Special Coordinates associated with Recursive Forward Dynamics Algorithm for Open Loop Rigid Multibody Systems

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Abstract

The recursive forward dynamics algorithm (RFDA) for a tree structured rigid multibody system has two stages. In the first stage, while going down the tree, certain equations are associated with each node. These equations are decoupled from the equations related to the node's descendants. We refer them as the equations of RFDA of the node and the current paper derives them in a new way. In the new derivation, associated with each node, we recursively obtain the coordinates which describe the system consisting of the node and all its descendants. The special property of these coordinates is that a portion of the equations of motion with respect to these coordinates are actually the equations of RFDA associated with the node. We first show the derivation for a two noded system and then extend to a general tree structure. Two examples are used to illustrate the derivation. While the derivation conclusively shows that equations of RFDA are part of equations of motion, it most importantly gives the associated coordinates and the left out portion of the equations of motion. These are significant insights into the RFDA.

1 Introduction

The forward dynamics of a tree structured or open-loop, rigid, multi-body system with n rigid bodies is efficiently done by the well known $\mathcal{O}(n)$ recursive forward dynamics algorithm (RFDA). Early contributions to this algorithm could be traced to Armstrong [Armstrong, 1979]. In Featherstone [Featherstone, 1983], this algorithm has been generalized and explained using the screw theory and the concept of articulated body inertia (AB Inertia) was also introduced. The same algorithm was explained using variational equations of motion, by Bae and Haug [Bae and Haug, 1987]. In the work by Rodriguez [Rodriguez, 1987], the algorithm was derived using the techniques similar to Kalman filtering and smoothing. Rodriguez and Kreutz-Delgado [Rodriguez and Kreutz-Delgado, 1992] used the spatial operator algebra to describe this algorithm. The concept of decoupled natural orthogonal coordinates and reverse Gaussian elimination was used to derive this algorithm by Saha [Saha, 1999]. Lubich

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and co-workers [Lubich et al, 1992] derived the recursive algorithm using constraint equations.

As described by Bae and Haug [Bae and Haug, 1987], Featherstone [Featherstone, 1983] and by Lubich and co-workers [Lubich et al, 1992], the RFDA consists of two sequential stages. In the first stage, certain equations are recursively associated with each node of the tree and they are decoupled from the equations related to the descendant nodes. We refer them as the equations of RFDA of the node and the current work presents a new way to derive them. It is well known that different coordinates [1] describing the same multibody system results in different equations of motion. The new derivation is based on recursively obtaining special coordinates for each node having following characterization - 1) it describes the system consisting of the node and all its descendants, 2) it is consistent with all joints in the node-descendants system, and 3) most importantly, a portion of equation of motion with respect to these coordinates is the equations of RFDA for the node. Henceforth, such coordinates are referred as coordinates of RFDA. In section 3.2.2, we examine that for a system as simple as two noded planar system with revolute joint, finding the coordinates of RFDA for the parent node is not straight forward.

In the paper, the derivation is first shown for a two noded system. For the terminal node, the coordinates of RFDA is same as the absolute coordinates for the node. For the parent node, the coordinates of RFDA is found in two stages. In the first stage, the coordinates describing terminal node and satisfying two conditions (see section 4) are found. In the second stage, using simple coordinate transformation, we obtain coordinates describing both nodes. The equation of motion in terms of the later coordinates has block diagonal mass matrix and the equations corresponding to one of the blocks is the equations of RFDA for the node. The originality of the paper lies in enunciating the two conditions for coordinates of the first stage and the methods used to obtain them. We later extend the derivation to a general tree structure.

This derivation conclusively shows that equations of RFDA is actually a part of equations of motion. Most importantly, it gives the associated coordinates and the left out portion of the equations of motion. These are important insights in this cornerstone algorithm in multibody dynamics. We don't make any claims on better computer implementation.

This paper is organized as follows: In section 2, we present a review of the equations of RFDA for a tree structured, rigid, multi-body system. In section 3, we explain motivation for the new derivation. In section 4, we present our method of obtaining equations of RFDA for a two noded tree. The details of the method are worked out in section 5. Section 6 extends the method to a general tree structure. We conclude in section 7.

2 Review of equations of recursive forward dynamics

The figure 1 shows the topological representation of a tree structured multibody system. Each node represents a rigid body and a line connecting two nodes represents the joint between the rigid bodies. The nodes are appropriately numbered. The joint between a node and its parent receives the same number as that of the node.

[1] We have used the term coordinates to mean the quantities that *independently and completely* describe a subset or all of the rigid bodies making up a multibody system.

