Instantaneous Properties of Multi-Degrees-of-Freedom Motions—Line Trajectories

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A general framework is presented for the study of the properties of trajectories generated by lines embedded in rigid bodies undergoing multi-degrees-of-freedom motions. Several new concepts, such as a line's angular and linear velocities and accelerations, are introduced and used to (1) characterize the differences between line trajectories generated by different mechanisms; (2) distinguish trajectories generated by different lines in the same rigid body; (3) distinguish properties at different positions in the same trajectory. Line trajectories are classified according to the number of degrees of freedom of the motion, and local and global properties are discussed. These techniques are illustrated in an example of a line trajectory generated by a two-degrees-of-freedom manipulator.

1 Introduction

In a companion paper [1], we have developed the properties of trajectories of points embedded in rigid bodies undergoing multi-degrees-of-freedom motions. In this paper, we present analogous results for trajectories generated by lines embedded in rigid bodies undergoing multi-degrees-of-freedom motions. Lines are important for several reasons, and it is advantageous to study them in their own right and not simply as collections of points. In rigid-body mechanics, one is often interested in the orientation and change of orientation of a moving body, while for a manipulator one is often interested in the orientation of the end-effector or the object which the end-effector is carrying. In the end-effector one speaks of the approach vector and the orientation vector, both of which can be represented as lines fixed in the end-effector. The study of line trajectories is natural when we are dealing with joint axes. The joint axes can be represented by lines, since typically, no single point on the axis is of interest. In addition, for the purpose of studying orientation or changes of orientation, a symmetric object such as a cylinder, can be modelled as a straight line parallel to the symmetry axis. Hence, if the end-effector is carrying a symmetric object, most of the orientation information about the moving body can be obtained by the study of the trajectory traced out by the line coinciding with the symmetry axis. In all of these foregoing problems, a line is the basic element of interest.

A line trajectory due to a one-degree-of-freedom motion is a ruled surface, and its properties are well known [2-5]. McCarthy [3] and McCarthy and Roth [4] have obtained the first-, second- and higher-order properties of line trajectories due to one-degree-of-freedom motions; however their method is not easily extendable to line trajectories due to two or higher degrees of freedom. First-order properties of line trajectories due to two-degrees-of-freedom motions were obtained by Blaschke [6], but they do not seem to have been studied for three- or higher-degrees-of-freedom motions. Very little is known about second- and higher-order properties of line trajectories due to two- or higher-degrees-of-freedom motions.

2.1 Mathematical Formulation

In most general terms, the trajectory generated by a line embedded in a rigid body undergoing m-degrees-of-freedom motion can be represented by a set of equations giving the coordinates of the line, in a fixed reference frame, as functions of m independent parameters. In manipulators or multi-degrees-of-freedom mechanisms the m independent parameters are typically the rotations or translations at the joints. The actual functions themselves depend on the mechanism's structural parameters, e.g., link lengths, offsets and twist angles. We can symbolically write these functions as

$$\mathbf{\Psi}:(\theta_1, \ldots, \theta_m) \rightarrow (\mathbf{S}_1;\mathbf{S}_{01}) \tag{1}$$

In equation (1), $(\theta_1, \ldots, \theta_m)$ are the m independent motion parameters, $(\mathbf{S}_1; \mathbf{S}_{01})$ are the *Plücker* vectors of the line [7], $\mathbf{S}_1$ is a unit vector along the line and $\mathbf{S}_{01}$ is the moment of the line about the origin of the fixed coordinate frame (out of the six scalars components of $\mathbf{S}_1$ and $\mathbf{S}_{01}$, only four are independent). $\mathbf{\Psi}$ represent the set of functions which give the *Plücker* line vectors as functions of the independent motion parameters.

For a one-degree-of-freedom motion ($n = 1$), the line trajectory in three-space is a ruled surface. For a two-degrees-of-freedom motion ($m = 2$), the line trajectory is a line congruence. For a three-degrees-of-freedom motion ($m = 3$), the line trajectory is called a line complex and for a four-degrees-of-freedom motion ($m = 4$), the line trajectory can be called a *line solid*. If m is greater than 4, the values of the independent

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motion parameters corresponding to any given *Plücker* vectors lie in an $m-4$ dimensional space. This is called a redundant motion, and the line trajectory is still a solid.

In this paper, we determine those properties, both local and global, which distinguish line trajectories generated by different mechanisms, and properties which distinguish lines at different positions in the same trajectory. We also find out special or interesting line trajectories. In addition, at each instant, we classify all lines in the moving rigid body.

2.2 Properties of Line Trajectories

It is possible to divide equation (1) into two parts, one dealing only with the orientation, the so-called spherical part of the motion

$$\Psi^s:(\theta_1,\theta_2,\ldots,\theta_m)\to \mathbf{S}_1 \tag{2}$$

and the other which gives $\mathbf{S}_{01}$,[1]

$$\Psi^*:(\theta_1,\theta_2,\ldots,\theta_m)\to \mathbf{S}_{01} \tag{3}$$

Equations (2) and (3) are highly nonlinear even for simple mechanisms and it is difficult to deduce much from them. In order to study the line trajectories due to multidegrees-of-freedom motions, we find their first-, second- and higher-order properties at a "point"[2] (the so-called local properties) and from these try to find global properties of the complete trajectory. In order to develop these properties we first introduce the concepts of angular and linear velocity and acceleration of a line.

3.1 Angular and Linear Velocity, and the Center of a Line

Let $\Psi[\theta_1(t),\theta_2(t),\ldots,\theta_m(t)]$, where t is the independent parameter time, denote a one-degree-of-freedom motion in the complete m-degrees-of-freedom motion, and let the moving rigid body containing the line $\mathcal{L}_1$ have an angular velocity $\mathbf{\Omega}$ with respect to a fixed reference frame *OXYZ*. We define the *angular velocity*, $\omega_{\mathcal{L}_1}$, of the line $\mathcal{L}_1$ as that component of $\mathbf{\Omega}$ which is not along the direction of the line, $\mathcal{L}_1$. This is consistent with the fact that in most of our applications the rotation about the line itself will be unspecified.

From the foregoing reasoning, we get

$$\omega_{\mathcal{L}_1}=\mathbf{S}_1\times\dot{\mathbf{S}}_1 \tag{4}$$

The magnitude of $\omega_{\mathcal{L}_1}$ is $|\dot{\mathbf{S}}_1|$. Introducing a unit vector $\mathbf{g}$ along $\mathbf{S}_1\times\dot{\mathbf{S}}_1$ we can write

$$\omega_{\mathcal{L}_1}=|\dot{\mathbf{S}}_1|\mathbf{g} \tag{5}$$

We introduce another unit vector $\mathbf{t}=\dot{\mathbf{S}}_1/|\dot{\mathbf{S}}_1|$. The three unit vectors $\mathbf{t}$, $\mathbf{g}$, and $\mathbf{S}_1$ are mutually perpendicular and may be used to form a moving reference frame attached to the line. In terms of the ruled surface generated by $\mathcal{L}_1$ during the motion, $\mathbf{S}_1$ is called the generator, $\mathbf{t}$ is called the central normal, and $\mathbf{g}$ is called the central tangent. The velocity of any point $\mathbf{r}$, on the line $\mathcal{L}_1$, in the *OXYZ* frame can be written as

$$\mathbf{v}=(\dot{\mathbf{r}}\cdot\mathbf{g})\mathbf{g}+(\dot{\mathbf{r}}\cdot\mathbf{t})\mathbf{t}+(\dot{\mathbf{r}}\cdot\mathbf{S}_l)\mathbf{S}_l \tag{6}$$

The first two terms of equation (6) can be rewritten in terms of the line coordinates and their derivatives. We get

$$\mathbf{v}=(\dot{\mathbf{S}}_{01}\cdot\mathbf{t})\mathbf{g}+[(\mathbf{r}\cdot\mathbf{S}_1)|\dot{\mathbf{S}}_1|-(\dot{\mathbf{S}}_{01}\cdot\mathbf{g})]\mathbf{t}+(\dot{\mathbf{r}}\cdot\mathbf{S}_1)\mathbf{S}_1 \tag{7}$$

We locate the origin of the moving frame ($\mathbf{t}$, $\mathbf{g}$, $\mathbf{S}_1$) at a point on the line $\mathcal{L}_1$ such that the point's velocity along $\mathbf{t}$ is zero. This gives

$$\mathbf{r}\cdot\mathbf{S}_1=\frac{\dot{\mathbf{S}}_{01}\cdot(\mathbf{S}_1\times\dot{\mathbf{S}}_1)}{\dot{\mathbf{S}}_1^2} \tag{8}$$

Equation (8), together with $\mathbf{r}\times\mathbf{S}_1=\mathbf{S}_{01}$, gives the location of the point whose velocity along $\mathbf{t}$ is zero. This point is called the *center* of the line, and is given by

$$\mathbf{r}_c=\frac{\dot{\mathbf{S}}_{01}\cdot(\mathbf{S}_1\times\dot{\mathbf{S}}_1)}{\dot{\mathbf{S}}_1^2}\mathbf{S}_1-\mathbf{S}_{01}\times\mathbf{S}_1 \tag{9}$$

Denoting $(\dot{\mathbf{S}}_{01}\cdot(\mathbf{S}_1\times\dot{\mathbf{S}}_1))/\dot{\mathbf{S}}_1^2$ by p^*, the velocity of the origin of the moving reference frame, called the *velocity of the center* of the line $\mathcal{L}_1$, can be written using equation (9) as

$$\mathbf{v}_c=(\dot{\mathbf{S}}_{01}\cdot\mathbf{t})\mathbf{g}+[\dot{p}^*+\mathbf{S}_{01}\cdot(\mathbf{S}_1\times\dot{\mathbf{S}}_1)]\mathbf{S}_1 \tag{10}$$

The coefficient of $\mathbf{g}$ is independent of the point chosen on the line. In fact for all points on the line $\mathcal{L}_1$, the velocity along $\mathbf{g}$ is the same. This can be shown as follows:

Let $\mathbf{r}_1$ be another point on the line; using a scalar λ,

$$\mathbf{r}_1=\mathbf{r}+\lambda\mathbf{S}_1,\ \dot{\mathbf{r}}_1=\dot{\mathbf{r}}+\dot{\lambda}\mathbf{S}_1+\lambda\dot{\mathbf{S}}_1 \tag{11}$$

and

$$\dot{\mathbf{r}}_1\cdot\mathbf{S}_1\times\dot{\mathbf{S}}_1/|\dot{\mathbf{S}}_1|=\dot{\mathbf{r}}\cdot\mathbf{S}_1\times\dot{\mathbf{S}}_1/|\dot{\mathbf{S}}_1| +\dot{\lambda}\mathbf{S}_1\cdot\mathbf{S}_1\times\dot{\mathbf{S}}_1/|\dot{\mathbf{S}}_1|+\lambda\dot{\mathbf{S}}_1\cdot\mathbf{S}_1\times\dot{\mathbf{S}}_1/|\dot{\mathbf{S}}_1| \tag{12}$$

The last two terms are zero, hence $\dot{\mathbf{r}}_1\cdot\mathbf{g}=\dot{\mathbf{r}}\cdot\mathbf{g}$.

