $\dot{\boldsymbol{\Theta}}$ with a quadratic relation of the form $\dot{\boldsymbol{\Theta}}^T\dot{\boldsymbol{\Theta}} = k^2$. When $k = 1$, we call the motion *unit speed*. By varying k it is possible to obtain all possible velocities at the point under consideration. We will use $\dot{\boldsymbol{\Theta}}^T\dot{\boldsymbol{\Theta}} = k^2$ as the normalization constraint on $\dot{\boldsymbol{\Theta}}$.

The second-order motion property considered will be the rate of change of velocity, i.e., the acceleration of the moving point at $\mathbf{p}$. In the cases where ($m < n$), we will have two components of acceleration: the normal and the tangential component. For point trajectories where the number of motion parameters is equal to or greater than the dimension of the space ($m \geq n$), there is no notion of a normal component. In the cases when m is equal to n, instead of the normal component, we consider the rate of change of certain area or volume (both defined later on) as a second-order property of interest. Where the number of parameters exceed the dimension of the space ($m > n$), we have free choices which can be used to modify the first- and second-order motion properties.

2.3 Classification of Point Trajectories

In general, point trajectories can be classified into three classes. First, we have the cases where the number of degrees of freedom, m, is less than the number of coordinates, n, required to specify a point. Second, we have cases where m is equal to n and, finally, we have cases where m is greater than n. When a point's trajectory is a curve ($m = 1$, $n = 2$ or 3) or a surfaces ($m = 2$, $n = 3$), we have the first class. Solid regions with boundaries come under the second class and redundant motions belong to the third category. We do not discuss curves since they have been widely studied, the reader is referred instead to [1-5]. We will however use the case $m = 2$ and $n = 3$ (i.e., the trajectory is a surface) to develop the concept of distribution of velocities.

3.1 Point Trajectories With $m < n$

In this section, we consider point trajectories with $m = 2$ and $n = 3$ (surfaces in three-dimensional space $\mathbf{R}^3$). The set of equations $x = \psi_1(\theta_1, \theta_2)$, $y = \psi_2(\theta_1, \theta_2)$, $z = \psi_3(\theta_1, \theta_2)$ defines a surface generated by a point (x, y, z) in terms of the motion parameters θ_1 and θ_2. We will represent a surface as a continuous, differentiable function $\boldsymbol{\Psi}: U \rightarrow \mathbf{R}^3$ where $U \subseteq \mathbf{R}^2$, is open with coordinates θ_1 and θ_2, and $(\partial\boldsymbol{\Psi}/\partial\theta_1) \times (\partial\boldsymbol{\Psi}/\partial\theta_2) \neq \mathbf{0}$. If there exist positions on the surface where this cross product of the partial derivatives is zero or not defined uniquely, we have a singularity and we do not at that position have a two-degrees-of-freedom motion. For convenience, we define symbols $\boldsymbol{\Psi}_i$ and $\boldsymbol{\Psi}_{ij}$ for the partial derivatives of the function $\boldsymbol{\Psi}$ with respect to the parameters θ_i; with $i, j = 1, 2$, we have

$$\boldsymbol{\Psi}_i = \partial\boldsymbol{\Psi}/\partial\theta_i$$
$$\boldsymbol{\Psi}_{ij} = \frac{\partial^2\boldsymbol{\Psi}}{\partial\theta_i\partial\theta_j} \tag{3}$$

3.2 First-Order Properties

Using the notation $\theta_{i,0}$ to denote a specific value of θ_i, it follows that the tangent plane to a surface $\boldsymbol{\Psi}: U \rightarrow \mathbf{R}^3$ at a point $\mathbf{p} = \boldsymbol{\Psi}(\theta_{1,0}, \theta_{2,0})$ is the plane through $\mathbf{p}$ perpendicular to $\boldsymbol{\Psi}_1(\theta_{1,0}, \theta_{2,0}) \times \boldsymbol{\Psi}_2(\theta_{1,0}, \theta_{2,0})$. The unit normal is given by

$$\mathbf{n} = \frac{(\boldsymbol{\Psi}_1 \times \boldsymbol{\Psi}_2)}{|\boldsymbol{\Psi}_1 \times \boldsymbol{\Psi}_2|} \tag{4}$$

where the expression on the right-hand side is evaluated at $(\theta_{1,0}, \theta_{2,0})$. The set $(\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \mathbf{n})$ at $\mathbf{p}$ is linearly independent, though in general not orthogonal, and serves as a local coordinate basis for the trajectory surface in $\mathbf{R}^3$. The tangent plane at $\mathbf{p}$ is independent of parametrization [5]. The Jacobian matrix $J(\boldsymbol{\Psi})_\mathbf{p} = (\boldsymbol{\Psi}_1(\theta_{1,0}, \theta_{2,0})\ \boldsymbol{\Psi}_2(\theta_{1,0}, \theta_{2,0}))$ is of rank two.

Any vector in the tangent plane can be expressed as a linear combination of Ψ_1 and Ψ_2. In particular, if the parameters θ_1 and θ_2 are given as functions of time t, the velocity of the point $\mathbf{p}$ moving on a curve Ψ $(\theta_1(t), \theta_2(t))$ can be expressed as

$$\mathbf{v} = \Psi_1\dot{\theta}_1 + \Psi_2\dot{\theta}_2 \tag{5}$$

From equations (4) and (5) $\mathbf{v} \cdot \mathbf{n} = 0$, i.e., all point-velocities lie in the tangent plane. Also, as $\dot{\theta}_1$ and $\dot{\theta}_2$ are varied without any constraint, all possible directions and magnitudes of $\mathbf{v}$ are obtained. To get a better understanding of the distribution of $\mathbf{v}$, we look at the magnitude of $\mathbf{v}$. The dot product of the velocity vector with itself may be written as

$$\mathbf{v}\cdot\mathbf{v} = g_{11}\dot{\theta}_1{}^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2{}^2 \tag{6}$$

where $g_{ij} = \Psi_i \cdot \Psi_j$, $i, j = 1, 2$. The matrix $[g]$, with elements g_{ij} is symmetric and positive-definite [10]. In the language of differential geometry of surfaces, the elements g_{ij}'s determine the *first fundamental form* of the surface. The dot product also defines a *metric*[1] in the tangent plane. We make the following observations from the definition of the g_{ij}'s and equation (6):

(1) The matrix $[g]$ is the same as J^TJ, where, for convenience, J is used to denote the Jacobian matrix, $J(\Psi)_{\mathbf{p}}$.

(2) The elements g_{11}, g_{12}, and g_{22} are in general functions of θ_1 and θ_2, and hence depend on the complete two-degrees-of-freedom motion.