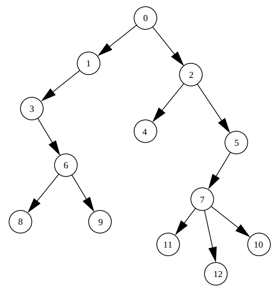


Figure 1: A typical tree structure

If $\boldsymbol{y}$ is absolute coordinates [2] for a n-noded multi-body system, then we can partition it as $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_0^T & \boldsymbol{y}_1^T & \dots & \boldsymbol{y}_n^T \end{bmatrix}^T$, where $\boldsymbol{y}_i$ describes [3] the rigid body i.

The constraint equation due to joint j, between body j and its parent i is represented as

$$\boldsymbol{Q}_j \dot{\boldsymbol{y}}_i + \boldsymbol{G}_j \dot{\boldsymbol{y}}_j = \boldsymbol{\nu}_j \tag{1}$$

If j is the root node, then the constraint equation has the form

$$\boldsymbol{G}_j \dot{\boldsymbol{y}}_j = \boldsymbol{\nu}_j \tag{2}$$

The differentiated form of equation (1) is

$$\boldsymbol{Q}_j \ddot{\boldsymbol{y}}_i + \boldsymbol{G}_j \ddot{\boldsymbol{y}}_j = \boldsymbol{\gamma}_j \tag{3}$$

where $\boldsymbol{\gamma}_j = -\dot{\boldsymbol{Q}}_j \dot{\boldsymbol{y}}_i - \dot{\boldsymbol{G}}_j \dot{\boldsymbol{y}}_j + \dot{\boldsymbol{\nu}}_j$. In [Haug, 1989], there is a detailed discussion on finding constraint equations in terms of absolute coordinates, for different kinds of joints.

Joint coordinates (also called relative coordinates) are also used to describe the multi-body system. If $\boldsymbol{q}$ represents the joint coordinates, then it can be partitioned as $\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_0^T & \boldsymbol{q}_1^T & \dots & \boldsymbol{q}_n^T \end{bmatrix}^T$, where $\boldsymbol{q}_i$ represent vector of joint variables of joint i. For example, if i^{th} joint is revolute joint, then $\boldsymbol{q}_i$ could be one dimensional vector containing joint angle $[\theta_i]$.

The absolute and joint coordinates are related. If body i is the parent of body j, then the relation is represented as

$$\dot{\boldsymbol{y}}_j = \boldsymbol{B}_j \dot{\boldsymbol{y}}_i + \boldsymbol{H}_j \dot{\boldsymbol{q}}_j + \boldsymbol{c}_j \tag{4}$$

One of the ways to obtain the above relation is given in Appendix A. The differentiated form of the equation (4) is

$$\ddot{\boldsymbol{y}}_j = \boldsymbol{B}_j \ddot{\boldsymbol{y}}_i + \boldsymbol{H}_j \ddot{\boldsymbol{q}}_j + \boldsymbol{d}_j \tag{5}$$

where $\boldsymbol{d}_j = \dot{\boldsymbol{B}}_j \dot{\boldsymbol{y}}_i + \dot{\boldsymbol{H}}_j \dot{\boldsymbol{q}}_j + \dot{\boldsymbol{c}}_j$. In [Bae and Haug, 1987] there is a detailed discussion on finding the above equations for revolute and translational joints.

[2] The symbol $\boldsymbol{y}$ actually represent coordinates stacked in the form of a *vector*.

[3] The coordinates could be pseudo-coordinates. So $\dot{\boldsymbol{y}}_i$ could be familiar translational and angular velocities, while $\boldsymbol{y}_i$ itself is symbolic.

The equation of motion for the unconstrained body i in terms of absolute coordinates $\boldsymbol{y}_i$, is given by

$$\boldsymbol{M}_i \ddot{\boldsymbol{y}}_i = \boldsymbol{f}_i \tag{6}$$

The mixed differential-algebraic equation (after differentiating constraints appropriately) for the constrained tree structured multibody system has the form.

$$\left\{ \begin{array}{c} \boldsymbol{M}_j \ddot{\boldsymbol{y}}_j + \boldsymbol{G}_j^T \boldsymbol{\lambda}_j = \boldsymbol{f}_j - \sum\limits_{k:j=\mathcal{P}(k)} \boldsymbol{Q}_k^T \boldsymbol{\lambda}_k \\ \boldsymbol{G}_j \ddot{\boldsymbol{y}}_j = \boldsymbol{\gamma}_j - \boldsymbol{Q}_j \ddot{\boldsymbol{y}}_{\mathcal{P}(j)} \end{array} \right\}, \quad j = 0, \ldots, n \tag{7}$$

where $\mathcal{P}(i)$ denote parent of node i, and $k : j = \mathcal{P}(k)$, indicates all k which has j as its parent.

