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An algebraic formulation of exact force-, moment-isotropy in spatial parallel manipulators

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Abstract—*In this paper, we present an algebraic method to study and design spatial parallel manipulators that demonstrate isotropy in the force and moment distributions. We use the force and moment transformation matrices separately, and derive conditions for their isotropy individually as well as in combination. The isotropy conditions are derived in closed-form in terms of the invariants of the quadratic forms associated with these matrices. The formulation has been applied to a class of Stewart platform manipulators. We obtain multi-parameter families of isotropic manipulator analytically. In addition to computing the isotropic configurations of an existing manipulator, we demonstrate a procedure for designing the manipulator for isotropy at a given configuration.*

I. Introduction

Isotropy is one of the common measures of performance of a manipulator. In the case of six-degrees-of-freedom (DOF) spatial manipulators, the term isotropy is generally used in the context of kinematics. However, in practice, the concept of twist-wrench duality is used to analyse the 6×6 wrench transformation matrix $\boldsymbol{H}$, to obtain conditions such that this matrix has identical singular values (see, e.g., [1]). A consequence of this approach is the concurrence of kinematic and static isotropy, where the later implies the ability of the manipulator end-effector to resist forces and moments *equally well* in *all spatial* directions. Among the spatial parallel manipulators, the Stewart platform manipulator (SPM) has been studied by several researchers for isotropy [1], [2], [3], [4]. However, to the best of our knowledge, no mechanically feasible, non-singular isotropic configuration has been obtained for a manipulator of this class. Further, it may be noted that the 3×6 submatrices of $\boldsymbol{H}$ pertaining to the force and moment parts have different physical dimensions for an SPM, therefore the physical significance of the singular values of $\boldsymbol{H}$ is not clear.

In this paper, we present a formulation for the study of static isotropy. Our approach is to analyse the above mentioned force and moment transformation matrices separately. We form the conditions for the force and moment isotropy in terms of algebraic equations involving the elements of the respective transformation matrices. We solve these equations in closed-form to obtain a multi-parameter family of kinematically valid configurations showing combined force and moment isotropy. We also present examples of isotropic configurations for an existing manipulator, as well as demonstrate the design for isotropy at a given configuration within the above mentioned family.

The paper is organised as follows: in section I, we present the general formulation of static isotropy of a spatial manipulator, followed by its application to the SRSPM in section III. In section IV, we present the analytical results, followed by a numerical example for an existing manipulator. In section V, we present the method of design for isotropy at a given configuration. Finally, in section VI, we conclude the paper.

II. Formulation

In this section, we derive the isotropy conditions of a general manipulator from its wrench transformation matrix. First we describe the formulation for obtaining the distributions of the force and moment resultants on the moving platform. We follow the approach presented in [5] in the context of the linear and angular velocity distributions of the moving platform. Using this approach, the said distributions are obtained from the solution of eigenproblems of two symmetric matrices. The conditions for force and moment isotropy are then derived in terms of algebraic equations involving the coefficients of the characteristic polynomials associated with the above eigenproblems. We assume in the following that the resultant force on the top platform, $\boldsymbol{F}$, and the corresponding moment (referred to the centre of the moving platform), $\boldsymbol{M}$, are available via linear maps of the actuator efforts (e.g., leg forces in the case of a platform-type parallel manipulator), $\boldsymbol{f}_i$.[1] Therefore we can write $\boldsymbol{F}$ and $\boldsymbol{M}$ in terms of the respective *equivalent*

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[1] Obtaining such a map is trivial for purely parallel manipulators. However, for hybrid manipulators, there can be significant difficulty in taking the reactions at the passive joints into account while computing the effect of the actuator effort on the end-effector.

transformation matrices:

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{H_F f} \\ \boldsymbol{M} &= \boldsymbol{H_M f} \end{aligned} \quad (1)$$

We analyse the properties of the above two linear maps using well know tools of linear algebra [6], [7]. This leads to the following eigenproblems respectively:

$$\boldsymbol{g_F f} = \lambda \boldsymbol{f} \quad (2)$$

$$\boldsymbol{g_M f} = \lambda_{\boldsymbol{M}} \boldsymbol{f} \quad (3)$$

where $\boldsymbol{g_F} = \boldsymbol{H_F^T H_F}$ and $\boldsymbol{g_M} = \boldsymbol{H_M^T H_M}$. These eigenproblems have the following characteristics:

- The eigenvalues λ,$\lambda_{\boldsymbol{M}}$ are real and nonnegative.
- At the most 3 of the eigenvalues are nonzero in each case, as the rank of $\boldsymbol{H_F}$ or $\boldsymbol{H_M}$ can not exceed three. Therefore, if $\dim(\boldsymbol{g_F}) = n$ with $n > 3$, *at least* $(n-3)$ eigenvalues of $\boldsymbol{g_F}$ are zeros– and the same applies to $\boldsymbol{g_M}$ as well.