(3) For any given value of g_{11}, g_{12}, and g_{22}, equation (6) describes a quadratic surface in the $(\dot{\theta}_1, \dot{\theta}_2, |\mathbf{v}|)$ space. If $\mathbf{v}^2$ is constant, equation (6) describes an ellipse in the $(\dot{\theta}_1, \dot{\theta}_2)$ space. If $\dot{\theta}_1$ or $\dot{\theta}_2$ is constant, equation (6) describes a hyperbola.

(4) Since $[g]$ is positive-definite, the minimum value of $|\mathbf{v}|$ is zero only when $\dot{\theta}_1$ and $\dot{\theta}_2$ are both zero. This case is ruled out as it implies no motion. The maximum value of $|\mathbf{v}|$ goes to infinity if $\dot{\theta}_1$ or $\dot{\theta}_2$ goes to infinity. As mentioned before, it is instructive to use the normalizing condition $\dot{\theta}_1^2 + \dot{\theta}_1^2 = 1$.

The maximum and minimum $\mathbf{v}^2$, subject to the unit speed condition $(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1)$, are obtained by solving $\partial \mathbf{v}^{*2}/\partial\dot{\theta}_i = 0$, $i = 1, 2$, where

$$\mathbf{v}^*\cdot\mathbf{v}^* = g_{11}\dot{\theta}_1{}^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2{}^2 - \lambda(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 - 1) \tag{7}$$

This reduces to solving the eigenvalue problem

$$[g]\dot{\mathbf{\Theta}} - \lambda\dot{\mathbf{\Theta}} = 0 \tag{8}$$

Since the matrix $[g]$ is symmetric and positive-definite, the two eigenvalues are real and positive. If λ_1 and λ_2 are the eigenvalues, we can write

$$\lambda_{1,2} = (1/2)\{(g_{11} + g_{22}) \pm [(g_{11} + g_{22})^2 - 4(g_{11}g_{22} - g_{12}^2)]^{1/2}\} \tag{9}$$

Assuming $\lambda_1 > \lambda_2$, the maximum and minimum values of $|\mathbf{v}|$ are

$$\begin{aligned} |\mathbf{v}|_{\max} &= \lambda_1^{1/2} \\ |\mathbf{v}|_{\min} &= \lambda_2^{1/2} \end{aligned} \tag{10}$$

and these occur at each point when the ratio $\dot{\theta}_2/\dot{\theta}_1$ are given by

$$\begin{aligned} (\delta)_1 &= (1/2)\tan^{-1}[2g_{12}/(g_{11} - g_{22})] \\ (\delta)_2 &= (1/2)\tan^{-1}[2g_{12}/(g_{11} - g_{22})] + \pi/2 \end{aligned} \tag{11}$$

where $\tan\delta = (\dot{\theta}_2/\dot{\theta}_1)$.

Equations (11) when used with $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$ give unique values of $\dot{\theta}_1$ and $\dot{\theta}_2$. When these values of $\dot{\theta}_1$, $\dot{\theta}_2$ are substituted in (5) we get the maximum and minimum $|\mathbf{v}|$ and the directions in the tangent plane along which they occur. It can be easily shown from (5) that, at a point $\mathbf{p}$, the direction of $\mathbf{v}$ depends only on the ratio of $\dot{\theta}_2$ and $\dot{\theta}_1$, and hence the directions of maximum and minimum $\mathbf{v}$ will not change if we use a nonunit speed condition, $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$ $(k \neq 1)$. However, the magnitude of the maximum and minimum $\mathbf{v}$ will be scaled by k.

Next, we show that at each point $\mathbf{p}$, as we vary $\dot{\mathbf{\Theta}}$, the tip of the velocity vector lies on an ellipse in the tangent plane. Equation (5) can be written as

$$\mathbf{v} = J\dot{\mathbf{\Theta}} \tag{12}$$

or

$$J^T\mathbf{v} = [g]\dot{\mathbf{\Theta}} \tag{13}$$

$[g]$ is nonsingular and hence we can take the inverse to obtain $\dot{\mathbf{\Theta}}$. From which it follows that

$$\mathbf{v}^TJ([g]^{-1})^T([g]^{-1})J^T\mathbf{v} = \dot{\mathbf{\Theta}}^T\dot{\mathbf{\Theta}} \tag{14}$$

As $[g]$ and $[g]^{-1}$ are symmetric we get

$$\dot{\mathbf{\Theta}}^T\dot{\mathbf{\Theta}} = \mathbf{v}^T(J[g]^{-1})(J[g]^{-1})^T\mathbf{v} \tag{15}$$

The quantity $(J[g]^{-1})(J[g]^{-1})^T$ is a symmetric 3×3 matrix of rank 2 (as both J and $[g]^{-1}$ are of rank 2). Hence, if the left-hand side, $\dot{\mathbf{\Theta}}^T\dot{\mathbf{\Theta}}$ is unity, the tip of the velocity vector describes an ellipse. If $\dot{\mathbf{\Theta}}^T\dot{\mathbf{\Theta}} = k^2$, the tip of the velocity vector *still* lies on an ellipse but the size of the ellipse is scaled by k.

When $\dot{\mathbf{\Theta}}^T\dot{\mathbf{\Theta}}$ is *unity, the area or size of the ellipse, denoted* by A_e, is given by

$$A_e = \pi(\lambda_1\lambda_2)^{1/2} = \pi(\det[g])^{1/2} \tag{16}$$

The quantity A_e/π or $(\det[g])^{1/2}$ is a measure of mean $\mathbf{v}^2$, where *the* (*mean* $\mathbf{v}^2)^{1/2}$ *is the radius of the circle with the same* area as the ellipse. (It is also the geometric mean of the maximum and minimum $|\mathbf{v}|$.) Hence as $\det[g]$ increases the mean $\mathbf{v}^2$ also increases.

In the context of manipulators the quantity $\dot{\theta}_1^2 + \dot{\theta}_2^2$ can be seen as an input effort ($\dot{\theta}_1$, $\dot{\theta}_2$ are the joint rates) and $\mathbf{v}^2$ as the output ($\mathbf{v}$ is the velocity of a point on the end-effector of the manipulator). If $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$ we can write

$$\begin{aligned} |\mathbf{v}|_{\max} &= k\lambda_1^{1/2} \\ |\mathbf{v}|_{\min} &= k\lambda_2^{1/2} \end{aligned} \tag{17}$$

Denoting the square of the geometric mean of $|\mathbf{v}|_{\max}$ and $|\mathbf{v}|_{\min}$ by $\bar{\mathbf{v}}^2$, we get

$$\bar{\mathbf{v}}^2/k^2 = (\lambda_1\lambda_2)^{1/2} = (\det[g])^{1/2} \tag{18}$$

Hence $(\det[g])^{1/2}$ can be thought of as a measure of "velocity transmission" at the point in the point trajectory under consideration.