Using the previous fact, we define the *linear velocity* of the line $\mathcal{L}_1$ as

$$\mathbf{v}_{\mathcal{L}_1}=(\dot{\mathbf{S}}_{01}\cdot\mathbf{t})\mathbf{g}=v_{\mathcal{L}_1}\mathbf{g} \tag{13}$$

Both the linear and angular velocity of the line are along $\mathbf{g}$, and we can talk of the ratio of the linear and angular velocity of the line $\mathcal{L}_1$. This is called the pitch of the (instantaneous) helicoidal motion of the line and is given by

$$p=v_{\mathcal{L}_1}/|\dot{\mathbf{S}}_1|=(\dot{\mathbf{S}}_{01}\cdot\dot{\mathbf{S}}_1)/(\omega_{\mathcal{L}_1})^2 \tag{14}$$

Equation (10) can be rewritten as

$$\mathbf{v}_c=p\omega_{\mathcal{L}_1}+(\dot{p}^*+\mathbf{S}_{01}\cdot\omega_{\mathcal{L}_1})\mathbf{S}_1 \tag{15}$$

where

$$p^*=\frac{\dot{\mathbf{S}}_{01}\cdot\omega_{\mathcal{L}_1}}{\omega^2_{\mathcal{L}_1}} \tag{16}$$

Finally, we will rewrite equations (4), (9), and (13) so that they become explicit expressions in terms of the mapping functions Ψ^S and Ψ^*. The angular velocity of the line $\mathcal{L}_1$, from (4), becomes

$$\omega_{\mathcal{L}_1}=\mathbf{S}_1\times\Psi_1{}^s\dot{\theta}_1+\ \ldots\ +\mathbf{S}_1\times\Psi_m{}^s\theta_m =\mathbf{S}_1\times J(\Psi^s)\dot{\mathbf{\Theta}} \tag{17}$$

where $\Psi_1{}^s$, $\Psi_2{}^s$, . . ,$\Psi_m{}^s$ are the partial derivatives of $\mathbf{S}_1$ with respect to $\theta_1,\theta_2,\ldots,\theta_m$, respectively. $J(\Psi^s)$ is the $3\times m$ Jacobian matrix for (2)[3] and $\dot{\mathbf{\Theta}}$ is the vector $(\dot{\theta}_1,\dot{\theta}_2,\ldots,\dot{\theta}_m)^T$. From equation (17), and from the definition of $\omega_{\mathcal{L}_1}$, we can see that the vector $\omega_{\mathcal{L}_1}$ always lies in a plane whose normal direction is $\mathbf{S}_1$ (see Fig. 1). This plane is analogous to the tangent plane for point trajectories [1] and will be denoted by $T_{\mathcal{L}_1}$.

Using the Jacobian matrices $J(\Psi^s)$ and $J(\dot{\Psi}^*)$ for (2) and (3), the center of the line as given in (9) may be rewritten as

$$\mathbf{r}_c=\frac{[J(\Psi^*)\dot{\mathbf{\Theta}}\cdot\mathbf{S}_1\times J(\Psi^s)\dot{\mathbf{\Theta}}]}{\omega^2_{\mathcal{L}_1}}\mathbf{S}_1-\Psi^*\times\Psi^s \tag{18}$$

Here, the coefficient of $\mathbf{S}_1$ is p^*.

The linear velocity of the line $\mathcal{L}_1$, $\mathbf{v}_{\mathcal{L}_1}$, given by (13) can be written as

$$\mathbf{v}_{\mathcal{L}_1}=\frac{[J(\Psi^*)\dot{\mathbf{\Theta}}\cdot J(\Psi^s)\dot{\mathbf{\Theta}}]}{\omega^2_{\mathcal{L}_1}}\omega_{\mathcal{L}_1} \tag{19}$$

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[1] $\mathbf{S}_{01}$ is also a function of $\mathbf{S}_1$.

[2] In this paper, we mean by a point a specified value of the *Plücker* vectors, ($\mathbf{S}_l$; $\mathbf{S}_{0l}$), which locates the line relative to a fixed reference frame, $\mathbf{R}^3$.

[3] The Jacobian matrix for (2) and (3) are $3\times m$ matrices formed, respectively, from the column vectors $\partial\mathbf{S}_1/\partial\theta_i$ and $\partial\mathbf{S}_{01}/\partial\theta_i$; ($i=1,2,\ldots,m$).

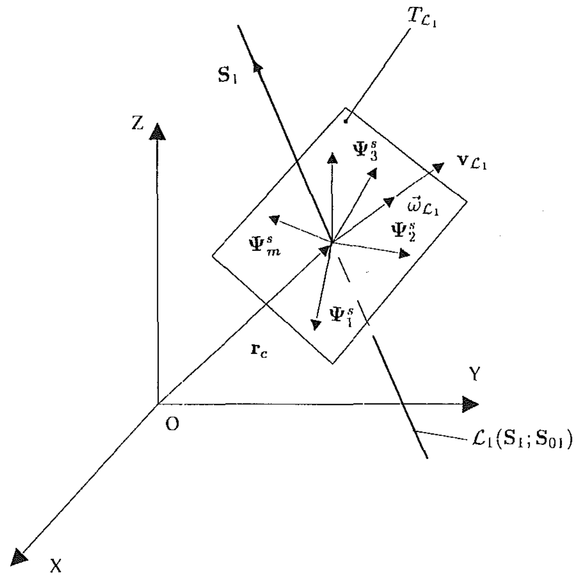

Fig. 1 Angular and linear velocity, and the center of a line

The pitch of the line's motion is the coefficient of $\omega_{\mathcal{L}_1}$ in equation (19). We note that p, p^* (and $\mathbf{r}_c$) depend only on the ratios of the $\dot{\theta}_i$'s but $\omega_{\mathcal{L}_1}$ (and $\mathbf{v}_{\mathcal{L}_1}$) depend on the complete vector $\dot{\mathbf{\Theta}}$.

The first-order properties of the line trajectory are the distributions of $\omega_{\mathcal{L}_1}$, $\mathbf{v}_{\mathcal{L}_1}$ and $\mathbf{r}_c$. We can make the following remarks about the distribution of $\omega_{\mathcal{L}_1}$, $\mathbf{v}_{\mathcal{L}_1}$ and $\mathbf{r}_c$:

(1) Both $\omega_{\mathcal{L}_1}$ and $\mathbf{v}_{\mathcal{L}_1}$ can have any direction and magnitude depending on $\dot{\mathbf{\Theta}}$. Hence, in order to study the distribution of $\omega_{\mathcal{L}_1}$ and $\mathbf{v}_{\mathcal{L}_1}$, we will use a normalizing condition in which we set $\dot{\mathbf{\Theta}}^T\dot{\mathbf{\Theta}}$ equal to a constant.

(2) When $\omega_{\mathcal{L}_1}$ and $\mathbf{v}_{\mathcal{L}_1}$ are treated individually without reference to the location of the center, $\mathbf{r}_c$, we have in each case a situation similar to point trajectories due to multi-degrees-of-freedom motions [1]. The angular velocity vector is essentially a two-dimensional quantity (it may be specified by a magnitude and a unit vector in the plane $T_{\mathcal{L}_1}$), and is very similar to the velocity of a point undergoing m-degrees-of-freedom motion in a plane. Only two out of the m $\mathbf{S}_1 \times \mathbf{\Psi}_i^s\dot{\theta}_i$ terms can be independent.

(3) Instead of dealing with three vectors, $\omega_{\mathcal{L}_1}$, $\mathbf{v}_{\mathcal{L}_1}$, and $\mathbf{r}_c$, it is easier to deal with the vector $\omega_{\mathcal{L}_1}$ and the two scalars p and p^*. Knowing the distribution of $\omega_{\mathcal{L}_1}$, p, and p^*, we can reconstruct the distribution of $\mathbf{v}_{\mathcal{L}_1}$ and $\mathbf{r}_c$ from (18) and (19).

In references [3, 4], McCarthy and McCarthy and Roth obtain second-order properties of line trajectories due to one-degree-of-freedom motions by finding the first-order properties of line trajectories generated by a line parallel to the central tangent $\mathbf{g}$. This is not a possible approach for trajectories due to multi-degrees-of-freedom motions as there is no central tangent. As the vector $\dot{\mathbf{\Theta}}$ changes, the direction of the central tangent (which is the same as the direction of $\omega_{\mathcal{L}_1}$) changes. We will develop the second-order properties of line trajectories due to m-degrees-of-freedom motions by finding the rate of change of $\omega_{\mathcal{L}_1}$, p, and p^*.

We define the *angular acceleration* of a line as the derivative of $\omega_{\mathcal{L}_1}$ with respect to time. Differentiating the expression for $\omega_{\mathcal{L}_1}$ in equation (4), we note that the angular acceleration vector, denoted by $\dot{\omega}_{\mathcal{L}_1}$ or $\alpha_{\mathcal{L}_1}$, lies in the plane $T_{\mathcal{L}_1}$. The *linear acceleration* of the line is defined as the derivative of $\mathbf{v}_{\mathcal{L}_1}$ with respect to time. It consists of two terms: one is $\dot{p}\omega_{\mathcal{L}_1}$ and the other is $p\dot{\omega}_{\mathcal{L}_1}$. We note that both of these vectors lie in the plane $T_{\mathcal{L}_1}$.

In analogy to point trajectories [1, 8], we will develop the differential equations whose solutions give paths of constant angular velocity. We will also define a characteristic area related to the orientation, and use the rate of change of this area as a second-order property.