The characteristic equation of $\boldsymbol{g_F}$ may be written with real coefficients a_i as:

$$0 = \begin{cases} \lambda^2 + a_1\lambda + a_2 & n = 2 \\ \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 & n = 3 \\ \lambda^{n-3}(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) & n > 3 \end{cases} \quad (4)$$

The characteristic equation of $\boldsymbol{g_M}$ has exactly the same form as above. However, we use the notations b_i for the coefficients, and $\lambda_{\boldsymbol{M}}$ for the eigenvalues in that case. From linear algebra isotropy of $\boldsymbol{H_F}$ and $\boldsymbol{H_M}$ imply, respectively:

$$\lambda_i = \|\boldsymbol{F}^*\|^2, \quad \lambda_{\boldsymbol{M}i} = \|\boldsymbol{M}^*\|^2 \quad i = 1, \ldots, n \quad (5)$$

where $(\cdot^*)$ indicates an extremal quantity. Under this condition, the *force ellipsoid*, (the ellipsoid corresponding to $\boldsymbol{F}$) to a sphere of radius $\|\boldsymbol{F}^*\|$. Similarly, the moment ellipsoid reduces to a sphere of radius $\|\boldsymbol{M}^*\|$. This implies that the nontrivial roots of equation (4) should be equal, and not all of a_1, a_2, a_3 can be zero.[2] The nontrivial roots of equation (4) can also be obtained explicitly in terms of a_i using Sridhar Acharya's and Cardan's formulæ for the quadratic and cubic cases respectively (see, e.g., [8]). However, it is not required to compute the roots explicitly in order to obtain the conditions for isotropy from their equality. Instead, those conditions can be easily formed as algebraic equations in the coefficients a_i etc. as follows. For the case $n = 2$, we equate the discriminant to zero and obtain the following condition:

$$a_1^2 - 4a_2 = 0 \quad (6)$$

For the case $n \geq 3$, we consider the nontrivial cubic part of equation (4):

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \quad (7)$$

[2]It may be noted here that the coefficients a_i, b_i can be computed in closed form using Newton identities [6], [7].

Using the standard transformation $\lambda = z - \frac{a_1}{3}$, the quadratic term may be removed to obtain the cubic in the form: $z^3 + Pz + Q = 0$ (see, e.g., [8]). It is obvious that if $P = Q = 0$, then $z = 0$, and hence equation (7) has the repeated roots $\lambda_i = -\frac{a_1}{3}$, $i = 1, 2, 3$. In terms of the coefficients of the original cubic equation (7), the conditions for equal roots are:

$$a_2 - \frac{a_1^2}{3} = 0, \quad \frac{2a_1^3}{27} - \frac{a_1 a_2}{3} + a_3 = 0 \quad (8)$$

Further, if we solve the above equations *exactly* in symbolic form, then the second of them can be simplified using the first, yielding the pair of equations below:

$$3a_2 - a_1^2 = 0, \quad 27a_3 - a_1^3 = 0 \quad (9)$$

The conditions for moment isotropy can be obtained in the same fashion. In the following, we list down the conditions for the different types of isotropy considered in this paper.