In conclusion, we have characterized the first-order property of the motion by the size and shape of the velocity ellipse. We have shown that there are directions (in the tangent plane) at $\mathbf{p}$ along which the point $\mathbf{p}$ can move with maximum and minimum velocity. We have shown that for nonunit speed motions $(\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2)$, at $\mathbf{p}$, the shape of the ellipse is the same as for the unit speed motion but the size of the ellipse and the maximum and minimum velocities are simply scaled by k. We have also shown that the area of the ellipse at $\mathbf{p}$ is a measure of the "velocity transmission" and the magnitude of the mean velocity vector at $\mathbf{p}$ is larger if the area of the velocity ellipse is larger.

At this stage we make the following remarks:

(1) In reference [11], Mason deals with the application of forces using manipulators. Our results combined with the duality of forces and velocities [4], immediately yield some of his result. For example, the principal directions of force application (i.e., the directions along which we can apply maximum and minimum force) are orthogonal to the directions of maximum and minimum velocities. In reference [12], Salisbury and Craig use the condition number of J^T to study forces in mechanical hands. The condition number of J is the ratio of its maximum and minimum eigenvalues, and would be a measure of the shape of the velocity ellipse in our analysis.

(2) Yoshikawa [13] has introduced the concept of

[1]A metric essentially defines distance and, in this case, angle in the tangent plane, for details see [5].

manipulability measure for redundant motions. The difference between our measure of velocity transmission and the manipulability measure of Yoshikawa is that we use $[\det(J^TJ)]^{1/2}$ rather than his $[\det(JJ^T)]^{1/2}$. In the particular case of $m = 2$ and $n = 3$, Yoshikawa's measure, $[\det(JJ^T)]^{1/2}$ is always zero and is not of much use.

(3) Asada [17] and Thomas, et al. [18], have studied the inertia ellipses for some two-degrees-of-freedom chains and developed dynamic criteria to synthesize linkage and actuator parameters. Our approach can be used in conjunction with theirs for design, thereby incorporating both kinematic and dynamic criteria.

Next, we look at some of the global properties of the point trajectory. These are obtained by examining the elements g_{11}, g_{12}, g_{22}, and the eigenvalues of the matrix $[g]$.

$g_{12} = 0$ is a curve in the (θ_1, θ_2) plane. At each point of this curve, the basis $(\mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{n})$ is orthogonal and the major and minor axes of the ellipse are along $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$; which of the two is the major axis depends on whether g_{11} is greater or less than g_{22}. For points on $g_{12} = 0$, the expressions for $\mathbf{v}^2$ does not contain any $\dot\theta_1\dot\theta_2$ term, and we say that the effects of $\dot\theta_1$ and $\dot\theta_2$ on the velocitiy are independent.

If the two eigenvalues of the $[g]$ matrix are equal, the ellipse reduces to a circle and it is equally "easy" to move in any direction. The points where the eigenvalues are equal are found by setting the quantity $(g_{11} - g_{22})^2 + 4g_{12}^2$ equal to zero, i.e., by setting $g_{11} = g_{22}$ and $g_{12} = 0$. For manipulators with revolute joints, these conditions are generally not satisfied since g_{11}, g_{12}, and g_{22} are only functions of θ_2. For some particular values of link lengths and θ_2 they may be satisfied, and the equal eigenvalues then lie on two circles in the workspace. The points where the eigenvalues are equal have been called isotropic points [12].

In general, the area of the velocity ellipse is a function of θ_1 and θ_2. The maximum value of $\det[g]$ is obtained by solving

$$\frac{\partial}{\partial\theta_i}\det[g] = 0 \quad i = 1,2 \tag{19}$$

In the case of a manipulator with revolute joints, equation (19) is always satisfied for $i=1$ and for two values of θ_2. Hence *equation (19) is satisfied along two circles in the workspace of* the manipulator.

If $\det[g]$ is zero, the degrees of freedom is no longer two. At such points, the velocity ellipse degenerates into a straight line or a point, and all possible velocity vectors at this point are parallel to a single straight line or are zero. The first situation happens at the boundary or at a singularity. The second situation is possible only if $\dot\theta_1$ and $\dot\theta_2$ are zero, and this is ruled out as it implies no motion. The equation, $\det[g] = 0$, can be used to find the boundary of a two-degrees-of-freedom point trajectory.

3.3 Second-Order Properties

To find the second-order *motion* properties of the point trajectory, we consider the acceleration of $\mathbf{p}$. The acceleration is obtained by differentiating the velocity equation, (5). We get

$$\mathbf{a} = \mathbf{\Psi}_1\ddot\theta_1 + \mathbf{\Psi}_2\ddot\theta_2 + \mathbf{\Psi}_{11}\dot\theta_1^2 + 2\mathbf{\Psi}_{12}\dot\theta_1\dot\theta_2 + \mathbf{\Psi}_{22}\dot\theta_2^2 \tag{20}$$

The introduction of the $\ddot\theta_1$ and $\ddot\theta_2$ terms means that the acceleration vector, at each point, depends upon four motion variables $(\dot\theta_1, \dot\theta_2, \ddot\theta_1, \ddot\theta_2)$. Hence, unlike the velocity vector, there is no single simple way to describe the distribution of the acceleration vector. To deal with this problem, we separate the acceleration vector into components. The normal component, a_n, is obtained by taking the dot product of $\mathbf{n}$ and $\mathbf{a}$, and is given as

$$a_n = \sum_{i,j=1}^{2} L_{ij}\dot\theta_i\dot\theta_j \tag{21}$$

where the L_{ij}'s are the dot products $\mathbf{\Psi}_{ij}\cdot\mathbf{n}$, they are called the coefficients of the *second fundamental form* [5]. Two tangential components a_{t_1} and a_{t_2} are obtained by taking the dot product with vectors $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$, respectively:

$$a_{t_k} = \ddot\theta_k + \sum_{i,j=1}^{2} \Gamma_{ij}^k\dot\theta_i\dot\theta_j \quad k = 1,2 \tag{22}$$

The Γ_{ij}^k's are known as the *Christoffel symbols* [5].

The maximum and minimum values of a_n, for the case $\dot\theta_1{}^2 + \dot\theta_2{}^2 = 1$, are the eigenvalues of the symmetric matrix $[L]$ whose elements are the L_{ij}'s. (Unlike the matrix $[g]$, $[L]$ is not necessarily positive-definite, and the eigenvalues could be positive, negative or zero.) The unit vectors corresponding to the eigenvectors of $[L]$ lie in the tangent plane, and give the direction of maximum and minimum a_n. If the eigenvalues of $[L]$ are equal, the a_n at that point are equal in all directions.