The second-order properties associated with the complete motion of the line are determined by the rate of change of $\omega_{\mathcal{L}_1}$, p, and p^*. In this paper, we analyze p, and p^* and their derivatives since they are easier to deal with than the linear acceleration and the velocity of the center. We can always find the linear acceleration and the velocity of the center from $\omega_{\mathcal{L}_1}$, p, p^* and their derivatives.

4.1 Line Trajectories Due to One-Degree-of-Freedom Motions

Line trajectories due to one-degree-of-freedom motion have been studied by many researchers. McCarthy [3] and McCarthy and Roth [4] present the first-, second-, and higher-order properties of such line trajectories. During a one-degree-of-freedom motion, a line generates a ruled surface, and their analysis of the line trajectory is from the point of view of obtaining the infinitesimal properties of this ruled surface. We use our newly defined quantities since they allow us to extend the same analysis to multi-degrees-of-freedom motions.

We first consider only the spherical part of the motion

$$\mathbf{\Psi}^s:(\theta_1)\rightarrow \mathbf{S}_1 \tag{20}$$

The angular velocity of the line is

$$\omega = (\mathbf{S}_1 \times \mathbf{\Psi}_1{}^s)\dot{\theta}_1 \tag{21}$$

The square of the magnitude of $\omega_{\mathcal{L}_1}$ is $g_{11}^s\,\dot{\theta}_1^2$, where g_{11}^s is $(\mathbf{\Psi}_1{}^s \cdot \mathbf{\Psi}_1{}^s)$, a function of θ_1. The linear velocity $\mathbf{v}_{\mathcal{L}_1}$ is given by

$$\mathbf{v}_{\mathcal{L}_1} = p\omega_{\mathcal{L}_1} \tag{22}$$

where p, the pitch of the motion, is given by $g_{11}^*\dot{\theta}_1^2/\omega_{\mathcal{L}_1}^2$, and g_{11}^* is $\mathbf{\Psi}_1{}^* \cdot \mathbf{\Psi}_1{}^s$.

The location of the center $\mathbf{r}_c$ is given by

$$\mathbf{r}_c = p^*\mathbf{S}_1 - \mathbf{S}_{01}\times\mathbf{S}_1 \tag{23}$$

where p^* is given by $(\mathbf{\Psi}_1{}^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_1{}^s)/g_{11}^s$. The center of the line is independent of $\dot{\theta}_1$ and is completely determined for a given $\mathbf{\Psi}$. It is a geometric quantity associated with the line $\mathcal{L}_1$ and does not depend on the speed along the trajectory. The pitch, p, of $\mathcal{L}_1$ is also independent of the rate $\dot{\theta}_1$ and is completely determined for a given $\mathbf{\Psi}$. p can be positive, negative, or zero depending only on the value of g_{11}^*. As the line $\mathcal{L}_1$ moves, the sign and magnitude of p change. The sign of p can thus be used to characterize different regions of the line trajectory.

We can also look at the quantity p from a different viewpoint. At any instant, during the motion, the lines in the rigid body have unique values of p. Thus the sign and magnitude of p characterize the lines in the rigid body. For example, the lines with p equal to zero have no linear velocity. These lines are moving so that in an infinitesimally separated instant they continue to intersect. Since setting p equal to zero imposes one constraint on the four independent scalars in $(\mathbf{S}_1; \mathbf{S}_{01})$, at any instant the dimension of the solution space (containing lines which have p equal to zero) is three. These lines, in general, form a line complex embedded in the moving body.

The angular acceleration of $\mathcal{L}_1$ is given by

$$\alpha_{\mathcal{L}_1} = \mathbf{S}_1 \times (\mathbf{\Psi}_{11}{}^s\dot{\theta}_1^2 + \mathbf{\Psi}_1{}^s\ddot{\theta}_1) \tag{24}$$

where $\mathbf{\Psi}_{11}{}^s$ is the second derivative of $\mathbf{\Psi}^s$ with respect to θ_1. As has been mentioned before, $\alpha_{\mathcal{L}_1}$ is normal to the direction $\mathbf{S}_1$, and both $\omega_{\mathcal{L}_1}$ and $\alpha_{\mathcal{L}_1}$ lie in $T_{\mathcal{L}_1}$. $\alpha_{\mathcal{L}_1}$ can also be written as

$$\alpha_{\mathcal{L}_1} = [(\mathbf{\Psi}_1{}^s\cdot\mathbf{\Psi}_1{}^s)\ddot{\theta}_1 + (\mathbf{\Psi}_{11}{}^s\cdot\mathbf{\Psi}_1{}^s)\dot{\theta}_1^2](\mathbf{S}_1\times\mathbf{\Psi}_1{}^s) - [\mathbf{\Psi}_{11}{}^s\cdot(\mathbf{S}_1\times\mathbf{\Psi}_1{}^s)]\dot{\theta}_1^2\mathbf{\Psi}_1{}^s \tag{25}$$

The last term in the foregoing equation is perpendicular to the

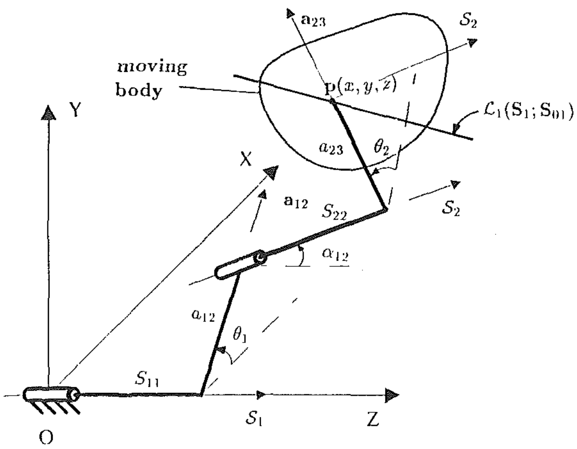

Fig. 2 2R manipulator

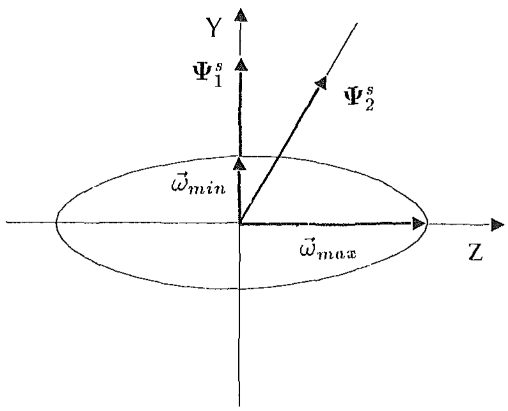

Fig. 3 Angular velocity ellipse at (0, 0)°

direction of $\omega_{\mathcal{L}_1}$, its coefficient, $\Psi_{11}{}^s \cdot (\mathbf{S}_1 \times \Psi_1{}^s)$, is analogous to the coefficient L_{11} for point trajectories [1, 8]. This coefficient can be thought of as the normal curvature of the spherical part of the line trajectory.

The first term in (25) is along the direction of the angular velocity and is the tangential component of the acceleration. Setting the coefficient of the first term (the entire expression in the square brackets) equal to zero yields a nonlinear differential equation whose solution gives a path of constant angular velocity. The coefficient $(\Psi_{11}{}^s \cdot \Psi_1{}^s)$ is analogous to the Christoffel symbol defined for point trajectories [1, 8]. The coefficient $\Psi_{11}{}^s \cdot (\mathbf{S}_1 \times \Psi_1{}^s)$, $\Psi_{11}{}^s \cdot \Psi_1{}^s$ and g_{11}^s determine the angular motion upto the second-order.

The linear acceleration of the line, $\mathbf{a}_{\mathcal{L}_1}$, is obtained by differentiating the line's linear velocity, $\mathbf{v}_{\mathcal{L}_1}$. Using the pitch p we can write

$$\begin{aligned} a_{\mathcal{L}_1} &= \dot{p}\omega_{\mathcal{L}_1} + p\dot{\omega}_{\mathcal{L}_1} \\ &= [(p' + p\Psi_{11}{}^s \cdot \Psi_1{}^s)\dot{\theta}_1^2 + p(\Psi_1{}^s \cdot \Psi_1{}^s)\ddot{\theta}_1](\mathbf{S}_1 \times \Psi_1{}^s) \\ &\quad - [p\dot{\theta}_1^2 \Psi_{11}{}^s \cdot (\mathbf{S}_1 \times \Psi_1{}^s)]\Psi_1{}^s \end{aligned} \tag{26}$$

where p' is the derivative of p with respect to θ_1. The normal component (the second term) is determined by the first-order property, p, and $\mathbf{S}_1 \times \Psi_1{}^s \cdot \Psi_{11}{}^s$. The tangential component depends on p'.

p' is a function of θ_1 and the line, $\mathcal{L}_1$, chosen in the rigid body. Hence setting p' equal to zero gives positions in the line trajectory (generated by $\mathcal{L}_1$) where linear and angular acceleration are simple multiples of each other. On the other hand, at a particular θ_1, p' equals zero identifies lines in the rigid body which at that instant have linear and angular acceleration along the same direction.

If the coefficient of $\mathbf{S}_1$ in equation (15) is zero there is no motion of the center along $\mathbf{S}_1$. This coefficient is zero if

$$(p^*)' + \mathbf{S}_{01} \cdot \mathbf{S}_1 \times \Psi_1{}^s = 0 \tag{27}$$

where

$$p^{*\prime} = \partial p^* / \partial \theta_1 \tag{28}$$

$p^{*\prime}$ is a second-order geometric (i.e., time-independent) property of the line trajectory. For a given line, the values of θ_1 satisfying the equation (27) map to positions in the line trajectory where the velocity of the center is zero along direction $\mathbf{S}_1$. For a given θ_1, equation (27) identifies lines in the moving rigid body which at that instant have zero velocity of the center along their directions.

The second-order geometric properties are determined by the quantities $\Psi_{11}{}^s \cdot (\mathbf{S}_1 \times \Psi_1{}^s)$, $\Psi_{11}{}^s \cdot \Psi_1{}^s$, p' and $p^{*\prime}$.