A. Force ($\boldsymbol{F}$-isotropy): $\boldsymbol{H_F}$ is isotropic.

$$\begin{aligned} &a_1^2 - 4a_2 = 0, && n = 2 \\ &\left.\begin{aligned} 3a_2 - a_1^2 &= 0 \\ 27a_3 - a_1^3 &= 0 \end{aligned}\right\} && n \geq 3 \end{aligned} \quad (10)$$

B. Moment ($\boldsymbol{M}$-isotropy): $\boldsymbol{H_M}$ is isotropic.

$$\begin{aligned} &b_1^2 - 4b_2 = 0, && n = 2 \\ &\left.\begin{aligned} 3b_2 - b_1^2 &= 0 \\ 27b_3 - b_1^3 &= 0 \end{aligned}\right\} && n \geq 3 \end{aligned} \quad (11)$$

C. Combined: Both $\boldsymbol{H_F}$, $\boldsymbol{H_M}$ are isotropic. The conditions that apply in this case are simply the union of the conditions in cases A and B.

$$\left.\begin{aligned} a_1^2 - 4a_2 &= 0 \\ b_1^2 - 4b_2 &= 0 \end{aligned}\right\} \quad n = 2 \quad (12)$$

$$\left.\begin{aligned} 3a_2 - a_1^2 &= 0 \\ 27a_3 - a_1^3 &= 0 \\ 3b_2 - b_1^2 &= 0 \\ 27b_3 - b_1^3 &= 0 \end{aligned}\right\} \quad n \geq 3 \quad (13)$$

III. Isotropy conditions of an SRSPM

In this section, we apply the theory developed in section II to formulate the isotropy conditions of an SRSPM. In addition to its wide-spread technical applications mentioned earlier, the other motivations to choose this manipulator as our example are: (a) it is probably the most well-studied spatial parallel manipulator (see section 1 for some of the references) (b) no feasible configuration of any Stewart platform manipulator demonstrating combined static isotropy is reported in literature to the best of our knowledge.

The manipulator along with the frames of reference used is shown in figure 1. The bottom platform, shown in figure 2, has the legs arranged in a circle, with each pair lying

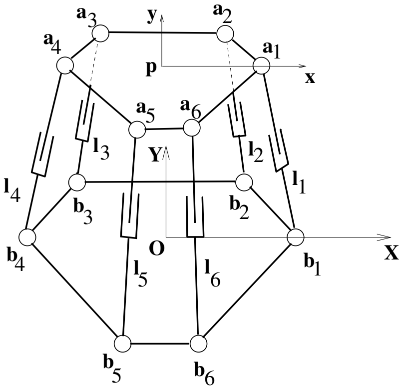


Fig. 1. The semi-regular Stewart platform manipulator

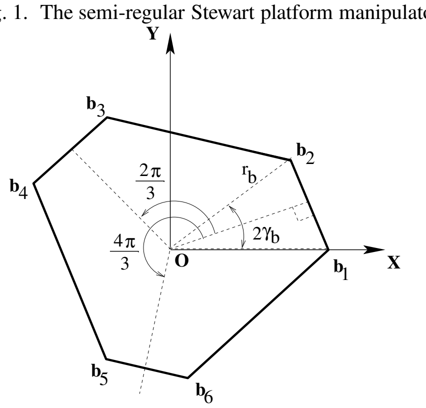


Fig. 2. Geometry of the bottom platform

symmetrically on either side of three axes of symmetry in the plane. The axes are $\frac{2\pi}{3}$ apart from each other, while the adjacent pair of legs have an angular spacing $2\gamma_b$. Without any loss of generality, we scale the circumradius of the bottom platform, r_b, to unity, thus eliminating one parameter from all subsequent analysis, and rendering all other length parameters used in this paper dimensionless[3]. The top platform geometry is similar, except that it has a circumradius r_t, and a leg spacing $2\gamma_t$.

The kinematic constraints defining the manipulator are written in the task-space variables. The center of the top platform is described in the base frame as $\boldsymbol{p} = (x, y, z)^T$. The top platform orientation is described by the matrix $\boldsymbol{R} \in SO(3)$, where $\boldsymbol{R} = \boldsymbol{R}_z(\phi)\boldsymbol{R}_x(\theta_x)\boldsymbol{R}_y(\theta_y)$[4]. The loop-closure equations are written as

$$\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_i - \boldsymbol{b}_i - l_i\boldsymbol{s}_i = \boldsymbol{0}, \qquad i = 1, ..., 6 \tag{14}$$

where l_i denotes the length of the ith leg and $\boldsymbol{a}_i, \boldsymbol{b}_i$ locate the leg connection points with respect to the platform centers in respective frames (see figure 1), and $\boldsymbol{s}_i$ denotes the ith screw axis along the respective leg. The screw axis can be written in terms of the task-space variables and actuated variables as:

$$\boldsymbol{s}_i = \frac{1}{l_i}(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_i - \boldsymbol{b}_i),\ i = 1, ..., 6 \tag{15}$$