From the differential geometry of surfaces, it is known that a curve $\mathbf{\Psi}(\theta_1(s), \theta_2(s))$, where s is the arc length, is a geodesic if the geodesic curvature, K_g, is zero [5]. The condition $K_g = 0$, gives two differential equations in $\theta_1(s)$ and $\theta_2(s)$. When the parametrization is not by arc length, but by time, t, we again obtain two differential equations when $K_g = 0$. These are

$$\ddot\theta_k + \sum_{i,j=1}^{2} \Gamma_{ij}^k\dot\theta_i\dot\theta_j = -\dot\theta_k\dot s^2 d^2t/ds^2 \quad k = 1,2 \tag{23}$$

Using the expressions for the tangential components of the acceleration, equation (22), we obtain that $K_g = 0$ only if $a_{t_1}/\dot\theta_1 = a_{t_2}/\dot\theta_2$. A specially interesting case is found by setting a_{t_1} and a_{t_2} equal to zero, we then get two coupled nonlinear differential equations in $\theta_1(t)$ and $\theta_2(t)$. A solution to these equations gives a curve C in the (θ_1, θ_2) space. If the point $\mathbf{p}$ under consideration is moving so that $\theta_1(t)$ and $\theta_2(t)$ are on the curve C, then the point has zero tangential acceleration, i.e., the velocity of the point has a constant magnitude and its component in the tangent plane remains parallel to itself. The locus of the moving point is a geodesic on the surface. In general the two equations obtained by setting a_{t_1} and a_{t_2} equal to zero are highly nonlinear and cannot be solved in closed form; however they can be integrated numerically.

For the point trajectory due to a two-degrees-of-freedom motion of a point, the time-dependent second-order motion properties are determined by $[L]$, $[g]$, the Christoffel symbols (which in turn can be expressed as functions of the coefficients g_{ij}'s and their derivatives with respect to θ_1 and θ_2 [5]) and the first and second rates of change of θ_1 and θ_2 with respect to time. The elements g_{ij}'s, L_{ij}'s and their partial derivatives are functions of θ_1, θ_2 and the mechanism's structural parameters. At different points in the trajectory these coefficients will be different. Also, at the same point in $\mathbf{R}^3$, the elements will be different if different mechanisms are used to generate the trajectories.

4.1 Point Trajectories With $n = m$

In this section, we consider the case where the number of coordinates required to specify a point is the same as the number of degrees of freedom of the motion. We have two cases: two-degrees-of-freedom motion of a point in a plane and three-degrees-of-freedom motion in space. Examples of such cases are the point trajectories generated by planar and spatial two- and three-degrees-of-freedom manipulators and mechanisms. The main difference in this section (from the previous section) is that there is no such thing as the normal space or the normal component of the acceleration vector. In order to study the second-order properties we have introduced scaler measures of "effectiveness" which depend on the possible locus of the tip of the velocity vector at each position. It is shown that these measures can be formulated in terms of areas for planar motions and volumes for motions in space. The rate of change of these scalar quantities give us the second-order characteristics of the motions.

4.2 Point Trajectories of Two-Degrees-of-Freedom Motion in a Plane

In a plane, a point is specified by two coordinates (x, y), and, in general any position in the plane may be reached by a two-degrees-of-freedom motion. However, when the motion is generated by a mechanism, there are physical constraints due to finite lengths. In such cases the trajectory can only fill a region of the plane, and this region has one or more boundary curves.

The first-order property of interest is the velocity distribution. At points inside the region (not at the boundary), $\det[g] > 0$ and the analysis of Section 3.2 can be applied to show that the velocity distribution can be pictured as an ellipse (for the case $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$) with the directions of maximum and minimum velocities along the major and minor axis of the ellipse. At the boundary, the velocity distribution is no longer an ellipse. All the velocities lie on a straight line.

To analyze the second-order properties, we introduce a scalar quantity which is proportional to the area of the velocity ellipse. The magnitude $|\boldsymbol{\Psi}_1 \times \boldsymbol{\Psi}_2|$ or $\det[g]$ is a differential invariant denoted by A. A is a scalar quantity which depends on θ_1, θ_2 and the function $\boldsymbol{\Psi}$. The quantity A is the square root of the product of the eigenvalues of $[g]$, and πA is the area of the velocity ellipse. A is zero at the boundary where there is only one effective motion parameter, and A is maximum when $\det[g]$ is maximum. From Section 3.2, A is also a measure of the "transmission ratio" at $\mathbf{p}$.

It is possible to generate curves of constant A in the point trajectory. We will call them "curves of constant A." The curves of constant A are fixed for a given two-degrees-of-freedom mechanism and, in general, exist for all values of A from 0 to some maximum value. The gradient of an A (a two-dimensional vector) is given by

$$\nabla A = (\partial A/\partial\theta_1, \partial A/\partial\theta_2) \tag{24}$$

The gradient vector gives the direction of the maximum change of the transmission ratio or the effectiveness. The constant A curves are similar to curves in contour maps describing the topography of a region. The gradient vector gives the direction in the trajectory in which the constant A curves are most bunched together.

For a Cartesian manipulator,[2] A is everywhere constant, since $\boldsymbol{\Psi}$ is linear in the parameters of the motion. The gradient vector is zero and all directions are equal. This is analogous to a flat region with no hills. For a planar manipulator with two revolute joints, A is independent of θ_1, and the constant A curves are circles. In Section 4.4, we present the first and second-order properties of trajectories generated by planar two-degrees-of-freedom mechanisms.

The boundary (as has been mentioned before) can be obtained from the equation $A = 0$. For a two-degrees-of-freedom motion this is a curve. There may be several such curves.

4.3 Point Trajectories of Three-Degrees-of-Freedom Motion in Three-Space

A point trajectory due to a three-degrees-of-freedom motion can be represented by a mapping of the form $\boldsymbol{\Psi}$: $(\theta_1, \theta_2, \theta_3) \rightarrow (x, y, z)$. θ_1, θ_2, and θ_3 are the three parameters of the motion of a rigid body which contains the point $\mathbf{p}$ with coordinates (x, y, z) in a fixed reference space $\mathbf{R}^3$. The function $\boldsymbol{\Psi}$ depends on the three-degrees-of-freedom mechanism, and the trajectory is a region in $\mathbf{R}^3$ with a boundary. The vectors, $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, and $\boldsymbol{\Psi}_3$, are independent except at the boundary of the region, and form a local basis.

The first-order property is the velocity distribution which is determined by the matrix $[g]$. In this case the matrix $[g]$ is 3×3 and $g_{ij} = \boldsymbol{\Psi}_i \cdot \boldsymbol{\Psi}_j$, $i, j = 1, 2, 3$. Except at the boundaries, the tip of the velocity vector for $\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 = k^2$ lies on an ellipsoid. The eigenvectors of $[g]$ when mapped to the space of the motion are along the axes of the ellipsoid. The maximum and minimum velocities at $\mathbf{p}$ are along the major and minor axes of the ellipsoid. The proofs of the foregoing statements are very similar to those presented for the velocity ellipses in Section 3.2.