5.1 Line Trajectories Due to Two-Degrees-of-Freedom Motions

It is known that a line's trajectory due to a two-degrees-of-freedom motion is a line congruence [9, 10]. Mathematically, under a two-degrees-of-freedom motion, the coordinates of a line $\mathcal{L}_1$ are given as

$$\Psi: (\theta_1, \theta_2) \rightarrow (\mathbf{S}_1; \mathbf{S}_{01}) \tag{29}$$

where (θ_1, θ_2), $(\mathbf{S}_1; \mathbf{S}_{01})$, and Ψ have their usual meaning. The spherical and the spatial part of equation (29) may be expressed as Ψ^s: $(\theta_1, \theta_2) \rightarrow (\mathbf{S}_1)$ and Ψ^*: $(\theta_1, \theta_2) \rightarrow (\mathbf{S}_{01})$, respectively. The angular velocity of the line $\mathcal{L}_1$ at a generic point corresponding to $(\theta_{1,0}, \theta_{2,0})$ is given by

$$\omega_{\mathcal{L}_1} = (\mathbf{S}_1 \times \Psi_1{}^s)\dot{\theta}_1 + (\mathbf{S}_1 \times \Psi_2{}^s)\dot{\theta}_2 \tag{30}$$

The linear velocity and the location of the center are given by

$$\begin{aligned} \mathbf{v}_{\mathcal{L}_1} &= p\omega_{\mathcal{L}_1} \\ \mathbf{r}_c &= p^* \mathbf{S}_1 - \mathbf{S}_{01} \times \mathbf{S}_1 \end{aligned} \tag{31}$$

where p and p^* are

$$\begin{aligned} p &= \frac{(\Psi_1{}^* \dot{\theta}_1 + \Psi_2{}^* \dot{\theta}_2) \cdot (\Psi_1{}^s \dot{\theta}_1 + \Psi_2{}^s \dot{\theta}_2)}{(\Psi_1{}^s \dot{\theta}_1 + \Psi_2{}^s \dot{\theta}_2)^2} \\ p^* &= \frac{(\Psi_1{}^* \dot{\theta}_1 + \Psi_2{}^* \dot{\theta}_2) \cdot \mathbf{S}_1 \times (\Psi_1{}^s \dot{\theta}_1 + \Psi_2{}^s \dot{\theta}_2)}{\omega_{\mathcal{L}_1}^2} \end{aligned} \tag{32}$$

All the possible $\omega_{\mathcal{L}_1}$'s, obtained by varying $\dot{\theta}_1$ and $\dot{\theta}_2$, lie in the plane, $T_{\mathcal{L}_1}$, defined by $(\Psi_1{}^s, \Psi_2{}^s)$. We use the normalizing relation, $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$ to obtain the distribution of the angular velocity vector. The maximum and minimum $|\omega_{\mathcal{L}_1}|$ for $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$ are obtained from the eigenvalues of matrix $[g^s]$. If the eigenvalues of $[g^s]$ are λ_1 and λ_2 $(\lambda_1 > \lambda_2)$

$$\begin{aligned} |\omega_{\mathcal{L}_1}|_{\max} &= (\lambda_1)^{1/2} \\ |\omega_{\mathcal{L}_1}|_{\min} &= (\lambda_2)^{1/2} \end{aligned} \tag{33}$$

These maximum and minimum values occur when

$$\begin{aligned} (\delta)_1 &= (1/2)\tan^{-1}[2g_{12}^s/(g_{11}^s - g_{12}^s)] \\ (\delta)_2 &= (1/2)\tan^{-1}[2g_{12}^s/(g_{11}^s - g_{22}^s)] + \pi/2 \end{aligned} \tag{34}$$

where g_{ij}^s $(i, j = 1, 2)$ are the elements of the symmetric 2×2 matrix $[g^s]$, g_{ij}^s is the dot product $\Psi_i{}^s \cdot \Psi_j{}^s$.

As in [1], if at a given position we let $\dot{\theta}_1$ and $\dot{\theta}_2$ take all possible values (subject to $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$), the tip of the vector $\omega_{\mathcal{L}_1}$ can be shown to lie on an ellipse in $T_{\mathcal{L}_1}$ with major and minor axes along the directions corresponding to the eigenvectors. The area of the ellipse is $\pi(\lambda_1\lambda_2)^{1/2}$. For non-unit speed motions, $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$, the shapes of the ellipses remain the same, but the lengths of the major and minor axes and the areas of the ellipses are scaled up or down depending on k. The square of the geometric mean of maximum and minimum angular velocity, $(\lambda_1\lambda_2)^{1/2}$, can be taken as a measure of the mean transmission ratio between $\dot{\theta}_1$, $\dot{\theta}_2$ and $\omega^2_{\mathcal{L}_1}$ at the point under consideration. The ratio (λ_1/λ_2) determines the shape of the ellipse and gives the distribution of the angular velocity vector, $\omega_{\mathcal{L}_1}$, of the line $\mathcal{L}_1$ at the point under consideration. In Fig. 3, we plot the distribution of $\omega_{\mathcal{L}_1}$ for an example worked out in Section 5.2.

To study the first-order properties of the positional as well as the directional aspects of the trajectory, we start with the expression for p. From the first equation in (32), we get

$$p=\frac{g^*_{11}\dot{\theta}_1^2+2g^*_{12}\dot{\theta}_1\dot{\theta}_2+g^*_{22}\dot{\theta}_2^2}{(g^s_{11}\dot{\theta}_1^2+2g^s_{12}\dot{\theta}_1\dot{\theta}_2+g^s_{22}\dot{\theta}_2^2)} \tag{35}$$

In equation (35), g^*_{11}, g^*_{12}, g^*_{22} denote $(\mathbf{\Psi}_1{}^* \cdot \mathbf{\Psi}_1{}^s)$, $(1/2)(\mathbf{\Psi}_1{}^* \cdot \mathbf{\Psi}_2{}^s + \mathbf{\Psi}_2{}^* \cdot \mathbf{\Psi}_1{}^s)$ and $(\mathbf{\Psi}_2{}^* \cdot \mathbf{\Psi}_2{}^s)$, respectively.

The sign of p is determined by the sign of the numerator, the denominator $(\omega^2_{\mathcal{L}_1})$ being always positive. The numerator is a quadratic form of the associated matrix $[g^*]$ with elements g^*_{ij}. If $[g^*]$ is positive-definite then p is positive and if $[g^*]$ is negative-definite then p is negative. The pitch is zero when the numerator of equation (35) is zero. This happens when

$$(\dot{\theta}_2/\dot{\theta}_1)=\frac{-g^*_{12}\pm(g^{*}_{12}{}^2-g^*_{11}g^*_{22})^{1/2}}{g^*_{22}} \tag{36}$$

We can see, from (36), that whenever $\det[g^*]$ is less than or equal to zero, there can be two real values of $(\dot{\theta}_2/\dot{\theta}_1)$. Since, in general, $\det[g^*]$ is a function of θ_1 and θ_2, there is a region in the (θ_1, θ_2) space where p can be zero. Furthermore, for any particular $(\dot{\theta}_2/\dot{\theta}_1)$, the values of (θ_1, θ_2), which satisfy equation (36), determine a curve in the (θ_1,θ_2) space. For each point on the curve, there is a position of $\mathcal{L}_1$ in the fixed space where it has instantaneously zero linear velocity. Taken together, all these positions of $\mathcal{L}_1$ form a ruled surface in the fixed space. This ruled surface is the locus of points in the line congruence of $\mathcal{L}_1$ where $\mathcal{L}_1$ is instantaneously undergoing a simple rotation.

In general, for a constant pitch, say p_0, we have $(\dot{\theta}_1, \dot{\theta}_2)([g^*] - p_0[g^s])(\dot{\theta}_1, \dot{\theta}_2)^T = 0$. For constant pitch, the angular and the linear velocities are proportional; for example, when p is equal to one, the linear and angular velocities are equal. We can solve for $(\dot{\theta}_2/\dot{\theta}_1)$ and we get

$$(\dot{\theta}_2/\dot{\theta}_1)=\frac{-(p_0g^s_{12}-g^*_{12})\pm[(p_0g^s_{12}-g^*_{12})^2-(p_0g^s_{11}-g^*_{11})(p_0g^s_{22}-g^*_{22})]^{1/2}}{(p_0g^s_{22}-g^*_{22})} \tag{37}$$

The g^*_{ij}'s also depend on the line $\mathcal{L}_1$ chosen in the rigid body. For every position of the rigid body, and any fixed value $(\dot{\theta}_2/\dot{\theta}_1)$, the lines in the rigid body satisfying equation (36), in general, form a line complex in the moving body. All the lines of the complex are undergoing a simple rotation at that instant. This fact and analogous results for p equal to other constants can be used to classify the lines in the rigid body.

In general, p as a function of $\dot{\theta}_2/\dot{\theta}_1$, is a rational cubic. Using $x_2 = \dot{\theta}_2/\dot{\theta}_1$ in equation (35) the rational cubic may be written as

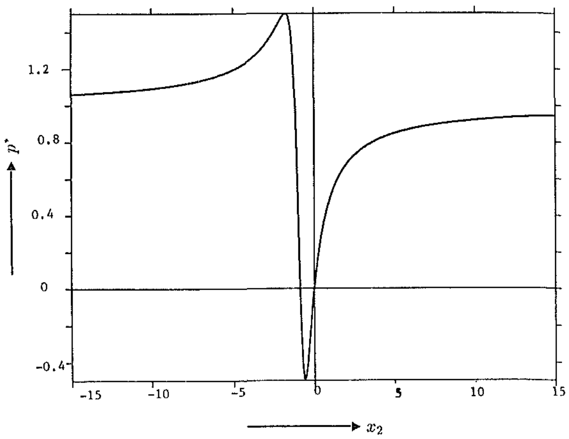

Fig. 4 Plot of p versus x_2

$$p=\frac{g^*_{11}+2g^*_{12}x_2+g^*_{22}x_2^2}{g^s_{11}+2g^s_{12}x_2+g^s_{22}x_2^2} \tag{38}$$

The magnitude of p is bounded, when $x_2 \to \infty$, $p = g^*_{22}/g^s_{22}$ and when $x_2 = 0$, $p = g^*_{11}/g^s_{11}$. The minimum and maximum value of p may be obtained by solving $\partial p/\partial x_2 = 0$. The values of x_2 for maximum and minimum p are

$$(x_2)_{1,2}=\frac{-(g^*_{11}g^s_{22}-g^*_{22}g^s_{11})\pm[(g^*_{11}g^s_{22}-g^*_{22}g^s_{11})^2-4(g^*_{11}g^s_{12}-g^*_{12}g^s_{11})(g^*_{12}g^s_{22}-g^s_{12}g^*_{22})]^{1/2}}{2(g^*_{12}g^s_{22}-g^s_{12}g^*_{22})} \tag{39}$$

The curve described in equation (38) can be plotted for known values of the g^s_{ij}'s and the g^*_{ij}'s. In Fig. 4, we give the plot for the example given in Section 5.2.