The actuation force along the ith leg can be written as $\boldsymbol{F}_i = \boldsymbol{s}_i f_i$, where f_i denotes the sense and magnitude of the force. In terms of the *force transformation matrix*, the resultant force on the top platform can be written as:

$$\boldsymbol{F} = \boldsymbol{H_F}\boldsymbol{f} \tag{16}$$

where, $\boldsymbol{f} = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ is the vector of leg forces, and the matrix $\boldsymbol{H_F}$ is given by:

$$\boldsymbol{H_F} = \begin{pmatrix} \frac{1}{l_1}(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_1 - \boldsymbol{b}_1)_1 & \dots & \frac{1}{l_6}(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_6 - \boldsymbol{b}_6)_1 \\ \frac{1}{l_1}(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_1 - \boldsymbol{b}_1)_2 & \dots & \frac{1}{l_6}(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_6 - \boldsymbol{b}_6)_2 \\ \frac{1}{l_1}(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_1 - \boldsymbol{b}_1)_3 & \dots & \frac{1}{l_6}(\boldsymbol{p} + \boldsymbol{R}\boldsymbol{a}_6 - \boldsymbol{b}_6)_3 \end{pmatrix}$$

where $(\cdot)_i$ denotes the ith component of the vector '$\cdot$'. Moment imparted on the top platform due to the force along the ith leg can be written as $\boldsymbol{M}_i = (\boldsymbol{R}\boldsymbol{a}_i) \times f_i\boldsymbol{s}_i$. Using the expression for $\boldsymbol{s}_i$ from equation (15), $\boldsymbol{M}_i$ may be written as $\boldsymbol{M}_i = \frac{f_i}{l_i}(((\boldsymbol{R}\boldsymbol{a}_i) \times (\boldsymbol{p} - \boldsymbol{b}_i)))$. Therefore the resultant moment $\boldsymbol{M}$ can be written in terms of the *moment transformation matrix* $\boldsymbol{H_M}$ as

$$\boldsymbol{M} = \boldsymbol{H_M}\boldsymbol{f} \tag{17}$$

with $\boldsymbol{H_M}$ given by

$$\begin{pmatrix} \frac{1}{l_1}(\boldsymbol{R}\boldsymbol{a}_1 \times (\boldsymbol{p} - \boldsymbol{b}_1))_1 & \dots & \frac{1}{l_6}(\boldsymbol{R}\boldsymbol{a}_6 \times (\boldsymbol{p} - \boldsymbol{b}_6))_1 \\ \frac{1}{l_1}(\boldsymbol{R}\boldsymbol{a}_1 \times (\boldsymbol{p} - \boldsymbol{b}_1))_2 & \dots & \frac{1}{l_6}(\boldsymbol{R}\boldsymbol{a}_6 \times (\boldsymbol{p} - \boldsymbol{b}_6))_2 \\ \frac{1}{l_1}(\boldsymbol{R}\boldsymbol{a}_1 \times (\boldsymbol{p} - \boldsymbol{b}_1))_3 & \dots & \frac{1}{l_6}(\boldsymbol{R}\boldsymbol{a}_6 \times (\boldsymbol{p} - \boldsymbol{b}_6))_3 \end{pmatrix}$$

It may be noted that the use of equation (15) ensures that expressions of $\boldsymbol{H_F}$ and $\boldsymbol{H_M}$ are kinematically consistent, i.e., the loop closure equations are automatically satisfied when they are cast in this form.

The conditions for static isotropy are obtained from $\boldsymbol{H_F}$, $\boldsymbol{H_M}$ following the previous section. The computational steps involved for all the cases A, B, and C are summarised below.

1. Form the symmetric matrix $\boldsymbol{g_F} = \boldsymbol{H_F^T}\boldsymbol{H_F}$
2. Form the symmetric matrix $\boldsymbol{g_M} = \boldsymbol{H_M^T}\boldsymbol{H_M}$
3. Compute the coefficients of the characteristic equations of $\boldsymbol{g_F}$, $\boldsymbol{g_M}$ using Newton identities.
4. Use equations (13) or any subset of the same, as appropriate for the different cases of static isotropy.