For the second-order properties, we introduce a differential invariant analogous to A. The scalar $\mathcal{V}$ is defined as $(\det[g])^{1/2}$ and is proportional to the volume of the ellipsoid described by the tip of the velocity vector $\mathbf{v}$. In general, $\mathcal{V}$ is a function of the three parameters θ_i, $i = 1, 2, 3$ and the dimensions of the physical mechanism. The scalar quantity $\mathcal{V}$ is a measure of the effectiveness of the three parameters – it has a maximum when $\det[g]$ is maximum while at the boundary, where at least one of the $\boldsymbol{\Psi}_i$ $(i = 1, 2, 3)$ is parallel to one of the other two, $\mathcal{V}$ is zero. It is also a measure of the transmission ratio at the point $\mathbf{p}$ under consideration. The gradient of $\mathcal{V}$ is a three-dimensional vector. It may be written as

$$\nabla\mathcal{V} = (\partial\mathcal{V}/\partial\theta_1, \partial\mathcal{V}/\partial\theta_2, \partial\mathcal{V}/\partial\theta_3) \tag{25}$$

$\mathcal{V}$ equals constant yields surfaces along which the measure of transmission ratio, or effectiveness, remain the same. The gradient of $\mathcal{V}$ gives the direction along which these measures are changing the fastest. For Cartesian manipulators $\mathcal{V}$ is the same everywhere, as $\boldsymbol{\Psi}$ is a linear function of the three joint displacements. The magnitude of the gradient of $\mathcal{V}$ is zero, and all directions are equal with respect to effectiveness. For a manipulator with three revolute joints $\mathcal{V}$ is independent of the first joint rotation, hence $\mathcal{V}$ is constant at all points on a surface of revolution. The gradient vector at a point is along the normal to the surface of revolution through that point.

The boundary is given by $\det[g] = 0$. This is the equation of a surface in $\mathbf{R}^3$. We can use the theory presented in Section 3.1 and 3.2 to find the first- and second-order properties of this surface.

4.4 Two-Degrees-of-Freedom Planar Mechanisms

In this section, we illustrate the foregoing concepts by presenting the first- and second-order properties of trajectories generated by two-degrees-of-freedom mechanisms. For a detailed development of these results the reader is referred to [14].

(1) The 2R Linkage. For a 2R linkage in a plane, shown in Fig. 1, the equations describing the kinematics, $\boldsymbol{\Psi}$: $(\theta_1, \theta_2) \rightarrow (x, y)$, are[3]

$$\begin{aligned} x &= a_{12}c_1 + a_{23}c_{1+2} \\ y &= a_{12}s_1 + a_{23}s_{1+2} \end{aligned} \tag{26}$$

Hence

$$\begin{aligned} g_{11} &= a_{12}^2 + a_{23}^2 + 2a_{12}a_{23}c_2 \\ g_{12} &= a_{23}^2 + a_{12}a_{23}c_2 \\ g_{22} &= a_{23}^2 \end{aligned} \tag{27}$$

Since the elements of $[g]$ do not contain θ_1, the shape of the velocity ellipse for a given a_{12} and a_{23}, depends only on the value of θ_2. In Fig. 1, the ellipse described by the tip of the velocity vector is shown for $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$, $a_{12} = 2$, $a_{23} = 1$, $\theta_1 = 0°$ and $\theta_2 = 45°$.

[2] In case of Cartesian manipulators, the independent motion parameters are the translations at the joints.

[3] We use the abbreviations of c_j for $\cos(\theta_j)$ and s_j for $\sin(\theta_j)$, the plus sign in the subscript indicates a sum of the two angles: $c_{1+2} = \cos(\theta_1 + \theta_2)$ and $s_{1+2} = \sin(\theta_1 + \theta_2)$.

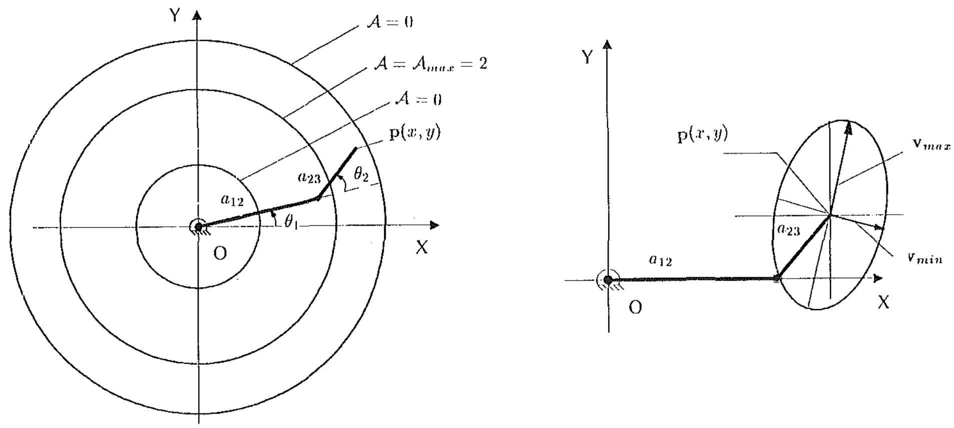

Fig. 1 Planar 2R linkage

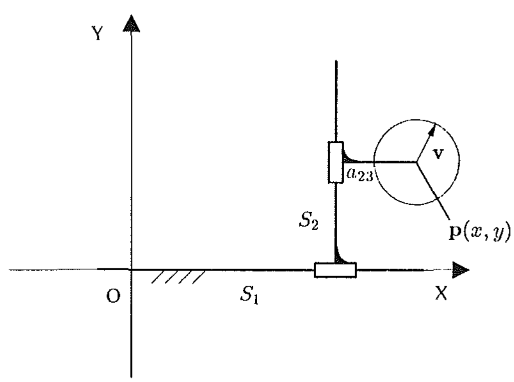

Fig. 2 2P linkage

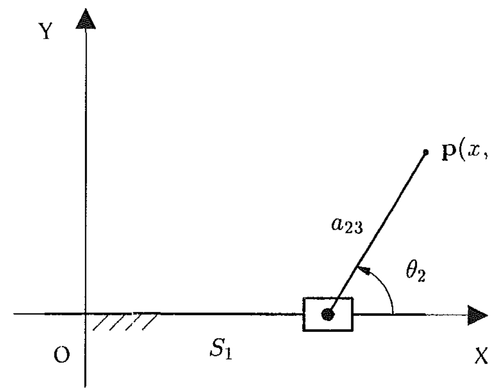

Fig. 3 PR linkage

The scalar, A, is given by $A = a_{23}a_{12}s_2$. The constant A curves are concentric circles about the origin. A is maximum when θ_2 is 90° and minimum when θ_2 is 0 or 180°. The magnitude of the gradient of A is maximum when θ_2 is 0 or 180°. The boundaries of the workspace are obtained from setting A equal to zero. The boundary curves are circles and are given by $x^2 + y^2 = (a_{12} \pm a_{23})^2$. These circles and the circle for $A_{\max}$, for $a_{12} = 2$ and $a_{23} = 1$ are shown in Fig. 1.