The center, $\mathbf{r}_c$, of the line $\mathcal{L}_1$ for the two-degrees-of-freedom motion under consideration, is determined by p^*; p^* is given by equation (32) which can be rewritten as

$$p^*=\frac{g^c_{11}+2g^c_{12}x_2+g^c_{22}x_2^2}{g^s_{11}+2g^s_{12}x_2+g^s_{22}x_2^2} \tag{40}$$

where

$$g^c_{11}=\mathbf{\Psi}_1{}^*\cdot(\mathbf{S}_1\times\mathbf{\Psi}_1{}^s)$$
$$g^c_{12}=(1/2)[\mathbf{\Psi}_1{}^*\cdot(\mathbf{S}_1\times\mathbf{\Psi}_2{}^s)+\mathbf{\Psi}_2{}^*\cdot(\mathbf{S}_1\times\mathbf{\Psi}_1{}^s)] \tag{41}$$
$$g^c_{22}=\mathbf{\Psi}_2{}^*\cdot(\mathbf{S}_1\times\mathbf{\Psi}_2{}^s)$$

p^* and hence the location of the center changes as x_2 is varied. p^* as a function of x_2 is a rational cubic; p^* is never infinity and hence p^* and $[\mathbf{r}_c]$ are bounded. The maximum and minimum values of p^* determine the extreme values of $\mathbf{r}_c$. p^* is maximum and minimum when the values of x_2 are given by

$$(x_2)_{1,2}=\frac{-(g^c_{11}g^s_{22}-g^c_{22}g^s_{11})\pm[(g^c_{11}g^s_{22}-g^c_{22}g^s_{11})^2-4(g^c_{11}g^s_{12}-g^c_{12}g^s_{11})(g^c_{12}g^s_{22}-g^s_{12}g^c_{22})]^{1/2}}{2(g^c_{12}g^s_{22}-g^s_{12}g^c_{22})} \tag{42}$$

The length of the segment of the line $\mathcal{L}_1$ on which $\mathbf{r}_c$ can lie is given by $p^*_{\max}-p^*_{\min}$. In Fig. 5, we plot (40) for the example in Section 5.2.

Since p, p^*, and $\omega_{\mathcal{L}_1}$ are bounded (for $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$), the tip of the velocity vector $\mathbf{v}_{\mathcal{L}_1}$ at any point in the line trajectory will describe a closed curve in space. If we visualize the vectors $\mathbf{v}_{\mathcal{L}_1}$ as arrows, their tails are attached at right angles to the line, $\mathcal{L}_1$, someplace between the extreme values of $\mathbf{r}_c$, and as $\dot{\theta}_1$, $\dot{\theta}_2$ vary, their tips lie on a space curve. The maximum and

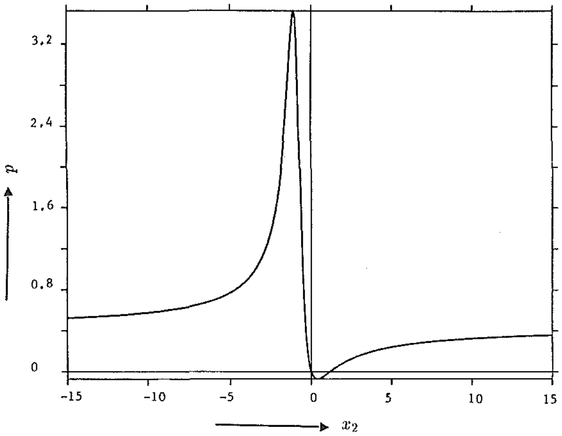

Fig. 5 Plot of p^* versus x_2

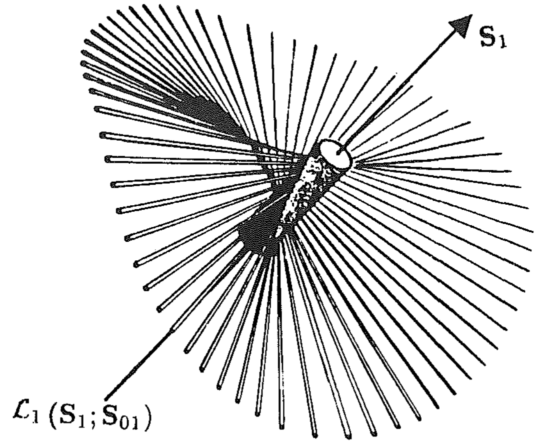

Fig. 6 Locus of the tip of $\mathbf{v}_{\mathcal{L}_1}$

minimum value of $|\mathbf{v}_{\mathcal{L}_1}|$ for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ will depend on the maximum and minimum value of p and $|\omega_{\mathcal{L}_1}|$. In general, the maximum and minimum of $|\mathbf{v}_{\mathcal{L}_1}|$ and $|\omega_{\mathcal{L}_1}|$ will not coincide. If we do not restrict $\omega_{\mathcal{L}_1}$ by $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$, the vector $\mathbf{v}_{\mathcal{L}_1}$ lies along the rulings of a ruled surface where each ruling is perpendicular to $\mathcal{L}_1$. The ruled surface extends between $(\mathbf{r}_c)_{\max}$ and $(\mathbf{r}_c)_{\min}$. This ruled surface is the well-known cylindroid [8, 11]. In Fig. 6, we show a schematic drawing of the space curve and the ruled surface.

The second-order properties of the line trajectory associated with the orientation of the line are obtained by differentiating $\omega_{\mathcal{L}_1}$. The angular acceleration is given by

$$\alpha_{\mathcal{L}_1} = (\mathbf{S}_1 \times \mathbf{\Psi}_1{}^s)\ddot{\theta}_1 + (\mathbf{S}_1 \times \mathbf{\Psi}_2{}^s)\ddot{\theta}_2 + (\mathbf{S}_1 \times \mathbf{\Psi}_{11}{}^s)\dot{\theta}_1^2 + 2(\mathbf{S}_1 \times \mathbf{\Psi}_{12}{}^s)\dot{\theta}_1\dot{\theta}_2 + (\mathbf{S}_1 \times \mathbf{\Psi}_{22}{}^s)\dot{\theta}_2^2 \quad (43)$$

The angular acceleration vector $\alpha_{\mathcal{L}_1}$ lies in the plane, $T_{\mathcal{L}_1}$, formed by $\mathbf{\Psi}_1{}^s$ and $\mathbf{\Psi}_2{}^s$. The components of the angular acceleration vector along $\mathbf{S}_1 \times \mathbf{\Psi}_1{}^s$ and $\mathbf{S}_1 \times \mathbf{\Psi}_2{}^s$ can be written as

$$\alpha_{\mathcal{L}_1} \cdot (\mathbf{S}_1 \times \mathbf{\Psi}_1{}^s) = g_{11}^s\ddot{\theta}_1 + g_{12}^s\ddot{\theta}_2 + \sum_{i,j=1}^{2} (\mathbf{\Psi}_{ij}{}^s \cdot \mathbf{\Psi}_1{}^s)\dot{\theta}_i\dot{\theta}_j$$
$$\alpha_{\mathcal{L}_1} \cdot (\mathbf{S}_1 \times \mathbf{\Psi}_2{}^s) = g_{12}^s\ddot{\theta}_1 + g_{22}^s\ddot{\theta}_2 + \sum_{i,j=1}^{2} (\mathbf{\Psi}_{ij}{}^s \cdot \mathbf{\Psi}_2{}^s)\dot{\theta}_i\dot{\theta}_j \quad (44)$$

Setting the left sides of equations (44) equal to zero, we get two differential equations in θ_1 and θ_2. A solution to these gives a one-degree-of-freedom time-dependent motion. Under this motion the line used to specify the coefficients in (44), will generate a trajectory with zero angular acceleration throughout the motion. These equations are similar to those for the tangential components of the acceleration of a point moving in a plane with two degrees of freedom. The coefficients $\mathbf{\Psi}_{ij}{}^s \cdot \mathbf{\Psi}_k{}^s$ determine the geometric second-order properties of the angular part of the motion of the line due to the two-degrees-of-freedom motion.

We define a "characteristic area" for the orientation part of the motion as

$$\mathcal{A} = |\mathbf{\Psi}_1{}^s \times \mathbf{\Psi}_2{}^s| = (g_{11}^s g_{22}^s - (g_{12}^s)^2)^{1/2} \quad (45)$$

If λ_1 and λ_2 are the eigenvalues of $[g^s]$, the characteristic area, given by $(\lambda_1\lambda_2)^{1/2}$, is a measure of the mean transmission ratio, or effectiveness, regarding the transformation of the input $(\dot{\theta}_1, \dot{\theta}_2)$ into the square of the angular velocity, $\omega_{\mathcal{L}_1}^2$. The gradient of $\mathcal{A}$ gives the direction and the magnitude of the maximum rate of change of this measure.

The linear acceleration of the line $\mathcal{L}_1$ can be written as

$$\mathbf{a}_{\mathcal{L}_1} = \dot{p}\omega_{\mathcal{L}_1} + p\dot{\omega}_{\mathcal{L}_1} \quad (46)$$

The second term on the right-hand side is a multiple of the angular acceleration. The first term depends on $\dot{p}$. If $\dot{p} = 0$, i.e., if p is constant, the acceleration of the line is completely along $\alpha_{\mathcal{L}_1}$ and is a multiple of the angular acceleration. In particular, the cases $p = 0$ and $p = 1$ which correspond, respectively, to the line $\mathcal{L}_1$ simply rotating and $\mathcal{L}_1$ moving with equal angular and linear velocity have been dealt with before.