Given time t, $\boldsymbol{y}$, and $\dot{\boldsymbol{y}}$, $\boldsymbol{M}_j$, $\boldsymbol{f}_j$, [4] $\boldsymbol{G}_j$, $\boldsymbol{Q}_j$ and $\boldsymbol{\gamma}_j$ could be found for $j = 0, \ldots, n$. So the equation (7) is essentially a set of linear equations in the unknowns $\ddot{\boldsymbol{y}}$ and $\boldsymbol{\lambda}$. The purpose of forward dynamics algorithm is to find $\ddot{\boldsymbol{y}}$, given t, $\boldsymbol{y}$, and $\dot{\boldsymbol{y}}$. One straight forward method to solve equations (7) is by using methods such as Gaussian elimination or LU decomposition. This straight forward method has $\mathcal{O}(n^3)$ complexity.

The $\mathcal{O}(n)$ recursive algorithm for forward dynamics of branched multibody system has two steps

1. Going from terminal bodies to root, forming new equations at parent nodes, along the way.
2. Going from root to terminal bodies, solving for $\ddot{\boldsymbol{y}}_j$ of each of the nodes j, along the way.

Step 1 : The new equation that is formed at a node, say j, is given by

$$\left\{ \begin{array}{l} \widehat{\boldsymbol{M}}_j \ddot{\boldsymbol{y}}_j + \boldsymbol{G}_j^T \boldsymbol{\lambda}_j = \widehat{\boldsymbol{f}}_j \\ \boldsymbol{G}_j \ddot{\boldsymbol{y}}_j = \boldsymbol{\gamma}_j - \boldsymbol{Q}_j \ddot{\boldsymbol{y}}_{\mathcal{P}(j)} \end{array} \right\}, \tag{8}$$

with the constraint part corresponding to node j remaining unchanged. In this paper, it is the first part of equation (8), that is referred to as the equations of RFDA for node j.

In reference [Lubich et al, 1992], the following expression for $\widehat{\boldsymbol{M}}_j$ and $\widehat{\boldsymbol{f}}_j$ has been derived.

$$\widehat{\boldsymbol{M}}_j = \boldsymbol{M}_j + \sum_{k:j=\mathcal{P}(k)} \boldsymbol{Q}_k^T \big(\boldsymbol{G}_k \widehat{\boldsymbol{M}}_k^{-1} \boldsymbol{G}_k^T\big)^{-1} \boldsymbol{Q}_k \tag{9}$$

$$\widehat{\boldsymbol{f}}_j = \boldsymbol{f}_j + \sum_{k:j=\mathcal{P}(k)} \boldsymbol{Q}_k^T \big(\boldsymbol{G}_k \widehat{\boldsymbol{M}}_k^{-1} \boldsymbol{G}_k^T\big)^{-1} \big(\gamma_k - \boldsymbol{G}_k \widehat{\boldsymbol{M}}_k^{-1} \widehat{\boldsymbol{f}}_k\big) \tag{10}$$

[4] There may be problems where $\boldsymbol{f}_j$ may depend linearly or nonlinearly on $\boldsymbol{\lambda}$. Those situations arise when dry friction is modeled into the equations. We will not consider such cases here.

An alternate expression for $\widehat{\boldsymbol{M}}_j$ and $\widehat{\boldsymbol{f}}_j$ is given in [Lubich et al, 1992], [Bae and Haug, 1987] and [Featherstone, 1983]. The expressions are as given below

$$\widehat{\boldsymbol{M}}_j = \boldsymbol{M}_j + \sum_{k:j=\mathcal{P}(k)} \boldsymbol{B}_k^T \big(\boldsymbol{I} - \widehat{\boldsymbol{M}}_k \boldsymbol{H}_k (\boldsymbol{H}_k^T \widehat{\boldsymbol{M}}_k \boldsymbol{H}_k)^{-1} \boldsymbol{H}_k^T\big) \widehat{\boldsymbol{M}}_k \boldsymbol{B}_k \tag{11}$$

$$\widehat{\boldsymbol{f}}_j = \boldsymbol{f}_j + \sum_{k:j=\mathcal{P}(k)} \boldsymbol{B}_k^T \big(\boldsymbol{I} - \widehat{\boldsymbol{M}}_k \boldsymbol{H}_k (\boldsymbol{H}_k^T \widehat{\boldsymbol{M}}_k \boldsymbol{H}_k)^{-1} \boldsymbol{H}_k^T\big)\big(\boldsymbol{f}_k - \widehat{\boldsymbol{M}}_k \boldsymbol{d}_k\big) \tag{12}$$

where $\boldsymbol{B}_k$, $\boldsymbol{H}_k$ and $\boldsymbol{d}_k$ are as given in equation (5).
Step 2 : The equation (8) could be solved for $\ddot{\boldsymbol{y}}_j$ and $\boldsymbol{\lambda}_j$ if $\ddot{\boldsymbol{y}}_{\mathcal{P}(j)}$ (corresponding to parent of j) is known. However if $j = 0$ (root), then the term $\boldsymbol{Q}_j \ddot{\boldsymbol{y}}_{\mathcal{P}(j)}$ doesn't exist (since root doesn't have parent). So initially for $j = 0$, $\ddot{\boldsymbol{y}}_j$ could be solved. Once $\ddot{\boldsymbol{y}}_0$ is known, step by step $\ddot{\boldsymbol{y}}_j$ for all the descendant nodes j could be solved. This constitute the second stage of the recursive algorithm.