(2) The 2P Linkage. For a 2P linkage in a plane, shown in Fig. 2, the equations describing the kinematics, Ψ: $(S_1, S_2) \rightarrow (x, y)$, are

$$x = S_1 + a_{23} \qquad y = S_2 \tag{28}$$

The elements g_{11}, g_{12}, and g_{22} are, respectively, 1, 0, and 1. For $\dot{S}_1{}^2 + \dot{S}_2{}^2 = 1$, the locus of the curve traced out by the tip of the velocity vector is the same at every point in the plane, and moreover, it is a circle. The scalar, A, is equal to 1 at every point in the plane except at the boundaries where it is zero. The gradient of A is maximum at the boundaries where A changes from 0 to 1.

(3) The PR Linkage. For the PR linkage, shown in Fig. 3, the equations describing the kinematics, Ψ: $(S_1, \theta_2) \rightarrow (x, y)$, are

$$x = S_1 + a_{23}c_2 \qquad y = a_{23}s_2 \tag{29}$$

Here, $g_{11} = 1$, $g_{12} = -a_{23}s_2$ and $g_{22} = a_{23}^2$. Hence, the shape of the ellipse traced out by the tip of the velocity vector for a given a_{23} and for the unit speed motion[4] depends only on the parameter θ_2. The scalar A is given by $a_{23}c_2|S_1|_{\max}$ and is maximum when θ_2 is zero or 180°. The constant area curves are straight lines parallel to the X-axis. The gradient of A is maximum when θ_2 is 90°.

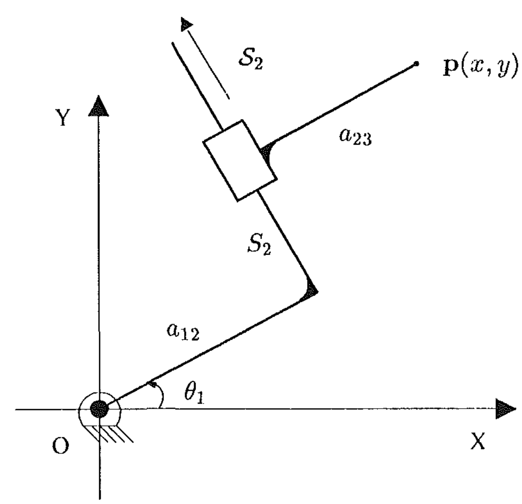

Fig. 4 RP linkage

(4) The RP Linkage. For the RP linkage, shown in Fig. 4, the equations describing the kinematics, Ψ: $(\theta_1, S_2) \rightarrow (x, y)$, are

$$x = (a_{12} + a_{23})c_1 - S_2s_1 \qquad y = (a_{12} + a_{23})s_1 + S_2c_1 \tag{30}$$

The shape of the ellipse described by the tip of the velocity vector for the unit speed motion $\dot{\theta}_1^2 + (\dot{S}_2/S_{2\,\max})^2 = 1$ and

[4] The unit-speed motion in the case of a PR linkage chain is taken as $(\dot{S}_1/(S_1)_{\max})^2 + \dot{\theta}_1^2 = 1$, where $(S_1)_{\max}$ is the maximum value which S_1 can take. This ensures that the two terms are dimensionally the same.

known a_{12} and a_{23} depends only on S_2. The scalar A, is $|S_2|_{\max}S_2$. It is zero when S_2 is zero and is maximum just before S_2 reaches its maximum value. At the boundary the area is again zero, as we have no motion along S_2. The constant area curves are circles.

5.1 Point Trajectories With $m > n$

In this section, we consider the case of point trajectories where the degree of freedom of the motion is larger than the number of coordinates required to specify the point. Examples of this case are point trajectories due to three- or higher-degrees-of-freedom motion in the plane and four- or higher-degrees-of-freedom motion in three-dimensional space. Mathematically, these motions can be represented by mappings which takes points in m-dimensional spaces to points in n-dimensional spaces, where $m > n$. The map is many-to-one. In the case of manipulators or multi-degrees-of-freedom mechanisms, the dimension of the space of inverse kinematic solutions is $m-n$, as there are $m-n$ more unknowns than equations. Such manipulators have redundant degrees of freedom when it comes to positioning a point. A number of researchers [13, 15, 16] have employed the pseudo-inverse of the Jacobian matrix to make use of this redundancy. Their work is concerned mainly with the control of redundant manipulators. In this section, we give a procedure to use the extra degrees of freedom for altering the velocity distribution at a point without using the pseudo-inverse. A desirable velocity capability is one in which the tip of the velocity vector can describe a circle (for motion in two-space) and a sphere (for motion in three-space), at every position in the point trajectory. We will show that by choosing a suitable rate of change of the extra $m-n$ parameters we can achieve a circular velocity distribution. We present the analysis for a three-degrees-of-freedom motion in a plane. A more general treatment is in [14].

5.2 Three-Degrees-of-Freedom Motion Point Trajectory in a Plane

Mathematically, a general point trajectory in a plane due to a three-degrees-of-freedom motion can be represented as $\mathbf{\Psi}$: $(\theta_1, \theta_2, \theta_3) \rightarrow (x, y)$ where θ_i, $i = 1, 2, 3$, are the motion parameters, (x, y) are the coordinates of a moving point as measured in a reference plane, $\mathbf{R}^2$, and the function $\mathbf{\Psi}$ depends on the actual mechanism. (The inverse function to $\mathbf{\Psi}$, which gives θ_i, $i = 1, 2, 3$, for known x and y has infinitely many solutions.) The velocity of the point is given by

$$\mathbf{v} = \sum_{i=1}^{3} \mathbf{\Psi}_i \dot{\theta}_i \tag{31}$$

For general points (except at the boundary), two out of three $\mathbf{\Psi}_i\dot{\theta}_i$'s are independent. Let $\mathbf{\Psi}_3\dot{\theta}_3$ be a linear combination of $\mathbf{\Psi}_i\dot{\theta}_i$, $i = 1, 2$. (We could just as well write $\mathbf{\Psi}_1\dot{\theta}_1$ or $\mathbf{\Psi}_2\dot{\theta}_2$ in terms of the remaining two.) We can write

$$\mathbf{\Psi}_3\dot{\theta}_3 = \sum_{i=1}^{2} \alpha_i \mathbf{\Psi}_i \dot{\theta}_i \tag{32}$$