Next, we look at the case when $\mathbf{a}_{\mathcal{L}_1} = 0$. In such a situation the line $\mathcal{L}_1$ has constant linear velocity. $\mathbf{a}_{\mathcal{L}_1} = 0$ if

$$\omega_{\mathcal{L}_1}\dot{p} = -p\dot{\omega}_{\mathcal{L}_1} \quad (47)$$

The aforementioned vector equation can be written as two scalar equations by taking components along $\mathbf{S}_1 \times \mathbf{\Psi}_1{}^s$ and $\mathbf{S}_1 \times \mathbf{\Psi}_2{}^s$. The solutions to these nonlinear differential equations will give a curve in the (θ_1, θ_2) space. This curve maps to a special ruled surface in the line trajectory—any time $\mathcal{L}_1$ coincides with a line in this ruled surface it has zero linear acceleration (provided $\dot{\theta}_1$ and $\dot{\theta}_2$ are in accordance with a solution of the differential equation).

The other second-order property of interest is the velocity of the center. The velocity of the center is given in equation (15) and can be rewritten as

$$\mathbf{v}_c = \dot{\mathbf{r}}_c = (\mathbf{r}_c)_1\dot{\theta}_1 + (\mathbf{r}_c)_2\dot{\theta}_2 + (\partial p^*/\partial x_2)\dot{x}_2\mathbf{S}_1 \quad (48)$$

In equation (48), $(\mathbf{r}_c)_1$ and $(\mathbf{r}_c)_2$ are the partial derivatives of $\mathbf{r}_c$ with respect to the parameters θ_1 and θ_2, and x_2 is $(\dot{\theta}_2/\dot{\theta}_1)$. The component of velocitiy of the center in $T_{\mathcal{L}_1}$ is independent of $\ddot{\theta}_1$ and $\ddot{\theta}_2$. As $\dot{\theta}_1$ and $\dot{\theta}_2$ are changed according to the relation $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$, the tip of the vector $\dot{\mathbf{r}}_c \times \mathbf{S}_1$ describes an ellipse in $T_{\mathcal{L}_1}$.

The component $\mathbf{v}_c$ along the line, $\mathcal{L}_1$, depends on the derivatives of p^* with respect to $(\dot{\theta}_2/\dot{\theta}_1)$, θ_1, θ_2, $\dot{\theta}_1$ and $\dot{\theta}_2$. It can be written as

$$\mathbf{v}_c \cdot \mathbf{S}_1 = \frac{\partial p^*}{\partial x^2}\dot{x}_2 + \sum_{i=1}^{2} \frac{\partial p^*}{\partial \theta_i}\dot{\theta}_i + \sum_{i=1}^{2} (\mathbf{S}_{01} \cdot \mathbf{S}_1 \times \mathbf{\Psi}_i{}^s\dot{\theta}_i) \quad (49)$$

For a given x_2 and $\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 = 1$, i.e., for a set of values of $\dot{\theta}_2$ and $\dot{\theta}_1$, and a chosen line $\mathcal{L}_1$ in the rigid body, $\mathbf{v}_c \cdot \mathbf{S}_1 = 0$ only if g_{11}^c, g_{12}^c, g_{22}^c, etc., take on appropriate values. Since g_{11}^c, g_{12}^c, and g_{22}^c, etc., are functions of θ_i, $i = 1, 2$, $\mathbf{v}_c \cdot \mathbf{S}_1 = 0$ gives a curve in the (θ_1, θ_2) space which maps to a ruled surface in the line trajectory. This special ruled surface is the locus of $\mathcal{L}_1$ where its velocity of the center along the direction $\mathbf{S}_1$ is equal to zero.

When p and $\mathbf{v}_c \cdot \mathbf{S}_1$ are both zero, we get two differential equations in θ_1 and θ_2. The solution of these differential equations gives a curve in the (θ_1, θ_2) space and the corresponding ruled surface in the line trajectory. This special ruled surface is generated by the line $\mathcal{L}_1$ simply rotating in such a way that its center point does not change.

Instead of $p = 0$, we may have $\dot{p} = 0$ and $\mathbf{v}_c \cdot \mathbf{S}_1 = 0$. Again, in this case, we will get a ruled surface in the line trajectory. This special ruled surface is generated by $\mathcal{L}_1$ moving so that its velocity of the center along $\mathbf{S}_1$ is zero and its linear acceleration is a multiple of its angular acceleration vector. It is analogous to a curve on a surface along which a point has zero tangential acceleration.

The second-order properties of the line trajectory due to two-degrees-of-freedom motions are determined by the coefficients $\mathbf{\Psi}_{ij}^s \cdot \mathbf{\Psi}_k^s$, $\mathbf{\Psi}_{ij}^s \cdot \mathbf{S}_1 \times \mathbf{\Psi}_k^s$, $i, j, k = 1, 2$ and the first partial derivatives of p and p^* with respect to θ_1, θ_2 and $\dot{\theta}_2/\dot{\theta}_1$. The third-order properties of the line trajectory are determined by the second partial derivatives of p, p^*, and the third partial derivatives of $\mathbf{S}_1$.

5.2 An Example—The 2R Manipulator

Figure 2 shows a rigid body attached to a 2R linkage chain. We are interested in the motion of a line $\mathcal{L}_1$ whose *Plücker* vectors in the reference frame $OXYZ$ are $(\mathbf{S}_1; \mathbf{S}_{01})$ and whose *Plücker* vectors in the reference frame $(\mathbf{a}_{23}, S_2 \times \mathbf{a}_{23}, S_2)$ attached to the moving body at $\mathbf{p}(x, y, z)$, have coordinates $(l, m, n; p, q, r)$. The kinematic equations, $\mathbf{\Psi}$: $(\theta_1, \theta_2) \rightarrow (\mathbf{S}_1; \mathbf{S}_{01})$, of motion can be written as

$$\mathbf{S}_1 = l\mathbf{a}_{23} + m(S_2 \times \mathbf{a}_{23}) + nS_2 \tag{50}$$

and

$$\mathbf{S}_{01} = (x,y,z) \times \mathbf{S}_1 + p\mathbf{a}_{23} + q(S_2 \times \mathbf{a}_{23}) + rS_2 \tag{51}$$

$(l, m, n; p, q, r)$ depends on the choice of the line $\mathcal{L}_1$ in the moving rigid body. For the sake of simplicity, we will take the line $\mathcal{L}_1$ along $\mathbf{a}_{23}$ through the point $\mathbf{p}(x, y, z)$, i.e., $(l, m, n; p, q, r) = (1, 0, 0; 0, 0, 0)$. Now since[4]

$$(x,y,z) = S_{11}(0,0,1)^T + a_{12}(c_1,s_1,0)^T + S_{22}(s_1s_{12},c_1s_{12},c_{12})^T + a_{23}\mathbf{a}_{23}$$

$$\mathbf{a}_{23} = (c_1c_2 - s_1s_2c_{12}, s_1c_2 + c_1s_2c_{12}, s_{12}s_2)^T \tag{53}$$

we get

$$\mathbf{S}_1 = [(c_1c_2 - s_1s_2c_{12}),(s_1c_2 + c_1s_2c_{12}),s_{12}s_2]^T$$

$$\mathbf{S}_{01} = S_{11}\begin{bmatrix} -s_1c_2 - c_1s_2c_{12} \\ c_1c_2 - s_1s_2c_{12} \\ 0 \end{bmatrix} + a_{12}\begin{bmatrix} s_1s_2s_{12} \\ -c_1s_2s_{12} \\ s_2c_{12} \end{bmatrix} + S_{22}\begin{bmatrix} -c_1s_2 - s_1c_2c_{12} \\ -s_1s_2 + c_1c_2c_{12} \\ c_2s_{12} \end{bmatrix} \tag{53}$$

The coefficients g_{11}^s, g_{12}^s, g_{22}^s which make up the symmetric matrix $[g^s]$ are

$$\begin{aligned} g_{11}^s &= c_2^2 + s_2^2c_{12}^2 \\ g_{12}^s &= c_{12} \\ g_{22}^s &= 1 \end{aligned} \tag{54}$$

In Fig. 3, we show the angular velocity ellipse for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ at $(\theta_1, \theta_2) = (0, 0)°$ and $\alpha_{12} = 30°$. The ellipse is in the plane $T_{\mathcal{L}_1}$ formed by $\mathbf{\Psi}_1^s$ and $\mathbf{\Psi}_2^s$ with normal direction along $\mathbf{S}_1$. The eigenvalues of $[g^s]$ are $(2 + \sqrt{2})/2$ and $(2 - \sqrt{2})/2$ and the maximum and minimum magnitude of the angular velocity are the square root of the eigenvalues.

Next, we evaluate the quantities g_{11}^*, g_{12}^*, g_{22}^* and the quantities g_{11}^c, g_{12}^c, g_{22}^c. The expressions for g_{11}^* etc., are

[4] $\cos(\theta_i)$, $\sin(\theta_i)$, $(i = 1, 2)$; $\cos(\alpha_{12})$, $\sin(\alpha_{12})$ are abbreviated as c_i, s_i, $(i = 1, 2)$; c_{12}, s_{12}, respectively.

$$\begin{aligned} g_{11}^* &= -a_{12}s_2^2c_{12}s_{12} - S_{22}c_2s_2s_{12}^2 \\ g_{12}^* &= -(1/2)a_{12}s_{12} \\ g_{22}^* &= a_{12}(c_2^2 + c_2s_2)c_{12}s_{12} \\ g_{11}^c &= S_{11}s_2s_{12}c_{12}(c_2^2 + s_2^2c_{12}^2) - a_{12}c_2s_2s_{12}^2 + S_{22}s_2c_{12}s_{12} \\ g_{12}^c &= S_{11}s_2s_{12}c_{12} + (1/2)(a_{12}c_2c_{12} + S_{22}s_2s_{12}) \\ g_{22}^c &= S_{11}s_2s_{12} + a_{12}c_2 \end{aligned} \tag{55}$$

p and p^* are given by

$$\begin{aligned} p &= \frac{g_{11}^* + 2g_{12}^*x_{12} + g_{22}^*x_2^2}{c_2^2 + s_2^2c_{12}^2 + 2c_{12}x_2 + x_2^2} \\ p^* &= \frac{g_{11}^c + 2g_{12}^cx_2 + g_{22}^cx_2^2}{c_2^2 + s_2^2c_{12}^2 + 2c_{12}x_2 + x_2^2} \end{aligned} \tag{56}$$

We make the following general observations about the cubic curves in (56).

(1) Both p and p^* do not depend on θ_1. This is because p and p^* are differential invariants to rotation of the coordinate system $OXYZ$, and because changing θ_1 is the same as rotating the coordiante system. p is also invariant to translation of the origin and hence does not depend on S_{11}.