IV. Analytical results on the isotropy of an SRSPM

We now describe some analytical results for the different cases of isotropy of the SRSPM using the formulation developed in the last section. The *independent* variables involved in the isotropy equations are the position of the top platform $\boldsymbol{p} = (x, y, z)^T$, the orientation variables α, β, ϕ, and the architectural variables r_t, γ_b and γ_t.

[3]We use *radians* for the angular unit in this paper, while the unit for the base radius can be chosen as convenient.

[4]In this paper, we denote the rotation about the axis $\boldsymbol{X}$ through an angle θ as $\boldsymbol{R}_x(\theta)$ etc.

A. Architecture, configuration constraints

The natural restrictions on the architectural parameters for mechanically feasible design would be the following:

- $\overline{r_t} \geq r_t \geq \underline{r_t}$ where $\overline{r_t}, \underline{r_t} > 0$ are two prescribed limits. We adopt in this work $\overline{r_t} = 1, \underline{r_t} = 1/4$.
- $\pi/3 \geq \gamma_b, \gamma_t \geq 0$. At both ends of these limits, the hexagonal platforms reduce to triangles, and beyond these limits the leg connection points with the platforms cross over, and the legs can interfere mutually.
- The moving platform is above the fixed one, i.e., $z > 0$.
- $\gamma_b \neq \gamma_t$. If the platforms are scaled versions of each other, the manipulator is architecturally singular [9], [10].

Any solution for the architecture within these restrictions would be termed as *feasible* or *valid*. Other mechanical constraints, such as joint limits, leg limits, and physical dimensions of the legs etc. are not considered in the present work. As a result, we do not impose any ranges on the values of the position and orientation variables, except $z > 0$. We start with the following assumptions which enable us to perform symbolic computations and obtain exact analytical expressions:

- Isotropic configurations and corresponding architectures are obtained only for the case when the manipulator is in its *home position*. The home position is defined as $x = y = 0$, $\alpha = \beta = 0$. In other words, displacement along and rotation about only the $\boldsymbol{Z}$ axis is considered.
- The leg lengths have special relationships among themselves. We consider a family of configurations in which alternate legs of the manipulator have equal lengths, i.e., length of the odd numbered legs is L_1, and that of the even numbered legs $L_2 = \rho L_1$, where $\rho > 0$ and in general $\rho \neq 1$. This choice is motivated by the 3-way symmetry inherent in the manipulator architecture, and the set of configurations is more general than those studied in [11], [12], [3], [1].

These restrictions by no means reflect any limitation of our formulation; relaxing these has only the effect of increasing the complexity of problem[5].

B. Isotropic configurations

To ensure the practical utility of the isotropy, we first check for the possible singularities within the target family of configurations. The singularities in statics occur when we have $\det\begin{pmatrix}\boldsymbol{H_F}\\ \boldsymbol{H_M}\end{pmatrix} = 0$. In this case the determinant is given by:

$$D_{\boldsymbol{H}} = \frac{54 r_t^3 z^3 \cos(\gamma - \phi)\sin(\gamma)}{L_1^6 \rho^3} \qquad \gamma = \gamma_b - \gamma_t$$

[5] Although we do not have a proof, we have not been able to find any other family of isotropic configuration (namely with *all unequal* leg lengths or at $x, y, \alpha, \beta \neq 0$) for the SRSPM's studied by us. This is in spite of extensive searches using various methods.

From equation (14), we obtain only two distinct equations defining the leg lengths:

$$L_1^2 = 1 + r_t^2 + z^2 - 2r_t\cos\phi$$
$$\rho^2 L_1^2 = 1 + r_t^2 + z^2 - 2r_t\cos(2\gamma - \phi) \tag{18}$$

Eliminating L_1 between the above equations, we get a linear equation in ρ^2, which gives the positive solution for ρ as

$$\rho = \sqrt{\frac{(r_t - \cos(2\gamma-\phi))^2 + z^2 + \sin^2(2\gamma-\phi)}{(r_t - \cos\phi)^2 + \sin^2\phi} + z^2}$$

The corresponding solution of L_1 is obtained as

$$L_1 = \sqrt{(r_t - \cos(2\gamma-\phi))^2 + z^2 + \sin^2(2\gamma-\phi)} \tag{19}$$

The expressions for ρ, L_1 indicate that there are five free parameters, namely $r_t, \gamma, \gamma_t, \phi$ and z, for which the kinematic constraints are valid. We now search for isotropic configurations within this 5-parameter family of kinematically valid configurations. First, we establish the conditions for isotropy in general.