We assume that α_1 and α_2 are finite and real. If α_1 (or α_2) is zero, then $\mathbf{\Psi}_3\dot{\theta}_3$ is parallel to $\mathbf{\Psi}_2\dot{\theta}_2$ (or $\mathbf{\Psi}_1\dot{\theta}_1$). α_1 and α_2 can be solved in terms of the $\dot{\theta}_i$'s and the dot products $\mathbf{\Psi}_i \bullet \mathbf{\Psi}_j$. Forming the dot product of (32) with $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ yields

$$\begin{aligned} (\mathbf{\Psi}_3 \bullet \mathbf{\Psi}_1)\dot{\theta}_3 &= \alpha_1 g_{11}\dot{\theta}_1 + \alpha_2 g_{12}\dot{\theta}_2 \\ (\mathbf{\Psi}_3 \bullet \mathbf{\Psi}_2)\dot{\theta}_3 &= \alpha_1 g_{12}\dot{\theta}_1 + \alpha_2 g_{22}\dot{\theta}_2 \end{aligned} \tag{33}$$

where g_{11}, g_{12}, and g_{22} are, respectively, the dot products $\mathbf{\Psi}_1 \bullet \mathbf{\Psi}_1$, $\mathbf{\Psi}_1 \bullet \mathbf{\Psi}_2$, and $\mathbf{\Psi}_2 \bullet \mathbf{\Psi}_2$.

From equations (33), we have

$$\begin{aligned} \alpha_1 &= \frac{[(\mathbf{\Psi}_3 \bullet \mathbf{\Psi}_1)g_{22} - (\mathbf{\Psi}_3 \bullet \mathbf{\Psi}_2)g_{12}]\dot{\theta}_3}{(g_{11}g_{22} - g_{12}^2)\dot{\theta}_1} = a_1(\dot{\theta}_3/\dot{\theta}_1) \\ \alpha_2 &= -\frac{[(\mathbf{\Psi}_3 \bullet \mathbf{\Psi}_1)g_{12} - (\mathbf{\Psi}_3 \bullet \mathbf{\Psi}_2)g_{11}]\dot{\theta}_3}{(g_{11}g_{22} - g_{12}^2)\dot{\theta}_2} = a_2(\dot{\theta}_3/\dot{\theta}_2) \end{aligned} \tag{34}$$

We need one more equation to solve for α_1, α_2, and $\dot{\theta}_3$. We obtain the equation from the desired circular velocity distribution.

Substituting (32) into (31), we get

$$\mathbf{v} = \sum_{i=1}^{2} (1 + \alpha_i)\mathbf{\Psi}_i\dot{\theta}_i \tag{35}$$

The values of $(\dot{\theta}_1, \dot{\theta}_2)$, which give the maximum and minimum $\mathbf{v}^2$, for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$, are the eigenvectors of the matrix $[g']$. The matrix $[g']$ is symmetric and its coefficients are given by

$$\begin{aligned} g'_{11} &= (1+\alpha_1)^2(\mathbf{\Psi}_1 \bullet \mathbf{\Psi}_1) = (1+\alpha_1)^2 g_{11} \\ g'_{12} &= (1+\alpha_1)(1+\alpha_2)(\mathbf{\Psi}_1 \bullet \mathbf{\Psi}_2) = (1+\alpha_1)(1+\alpha_2)g_{12} \\ g'_{22} &= (1+\alpha_2)^2(\mathbf{\Psi}_2 \bullet \mathbf{\Psi}_2) = (1+\alpha_2)^2 g_{22} \end{aligned} \tag{36}$$

The eigenvalues are real and are functions of g_{11}, g_{12}, g_{22}, α_1, and α_2. The velocity distribution at a point is always an ellipse, but its shape can be changed by choosing some velocity criterion and then solving for $\dot{\theta}_3$, α_1 and α_2. We give the solution procedure for the case when the eigenvalues of $[g']$ are equal and the point being isotropic [12]. This corresponds to the velocity distribution (at a point) being a circle.

The condition for equal eigenvalues is given by

$$\begin{aligned} &[(1+\alpha_2)^2 g_{22} - (1+\alpha_1)^2 g_{11}]^2 \\ &\qquad + 4(1+\alpha_1)^2(1+\alpha_2)^2 g_{12}^2 = 0 \end{aligned} \tag{37}$$

Since the left-hand side of the foregoing equation is the sum of two squares, it follows that both terms must be zero for the eigenvalues to be equal. If $g_{12} \neq 0$, then

$$\alpha_1 = \alpha_2 = -1 \tag{38}$$

Otherwise, we require $g_{12} = 0$. Then

$$\frac{(1+\alpha_1)}{(1+\alpha_2)} = \pm (g_{22}/g_{11})^{1/2} \tag{39}$$

The first case results in $\mathbf{v}$ always equal to zero, from (35), and hence is not of much interest. In the second case, equation (39), α_1 and α_2 are

$$\begin{aligned} \alpha_2 &= \frac{\pm (g_{11}/g_{22})^{1/2} - 1}{1 \mp (g_{11}/g_{22})^{1/2}[a_1\dot{\theta}_2/(a_2\dot{\theta}_1)]} \\ \alpha_1 &= \frac{a_1\dot{\theta}_2}{a_2\dot{\theta}_1}\alpha_2 \end{aligned} \tag{40}$$

where a_1 and a_2 are defined in (34), and by using (34) and $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$, we get

$$\dot{\theta}_3^2 = 1/[(a_1/\alpha_1)^2 + (a_2/\alpha_2)^2] \tag{41}$$

The aforementioned procedure (to compute $\dot{\theta}_3$) can be used as long as we can solve for a finite α_1 and α_2 from equations (34). This cannot be done when α_1 and α_2 are in the indeterminate form zero divided by zero. α_1 and α_2 are indeterminate where the degrees of freedom is less than three. Setting the numerator and denominator (of, say, α_1) in (34) to zero yields equations $g_{11}g_{22} - g_{12}^2 = 0$ and $(\mathbf{\Psi}_1 \bullet \mathbf{\Psi}_3)g_{22} - (\mathbf{\Psi}_2 \bullet \mathbf{\Psi}_3)g_{12} = 0$. These equations represent curves, and when $\mathbf{p}$ is on these curves its velocity distribution cannot be altered. It is interesting to note that at boundaries, the degrees of freedom is less than three and hence the boundaries of the trajectory are included in these curves.