(2) For a given value of $\dot{\theta}_2/\dot{\theta}_1$, p and p^* are functions of θ_2. If we use the tangent half-angle substitution for s_2 and c_2, we will get a fifth-degree equation for p and p^* in terms of $\tan(\theta_2/2)$. Figure 4 shows the plot of p as a function of $\dot{\theta}_2/\dot{\theta}_1$ for the case $\alpha_{12} = 30°$, $a_{12} = S_{22} = a_{23} = 1$, and $(\theta_1, \theta_2) = (0, 0)°$.

(3) The center $\mathbf{r}_c$, of the line $\mathcal{L}_1$ is determined by p^*. In Fig. 5, we show a plot of p^* as a function of x_2. We use the same set of values for the linkage dimensions as for the plot of p. The maximum and minimum values of p^* with respect to $\dot{\theta}_2/\dot{\theta}_1$ can be found analytically from equation (56) and also from Fig. 5. They are approximately 1.5 and -0.5, respectively. $p_{max} - p_{min}$ is the range of possible positions of $\mathbf{r}_c$ on the line $\mathcal{L}_1$.

The linear velocity distribution of the line can be obtained from the distribution of p, p^*, and $\omega_{\mathcal{L}_1}$. In Fig. 6, we present a schematic drawing, showing the space curve traced by the tip of the vector $\mathbf{v}_{\mathcal{L}_1}$ for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ and the cylindroid composed of the lines parallel to $\mathbf{v}_{\mathcal{L}_1}$.

6.1 Line Trajectories Due to Three-Degrees-of-Freedom Motions

It is known that a line moving with three-degrees-of-freedom generates a line complex [9, 10]. Reference [9] deals extensively with linear complexes. Since the equations of the line trajectories generated by mechanisms are rarely linear, the results in [9] are of not much use. In this section, we briefly present the first- and second-order properties of the line trajectory in terms of the quantities defined in Section 3.1.

The equations representing a three-degrees-of-freedom motion of a line can be written as

$$\mathbf{\Psi}:(\theta_1,\theta_2,\theta_3) \rightarrow (\mathbf{S}_1;\mathbf{S}_{01}) \tag{57}$$

The angular velocity, $\omega_{\mathcal{L}_1}$, the linear velocity, $\mathbf{v}_{\mathcal{L}_1}$, and the center of the line, $\mathbf{r}_c$, at a generic point corresponding to $(\theta_{1,0}, \theta_{2,0}, \theta_{3,0})$ can be written as

$$\begin{aligned} \omega_{\mathcal{L}_1} &= \sum_{i=1}^{3} (\mathbf{S}_1 \times \mathbf{\Psi}_i^s)\dot{\theta}_i \\ \mathbf{v}_{\mathcal{L}_1} &= p\omega_{\mathcal{L}_1} \\ \mathbf{r}_c &= p^*\mathbf{S}_1 - \mathbf{S}_{01} \times \mathbf{S}_1 \end{aligned} \tag{58}$$

where p and p^* are

$$p = \frac{\dot{\mathbf{S}}_{01} \cdot \dot{\mathbf{S}}_1}{\dot{\mathbf{S}}_1^2}, \quad p^* = \frac{\dot{\mathbf{S}}_{01} \cdot \omega_{\mathcal{L}_1}}{\omega_{\mathcal{L}_1}^2} \tag{59}$$

The angular velocity, $\omega_{\mathcal{L}_1}$, lies in the plane $T_{\mathcal{L}_1}$. Hence, out of the three $(\mathbf{S}_1 \times \mathbf{\Psi}_i^s)\dot{\theta}_i$'s only two are independent. If we are only interested in the line's orientation, during the motion, and wish to utilize the redundant parameter to alter the distribution of $\omega_{\mathcal{L}_1}$, we can proceed in the following manner. We write one of the vectors, say $(\mathbf{S}_1 \times \mathbf{\Psi}_3^s)\dot{\theta}_3$, as a linear combination of the other two:

$$(\mathbf{S}_1 \times \mathbf{\Psi}_3^s)\dot{\theta}_3 = \sum_{i=1}^{2} \alpha_i (\mathbf{S}_1 \times \mathbf{\Psi}_i^s \dot{\theta}_i) \tag{60}$$

The angular velocity can then be written as

$$\omega_{\mathcal{L}_1} = \sum_{i=1}^{2} (1+\alpha_i)(\mathbf{S}_1 \times \mathbf{\Psi}_i^s)\dot{\theta}_i \tag{61}$$

α_1 and α_2 can be found easily from equation (60) as

$$\begin{aligned} \alpha_1 &= \frac{[(\mathbf{\Psi}_3^s \cdot \mathbf{\Psi}_1^s) g_{22}^s - (\mathbf{\Psi}_3^s \cdot \mathbf{\Psi}_2^s) g_{12}^s]\dot{\theta}_3}{(g_{11}^s g_{22}^s - g_{12}^{s\,2})\dot{\theta}_1} \\ \alpha_2 &= \frac{[(\mathbf{\Psi}_3^s \cdot \mathbf{\Psi}_1^s) g_{12}^s - (\mathbf{\Psi}_3^s \cdot \mathbf{\Psi}_2^s) g_{11}^s]\dot{\theta}_3}{(g_{11}^s g_{22}^s - g_{12}^{s\,2}]\dot{\theta}_2} \end{aligned} \tag{62}$$

α_1 and α_2 can be evaluated so long as the denominators and the numerators of (62) are not zero. $(g_{11}^s g_{22}^s - g_{12}^{s\,2}) = 0$ and $[(\mathbf{\Psi}_3^s \cdot \mathbf{\Psi}_1^s) g_{22}^s - (\mathbf{\Psi}_3^s \cdot \mathbf{\Psi}_2^s) g_{12}^s] = 0$ define a region in the line trajectory where α_1 and α_2 are indeterminate and the distribution of $\omega_{\mathcal{L}_1}$ cannot be altered.

To find $\dot{\theta}_3$, we need to specify the required distribution of $\omega_{\mathcal{L}_1}$. For example, if we want the distribution to be a circle (magnitude of $\omega_{\mathcal{L}_1}$ equal in all directions as $\dot{\theta}_1$ and $\dot{\theta}_2$ are varied, subject to $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$), we may use equations resulting from forcing the eigenvalues of the matrix $[g^{s\prime}]$ to be equal. The elements of $[g^{s\prime}]$ are, respectively, $g_{11}^{s\prime} = (1 + \alpha_1)^2 g_{11}^s$, $g_{12}^{s\prime} = (1 + \alpha_1)(1 + \alpha_2) g_{12}^s$, and $g_{22}^{s\prime} = (1 + \alpha_2)^2 g_{22}^s$. If we use this computed value of θ_3 we will get the distribution of $\omega_{\mathcal{L}_1}$ described by a circle, but the translational motion of the line will be completely determined.

Alternatively, we could get the additional equations in $\dot{\theta}_3$ from the equations governing the translational motion of the line. In [8], we present equations resulting from considering the translational aspects of the motion.

The first-order properties of the complete motion, involve $\omega_{\mathcal{L}_1}$, p, and p^*. We start with p. From equation (59), the pitch p may be rewritten as a function of the ratios $\dot{\theta}_2/\dot{\theta}_1$ and $\dot{\theta}_3/\dot{\theta}_1$ (denoted for convenience as x_2 and x_3)

$$p = \frac{g_{11}^* + 2g_{12}^* x_2 + 2g_{13}^* x_3 + g_{22}^* x_2^2 + 2g_{23}^* x_2 x_3 + g_{33}^* x_3^2}{g_{11}^s + 2g_{12}^s x_2 + 2g_{13}^s x_3 + g_{22}^s x_2^2 + 2g_{23}^s x_2 x_3 + g_{33}^s x_3^2} \tag{63}$$

where

$$\begin{aligned} g_{11}^* &= \mathbf{\Psi}_1^* \cdot \mathbf{\Psi}_1^s \\ g_{12}^* &= (1/2)(\mathbf{\Psi}_1^* \cdot \mathbf{\Psi}_2^s + \mathbf{\Psi}_2^* \cdot \mathbf{\Psi}_1^s) \\ g_{13}^* &= (1/2)(\mathbf{\Psi}_1^* \cdot \mathbf{\Psi}_3^s + \mathbf{\Psi}_3^* \cdot \mathbf{\Psi}_1^s) \\ g_{22}^* &= \mathbf{\Psi}_2^* \cdot \mathbf{\Psi}_2^s \\ g_{23}^* &= (1/2)(\mathbf{\Psi}_2^* \cdot \mathbf{\Psi}_3^s + \mathbf{\Psi}_3^* \cdot \mathbf{\Psi}_2^*) \\ g_{33}^* &= \mathbf{\Psi}_3^* \cdot \mathbf{\Psi}_3^s \end{aligned} \tag{64}$$

and

$$\begin{aligned} g_{11}^s &= \mathbf{\Psi}_1^s \cdot \mathbf{\Psi}_1^s, g_{12}^s = \mathbf{\Psi}_1^s \cdot \mathbf{\Psi}_2^s, g_{13}^s = \mathbf{\Psi}_1^s \cdot \mathbf{\Psi}_3^s \\ g_{22}^s &= \mathbf{\Psi}_2^s \cdot \mathbf{\Psi}_2^s, g_{23}^s = \mathbf{\Psi}_2^s \cdot \mathbf{\Psi}_3^s, g_{33}^s = \mathbf{\Psi}_3^s \cdot \mathbf{\Psi}_3^s \end{aligned} \tag{65}$$

Equation (63) describes a cubic surface in the (p, x_2, x_3) space. We can, in general, make the following observations about this surface:

(1) For a constant value of p, we have a conic in the (x_2, x_3)-plane. This conic can be an ellipse, parabola, or hyperbola depending on whether $(g_{23}^* - g_{23}^s)^2 - (g_{22}^* - g_{22}^s)(g_{33}^* - g_{33}^s)$ is greater than, equal to or less than zero. When the pitch is constant, the linear and the angular velocities are proportional. p is a constant, say p_0, when at any point $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ satisfy

$$(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)([g^*] - p_0[g^s])(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T = 0 \tag{66}$$

If we fix any two of the ratios, say x_2 and x_3, then equation (66) is a function of $(\theta_1, \theta_2, \theta_3)$ and represents a surface in the $(\theta_1, \theta_2, \theta_3)$ space. The corresponding two-degrees-of-freedom line trajectory in the fixed reference frame consists of lines which have constant p.