B.1 $\boldsymbol{F}$-isotropy.

The kinematically consistent $\boldsymbol{F}$-isotropy conditions computed from equation (10) are found to share a common factor, which can be written as a polynomial in $z_{\boldsymbol{F}}$:

$$c_0 z_{\boldsymbol{F}}^4 + c_1 z_{\boldsymbol{F}}^2 + c_2 = 0, \text{ where } c_0 = 2, \tag{20}$$
$$c_1 = 2r_t^2 - 4\cos(\gamma)\cos(\gamma-\phi)r_t + \cos(4\gamma - 2\phi) + \cos(2\phi)$$
$$c_2 = (\cos(4\gamma-2\phi) + \cos(2\phi) - 2)r_t^2 + 4(\cos(\phi)\sin^2(2\gamma-\phi)$$
$$+ \cos(2\gamma-\phi)\sin^2(\phi))r_t + \cos(4\gamma-2\phi) + \cos(2\phi) - 2$$

B.2 $\boldsymbol{M}$-isotropy.

In this case also, the isotropy conditions in equation (11) have a common factor, which is a quadratic in $z_{\boldsymbol{M}}^2$:

$$d_0 z_{\boldsymbol{M}}^4 + d_1 z_{\boldsymbol{M}}^2 + d_2 = 0, \text{ where } d_0 = 2 \tag{21}$$
$$d_1 = r_t^2 - 2\cos(\gamma)\cos(\gamma-\phi)r_t + 1$$
$$d_2 = -r_t^4 + 4\cos(\gamma)\cos(\gamma-\phi)r_t^3 - 2(\cos(2\gamma)$$
$$+ \cos(2(\gamma-\phi)) + 1)r_t^2 + 4\cos(\gamma)\cos(\gamma-\phi)r_t - 1$$

B.3 Combined static isotropy.

The condition for combined static isotropy is simply the intersection of the above two conditions, i.e., $z_{\boldsymbol{F}} = z_{\boldsymbol{M}}$. In other words, equations (20,21) should have common root(s) in z^2. The condition for the same can be obtained in *closed form* by eliminating z from these equations:

$$c_2^2 d_0^2 + c_1 c_2 d_0 d_1 + c_1^2 d_0 d_2 - c_1 d_1 d_2$$
$$+ d_2^2 + c_2(d_1^2 - 2d_0 d_2) = 0 \tag{22}$$

The eliminant is of degree 6 in r_t, but it is possible to write it as $r_t \sin^2(\gamma)\sin^2(\gamma-\phi)P_5(r_t)$, where the quintic $P_5(r_t)=0$, as the vanishing of the other factor leads to singularity. The coefficients of the quintic are functions of the parameters γ, ϕ, and can be derived in closed-form. However, due to their large size, we do not include them here. When the quintic has a real solution, equations (20,21) share a common root, and the corresponding positive value of z can be obtained as:

$$z = \sqrt{\frac{N_z}{D_z}} \tag{23}$$

$$N_z = -r_t^4 + 4\cos(\gamma)\cos(\gamma-\phi)r_t^3 - 2(2\cos(2(\gamma-\phi))$$
$$\times \cos^2(\gamma) + \cos(2\gamma))r_t^2 + 2\cos(\gamma)((2\cos(2\gamma)-1)\cos(\gamma-\phi)$$
$$+\cos(3(\gamma-\phi)))r_t + 2\sin^2(\phi) - \cos(4\gamma - 2\phi)$$
$$D_z = r_t^2 - 2\cos(\gamma)\cos(\gamma-\phi)r_t + \cos(4\gamma-2\phi)$$
$$+\cos(2\phi) - 1$$

C. Examples of combined static isotropy

We choose the free parameters as $\gamma_b = \frac{\pi}{5}$, $\gamma_t = \frac{\pi}{10}$, and $\phi = \frac{\pi}{18}$. The numerical solutions for r_t are obtained as

$$r_t = (0.3789, 0.9828 \pm 0.1866i, 1.4795, 5.5939)$$

Of these, only $r_t = 0.3789$ is *feasible* for the ranges of parameters we have chosen. The resulting value of z from equation (23) is obtained as $z = 0.4627$. The corresponding configuration is shown below in figure 3.