From the previous analysis, we can make the following general statement for point trajectories due to three-degrees-of-freedom motion in $\mathbf{R}^2$: in general, except for regions given

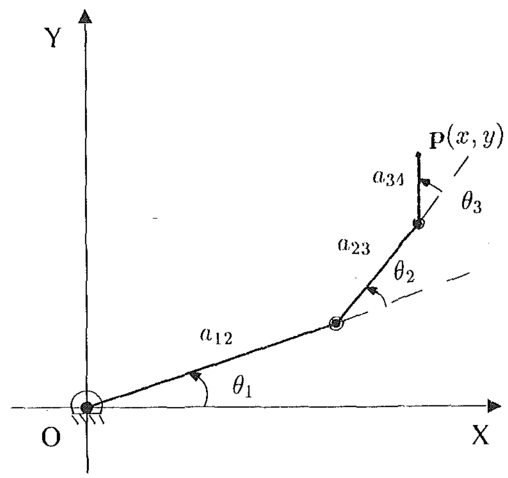

Fig. 5 Planar 3R manipulator

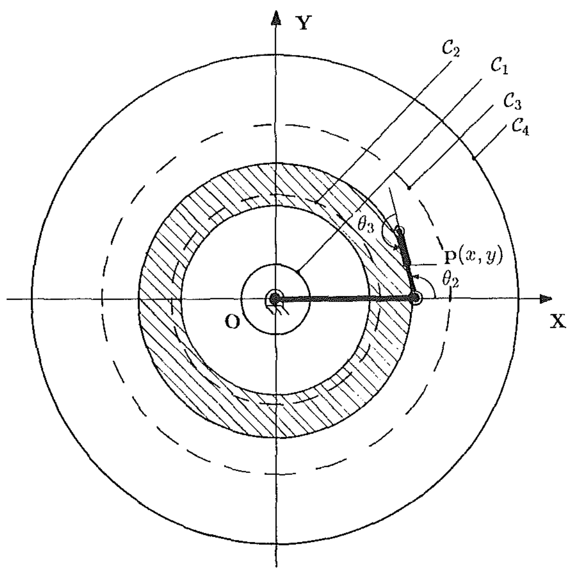

Fig. 6 Regions in the workspace of the 3R manipulator

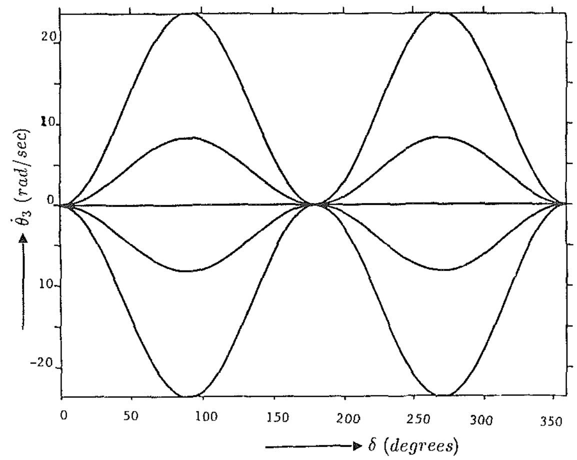

Fig. 7 Plot of $\dot{\theta}_3$ with respect to δ

by det[g] = 0, $\alpha_1 = 0$ and $\alpha_2 = 0$, we can at each point determine the values of $\dot{\theta}_3$, as a function of $g_{11}, g_{12}, g_{22}, \dot{\theta}_1, \dot{\theta}_2, \Psi_3 \cdot \Psi_1$, and $\Psi_3 \cdot \Psi_2$, which will give any required velocity distribution. $\dot{\theta}_3$ as a function of the foregoing quantities can be pictured, in general, in the $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$ space as a curve on the surface of a cylinder. Furthermore, for equal eigenvalues g_{12} must be zero. The regions where alterations in the velocity distribution are possible will be called the *alterable* regions of the point trajectory.

The previous procedure can be easily modified for three-degrees-of-freedom motion in three-space. For motions with more degrees of redundancy ($m - n > 1$), a similar procedure can be used to alter first- and second-order properties. For more details the reader is referred to [14].

5.3 Example: A 3R Manipulator in a Plane

Figure 5 shows a three-degrees-of-freedom manipulator in the plane XY. There are three revolute joints with rotations θ_1, θ_2, and θ_3. The link lengths are a_{12}, a_{23}, and a_{34}. We are interested in the motion of the point $\mathbf{p}(x, y)$. The kinematic equations, $\Psi: (\theta_1, \theta_2, \theta_3) \rightarrow (x, y)$, are

$$\begin{aligned} x &= a_{12}c_1 + a_{23}c_{1+2} + a_{34}c_{1+2+3} \\ y &= a_{12}s_1 + a_{23}s_{1+2} + a_{34}s_{1+2+3} \end{aligned} \tag{42}$$

We use numerical values of $a_{12} = 4$, $a_{23} = 2$, $a_{34} = 1$ in this example. For equal eigenvalues θ_2 and θ_3 have to be such that $g_{12} = 0$. The condition $g_{12} = 0$, reduces to

$$5 + 8c_2 + 4c_{2+3} + 4c_3 = 0 \tag{43}$$

From the aforementioned equation it follows that for each value of θ_3 we have two values of θ_2 given by

$$\tan(\theta_2/2) = (1/3)[-4s_3 \pm (55 + 24c_3 - 16c_3^2)^{1/2}] \tag{44}$$

In this case, θ_3 can take any value between 0 and 360° and the extreme values of θ_2 are ±138.59° and ±104.47°. Since θ_1 can taken any value, the velocity distribution can be altered in the annular shaded region shown in Fig. 6. In this region, on one circle, $x^2 + y^2 = 9$, α_1 and α_2 are indeterminate and the velocity distribution cannot be altered. The conditions of indeterminate α_1 and α_2 yields curves C_1, C_2, C_3 and C_4 – four concentric circles of radius 1, 3, 5, and 7, respectively. The circles with radius 1 and 7 are the inner and the outer boundaries, respectively. The circles are shown in Fig. 6.

One point where the velocity distribution can be altered is when $\theta_2 = \cos^{-1}(-1/4)$ and $\theta_3 = 180°$. We plot $\dot{\theta}_3$ (for this point) as a function of the angle δ, defined by tan $\delta = \dot{\theta}_2/\dot{\theta}_1$, in Fig. 7. The angle δ varies from 0 to 360°.

6.1 Conclusion

A general framework, based on concepts from differential geometry, has been presented to facilitate the study the properties of trajectories of points embedded in rigid bodies undergoing *multi-degrees-of-freedom motions*. Quantities such as the elements of the matrices [g], [L] and the Γ_{ij}^k have been developed to characterize and distinguish between trajectories generated by different multi-degrees-of-freedom mechanisms. For motions where the degrees of freedom are equal to or less than the dimension of the space in which the motion takes place, scalar measures of effectiveness of velocity transmission were developed. For motions where the degrees of freedom are greater than the dimension of the space, a new approach has been developed for using the redundancy to alter the first-order properties of the trajectory. The theory has been applied to several examples of motions generated by two- and three-degrees-of-freedom open-loop mechanisms containing revolute and prismatic joints.

In a companion paper [19], we developed analogous results for the trajectories generated by lines under multi-degrees-of-freedom motions.

Acknowledgments

The financial support of the National Science Foundation (through Grant MEA 8207694) and the Systems Development Foundation are gratefully acknowledged.

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