(2) Since the denominator is never zero (except when all the $\dot{\theta}_i$'s are zero), p is bounded.

(3) For a given value of x_2 and x_3, the right-hand side of equation (63) is a function of θ_i, $i = 1, 2, 3$, and the mechanism parameters. Hence for a given value of p, say p equal to p_0, we get a surface in the $(\theta_1, \theta_2, \theta_3)$ space. The corresponding line trajectory in the fixed reference frame is a congruence. Also, at any given instant ($\theta_i(t)$ given) p equals p_0 is satisfied by a line complex in the rigid body.

p^*, which locates the center of the line, may be rewritten as

$$p^* = \frac{g_{11}^c + 2g_{12}^c x_2 + 2g_{13}^c x_3 + g_{22}^c x_2^2 + 2g_{23}^c x_2 x_3 + g_{33}^c x_3^2}{g_{11}^s + 2g_{12}^s x_2 + 2g_{13}^s x_3 + g_{22}^s x_2^2 + 2g_{23}^s x_2 x_3 + g_{33}^s x_3^2} \tag{67}$$

where

$$\begin{aligned} g_{11}^c &= \mathbf{\Psi}_1^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_1^s \\ g_{12}^c &= (1/2)(\mathbf{\Psi}_1^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_2^s + \mathbf{\Psi}_2^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_1^s) \\ g_{13}^c &= (1/2)(\mathbf{\Psi}_1^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_3^s + \mathbf{\Psi}_3^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_1^s) \\ g_{22}^c &= \mathbf{\Psi}_2^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_2^s \\ g_{23}^c &= (1/2)(\mathbf{\Psi}_2^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_3^s + \mathbf{\Psi}_3^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_2^s) \\ g_{33}^c &= \mathbf{\Psi}_3^* \cdot \mathbf{S}_1 \times \mathbf{\Psi}_3^s \end{aligned} \tag{68}$$

Equation (67) describes a cubic surface in the (p^*, x_2, x_3) space. The maximum and minimum values of p^* may be obtained by solving the equations obtained by setting the partial derivatives of p^* with respect to x_2 and x_3 equal to zero. This gives us the range along the line $\mathcal{L}_1$, over which the center may lie. With the foregoing facts in mind, we can now visualize the distribution of $\mathbf{v}_{\mathcal{L}_1}$ for $\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 = 1$.

For a particular p^*, i.e., a known center of the line, we have a range of values of p and $\omega_{\mathcal{L}_1}$. The tail of the vector $\mathbf{v}_{\mathcal{L}_1}$ lies in the plane $T_{\mathcal{L}_1}$ at the point located by $\mathbf{r}_c$. The tip lies on a curve in the same plane. Hence, as p^* varies between $p^*_{\max}$ and $p^*_{\min}$, the tip lies on a surface in three-space. If we assume nonunit speed motions; viz., $\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 = k^2$, and vary k, the tip of $\mathbf{v}_{\mathcal{L}_1}$ lies in a solid.

The second-order properties involve the derivatives of $\omega_{\mathcal{L}_1}$, p and p^*. These in turn are determined by the coefficients $\mathbf{\Psi}_{ij}^s \cdot \mathbf{\Psi}_k^s$, $\mathbf{\Psi}_{ij}^s \cdot \mathbf{S}_1 \times \mathbf{\Psi}_k^s$, and the ten partial derivatives of p and p^* and the first and second rate of change of θ_i ($i = 1, 2, 3$) with respect to time. A more detailed discussion on the second-order properties is in [8]; however more work needs to be done.

7.1 Line Trajectories Due to Four-Degrees-of-Freedom Motions

Mathematically, a line trajectory due to a four-degrees-of-freedom motion can be represented as

$$\mathbf{\Psi}: (\theta_1, \theta_2, \theta_3, \theta_4) \rightarrow (\mathbf{S}_1; \mathbf{S}_{01}) \tag{69}$$

Without any constraint, the line trajectory fills the entire space. (This is analogous to the case of three-degrees-of-freedom motion of a point in $\mathbf{R}^3$). There are however constraints of finite link lengths and limited joint displacements in a manipulator. Hence, the line trajectory will be a *line solid* with a boundary. The equations for the angular velocity, the linear velocity and the center of the line $\mathcal{L}_1$ are similar to equations (58) and (59). The only difference is that the summation is from $i = 1$ to $i = 4$. Mathematically, the angular velocity vector is similar to the velocity of a point undergoing four-degrees-of-freedom motion in the plane. We give the following result and refer the reader to [8] for details of the development.

The vector $\omega_{\mathcal{L}_1}$ lies in the plane $T_{\mathcal{L}_1}$. Hence, out of the four $\mathbf{S}_1 \times \mathbf{\Psi}_i^s \dot{\theta}_i$, $i = 1, \ldots, 4$, only two can be independent. We can write the other two as a linear combination of the two independent $\mathbf{S}_1 \times \mathbf{\Psi}_i^s \dot{\theta}_i$'s. The four coefficients α_{ji}, $i, j = 1, 2$, can be found in terms of the $\dot{\theta}_i$'s, g_{11}^s, g_{12}^s, and g_{22}^s. By making appropriate use of the two free choices, the shape and size of the ellipse can be controlled except in certain regions.

The direction of $\omega_{\mathcal{L}_1}$, and the values of p and p^* depend only on the three ratios ($\dot{\theta}_i/\dot{\theta}_1$, $i = 2, 3, 4$). However, the magnitude of $\omega_{\mathcal{L}_1}$ depends on all four $\dot{\theta}_i$, $i = 1, \ldots, 4$. p and p^*, are bounded. We have the following description for the motion of the line up to first-order for $|\dot{\mathbf{\Theta}}| = 1$:

For particular $\dot{\theta}_i/\dot{\theta}_1$, $i = 2, 3, 4$, the direction of $\omega_{\mathcal{L}_1}$, and the scalars p and p^* are determined. On the line $\mathcal{L}_1$ we can locate the center (using p^*) and draw a line perpendicular to $\mathcal{L}_1$ along the direction of the vector $\omega_{\mathcal{L}_1}$. The tips of the linear and angular velocity vectors lie on this line. As we vary the ratios, $\dot{\theta}_i/\dot{\theta}_1$, we get different lines at new locations. All these lines lie between two points obtained from the maximum and minimum values of p^*. For a given direction and p^*, p varies between a maximum and minimum. The tip of the linear velocity vector lies on a bounded solid and the tip of the angular velocity vector lies on an ellipse in the plane $T_{\mathcal{L}_1}$. If we vary k, the tips of the linear velocity vectors describe a family of solids.

The second-order properties of a line trajectory due to four-degrees-of-freedom motion have not been worked out in detail. We can, however, say that they would depend on the second derivatives of $\mathbf{\Psi}^s$ with respect to the motion parameters and the derivatives of p and p^* with respect to both the parameters θ_i, $i = 1, \ldots, 4$ and the ratios $\dot{\theta}_i/\dot{\theta}_1$, $i = 2, 3, 4$. Since $\mathbf{\Psi}_{ij}^s = \mathbf{\Psi}_{ji}^s$, we have in general 54 quantities which determine the second-order properties of the line trajectory due to four-degrees-of-freedom motions.

8.1 Line Trajectories Due to *m*-Degrees-of-Freedom Motions

Line trajectories in $\mathbf{R}^3$ due to five- or higher-degrees-of-freedom motions are analogous to point trajectories in $\mathbf{R}^3$ due to four- or higher-degrees-of-freedom motions. In the case of mechanisms with m-degrees-of-freedom, the inverse kinematic solution is not unique. The dimension of the solution space is $m - 4$. The line trajectory can be thought of as a *line solid* with a boundary.

The angular velocity vector $\omega_{\mathcal{L}_1} = \mathbf{S}_1 \times \mathrm{J}(\mathbf{\Psi}^s)\dot{\mathbf{\Theta}}$ at a point lies in the plane $T_{\mathcal{L}_1}$. With a normalizing condition on $\dot{\mathbf{\Theta}}$, the tip of the angular velocity vector lies on an ellipse. Also, out of the m-terms, $\mathbf{S}_1 \times \mathbf{\Psi}_i^s \dot{\theta}_i$ $(i = 1, \ldots, m)$ making up the angular velocity vector, only two are independent. We can write $m - 2$ of these terms as linear combinations of the two independent ones. Then following the same procedure as in previous cases, the shape and size of the angular velocity ellipse can be altered by specifying the required constraints involving angular velocity, the pitch of the line, the center, and linear and angular accelerations, or a combination of these [8].

The first-order properties of the complete motion depend on the scalars, p, p^*, the direction (in the plane $T_{\mathcal{L}_1}$) and the magnitude of the angular velocity vector. For given values of these four quantities, and when m is greater than 4, the $\dot{\theta}_i$'s cannot be uniquely determined and we have $m - 4$ extra or redundant motion parameters. We can add $m - 4$ additional constraints to solve for all the $\dot{\theta}_i$'s and alter the first-order properties to our choice. We could, for example, change the ranges for p, p^* and alter the shape and size of the ellipse by appropriately using the $m - 4$ free choices. We could also alter the second-order properties (e.g., the angular acceleration of the line, its linear acceleration and the velocity of its center) of the motion of the line. As in the case of point trajectories [1, 8], there would be regions where such alterations are not possible.

9.1 Conclusion

A general framework has been presented to study the properties of trajectories of lines embedded in rigid bodies undergoing multi-degrees-of-freedom motions. Quantities such as the angular velocity, pitch, and the center of the line have been developed to characterize different trajectories generated by different multi-degrees-of-freedom mechanisms, to distinguish properties at different positions of the trajectory, and to classify the lines in the moving rigid body. The analysis was done by first studying only the orientation aspects of the motion and then by studying the actual motion. First-order properties were discussed in detail and some second-order properties were presented. In the case of redundant motion, criteria were presented for making use of the redundancy. Finally, the theory was used for the analysis of trajectories generated by a two-degrees-of-freedom open-loop manipulator containing revolute joints.

It is expected that the results in [1] and in this paper will give a better understanding of the kinematics of multi-degrees-of-freedom motions, and will facilitate the analysis, control and operation of multi-degrees-of-freedom mechanisms.

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