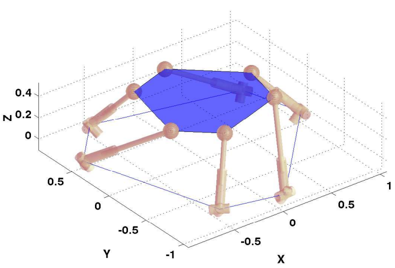


Fig. 3. Combined static isotropic configurations of the SRSPM

V. Design of an SRSPM for combined static isotropy

The formulation presented in this paper allow us to solve the problems of analysis and synthesis within the same setup, in addition to studying the isotropic configurations in general. In this context, by *analysis* we mean obtaining the isotropic configurations of a manipulator with a *given* architecture, and by *synthesis*, the determination of the architectural parameters such that the manipulator is isotropic in a given configuration. We present a few case studies below.

A. Synthesis of an SRSPM for combined static isotropy at a given position z_0 and orientation ϕ_0

In this case we assume that the top platform location and orientation have been completely specified by $z_{\boldsymbol{F}} = z_{\boldsymbol{M}} = z_0$ and $\phi = \phi_0$ (in conjunction with the assumptions defining the isotropic family). The task is to obtain γ and r_t such that the manipulator exhibits combined static isotropy.

We start with the $\boldsymbol{F}$-isotropy equation (20) and the $\boldsymbol{M}$-isotropy equation (21). Substituting the actual expressions of c_i, d_i in these equations, and rewriting them as polynomial equations in r_t, we get a quartic and a quadratic respectively:

$$g_0 r_t^4 + g_1 r_t^3 + g_2 r_t^2 + g_3 r_t + g_4 = 0$$
$$h_0 r_t^2 + h_1 r_t + h_2 = 0 \tag{24}$$

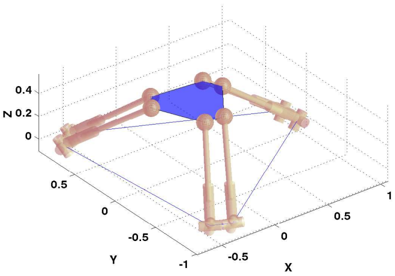
The common root of these two equations can be obtained in terms of the coefficients g_i, h_i when the resultant with respect to r_t vanishes. The resultant is a complicated expression involving trigonometric terms in γ, and algebraic terms in z_0. We transform it to a polynomial in $t = \tan(\gamma/2)$ and simplify its coefficients using the algorithms described in [10]. This results in a 32nd degree polynomial in t. Extracting the real values of t such that the corresponding values of γ are within the prescribed limits, we compute r_t numerically. For every positive solution for r_t within the specified range, the free parameter γ_t can be chosen as convenient, and the architecture of the manipulator can be completely prescribed. We illustrate this synthesis procedure with an example below.

We choose the configuration as $z_0 = 1/2$, $\phi_0 = \pi/20$, and the free architectural parameter as $\gamma_t = \pi/12$. Corresponding to these numerical values, there are 24 real solutions for t, of which, however, only 2 turn out to be feasible. The feasible values of γ are $(-0.1750, 0.3321)$ and the corresponding values of r_t are $(0.3239, 0.3239)$ respectively. The configurations are shown in figures 4(a)-4(b).

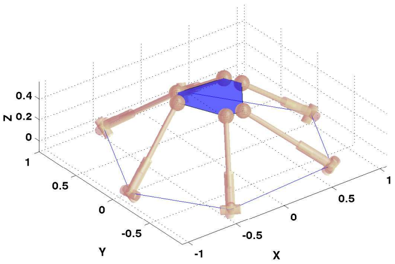
B. Combined static isotropic configurations of an SRSPM of given architecture

In this section, we find out the configurations of an SRSPM of given architecture showing combined static isotropy. The manipulator geometry is completely specified in terms of the architectural variables, r_t, γ and γ_t. We need to find the configuration variables z and ϕ such that the conditions for combined static isotropy are met.

We refer to the condition for combined static isotropy in equation (22), which is a function of ϕ alone. We convert this equation into a polynomial in $u = \tan(\phi/2)$ using the symbolic simplification tools as in the case of the synthesis. In this case we end up with a 8-degree polynomial in u. For each of the *feasible* values of ϕ arising from the solutions for u, the corresponding value of z can be computed *uniquely* from equation (23), thereby completing the defini-



(a) $\gamma = -0.1750,\ r_t = 0.3239$



(b) $\gamma = 0.3321,\ r_t = 0.3239$

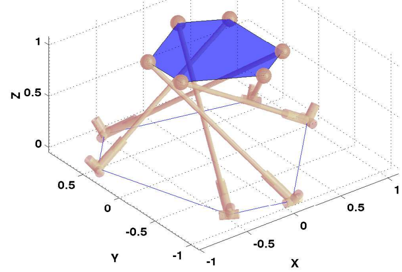
Fig. 4. Combined static isotropy of an SRSPM at a given location and orientation

tion of the manipulator configuration. We demonstrate the solution procedure with an example below.

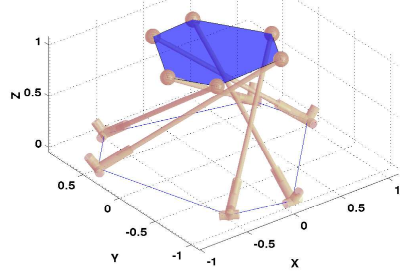
We use an architecture based on the INRIA prototype of the SRSPM (data taken from [13]). The isotropic configurations are shown in figures 5(a)-5(b).

VI. Conclusion

In this paper, we have developed an algebraic formulation for the study of static isotropy of spatial manipulators. We have applied the theory to SRSPM's, and obtained in closed-form a family of configurations showing force and moment isotropy. The formulation allows us to design an SRSPM for combined static isotropy at a given configuration within this family. Also, we can obtain such configurations for an SRSPM with existing architecture. The analytical procedures and results presented in the paper have been numerically illustrated with examples of both analysis and design.



(a) $z = 1.0216,\ \phi = -2.7254$



(b) $z = 1.0216,\ \phi = 2.0078$

Fig. 5. Combined static isotropic configurations of the SRSPM with INRIA geometry

References

[1] K. Y. Tsai and K. D. Huang, "The design of isotropic 6-DOF parallel manipulators using isotropy generators," *Mechanism and Machine Theory*, vol. 38, pp. 1199–1214, 2003.

[2] K. E. Zanganeh and J. Angeles, "Kinematic isotropy and the optimum design of parallel manipulators," *International Journal of Robotics Research*, vol. 16, pp. 185–197, April 1997.

[3] A. Fattah and A. M. H. Ghasemi, "Isotropic design of spatial parallel manipulators," *International Journal of Robotics Research*, vol. 21, pp. 811–824, September 2002.

[4] Y. X. Su, B. Y. Duan, and C. H. Zheng, "Genetic design of kinematically optimal fine tuning Stewart platform for large spherical radio telescope," *Mechatronics*, vol. 11, pp. 821–835, 2001.

[5] S. Bandyopadhyay and A. Ghosal, "An algebraic formulation of kinematic isotropy and design of exactly isotropic 6-6 Stewart platform manipulators." Submitted to *Mechanism and Machine Theory*.

[6] R. A. Horn and C. A. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[7] S. Bandyopadhyay, *Analysis and design of spatial manipulators: an exact algebraic approach using dual numbers and symbolic computations*. PhD thesis, Indian Institute of Science, Bangalore, 2006.

[8] I. N. Herstein, *Topics in Algebra*. New York: John Wiley & Sons, 1975.

[9] O. Ma and J. Angeles, "Architectural singularities of platform manipulators," in *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, pp. 1542–1547, 1991.

[10] S. Bandyopadhyay and A. Ghosal, "Geometric characterization and parametric representation of the singularity manifold of a 6-6 Stewart platform manipulator," *Mechanism and Machine Theory*, vol. 41, pp. 1377–1400, Nov. 2006.

[11] C. A. Klein and T. A. Miklos, "Spatial robotic isotropy," *International Journal of Robotics Research*, vol. 10, pp. 426–437, August 1991.

[12] J. Angeles, "The design of isotropic manipulator architectures in the presence of redundancies," *International Journal of Robotics Research*, vol. 11, pp. 196–202, June 1992.

[13] H. Li, C. Gosselin, M. J. Richard, and B. M. St-Onge, "Analytic form of the six-dimensional singularity locus of the general Gough-Stewart platform," in *Proceedings of ASME 28th Biennial Mechanisms and Robotics Conference, Salt Lake City, Utah, USA*, September 2004.