$\boldsymbol{M}_j$ is positive definite for all $j = 0, \ldots, n$. The term $\sum_{k:j=\mathcal{P}(k)} \boldsymbol{Q}_k^T \big(\boldsymbol{G}_k \widehat{\boldsymbol{M}}_k^{-1} \boldsymbol{G}_k^T\big)^{-1} \boldsymbol{Q}_k$ in equation (9) is positive or positive-semi definite. Thus $\widehat{\boldsymbol{M}}_j$ is also positive definite. Similar comments could be shown to hold true for $\widehat{\boldsymbol{M}}_j$ in equation (11) also. With $\widehat{\boldsymbol{M}}_j$ being positive definite for $j = 0, \ldots, n$, the terms $\big(\boldsymbol{G}_k \widehat{\boldsymbol{M}}_k^{-1} \boldsymbol{G}_k^T\big)^{-1}$ and $\big(\boldsymbol{H}_k^T \widehat{\boldsymbol{M}}_k \boldsymbol{H}_k\big)^{-1}$ are defined only if $\boldsymbol{G}_k$ is full row rank and $\boldsymbol{H}_k$ is full column rank. We assume that $\boldsymbol{G}_k$ is full row rank. This assumption also ensures that $\boldsymbol{H}_k$ is of full column rank (see equations (60) and (61)). Further, the assumption $\boldsymbol{G}_k$ being full row rank for $k = 0, \ldots, n$, would render constraint Jacobian in the overall system equation (7), to be of full row rank and the existence and uniqueness of $\ddot{\boldsymbol{y}}$ and $\boldsymbol{\lambda}$ as solution to equation (7), follows form Constrained Dynamic Existence Theorem [Haug, 1989].

In this paper, we present a new approach to derive equations of RFDA. Our approach involves finding new coordinates with special properties. This is described next.

3 Motivation

In this section, we consider a two noded multibody system described in [Featherstone, 1983] and, using intuition, give coordinates of RFDA for it. We also consider a two noded planar multibody system with revolute joint and realize that finding coordinates of RFDA is not straight-forward. Finding coordinates of RFDA for general multibody system has been the motivation for the new derivation given in section 4.

3.1 Featherstone's example

This system has been described in [Featherstone, 1983]. It is a planar system where the body 1 slides on the horizontal rail fixed to the base and the body 2 slides on a the vertical rail fixed to the body 1.

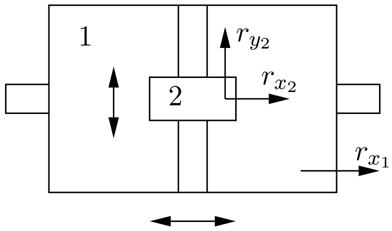


Figure 2: A simple example given in [Featherstone, 1983]

3.1.1 Absolute coordinate - equations of recursive algorithm

Since the two bodies can only translate, take absolute coordinates to be $\boldsymbol{y} = \begin{bmatrix}\boldsymbol{y}_1^T & \boldsymbol{y}_2^T\end{bmatrix}^T$, where $\boldsymbol{y}_1 = \begin{bmatrix}r_{x_1} & r_{y_1}\end{bmatrix}^T$, $\boldsymbol{y}_2 = \begin{bmatrix}r_{x_2} & r_{y_2}\end{bmatrix}^T$. The two constraints on this coordinate are $r_{y_1} = 0$ and $r_{x_2} - r_{x_1} = 0$. The mixed differential algebraic equations for the system has the following form.

$$\text{node 1}: \begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix} \begin{bmatrix} \ddot{r}_{x_1} \\ \ddot{r}_{y_1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda_1 = \begin{bmatrix} f_{x_1} \\ f_{y_1} \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \lambda_2 \tag{13a}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{r}_{x_1} \\ \ddot{r}_{y_1} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{13b}$$

$$\text{node 2}: \quad \begin{bmatrix} m_2 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{r}_{x_2} \\ \ddot{r}_{y_2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \lambda_2 = \begin{bmatrix} f_{x_2} \\ f_{y_2} \end{bmatrix} \tag{14a}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{r}_{x_2} \\ \ddot{r}_{y_2} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{r}_{x_1} \\ \ddot{r}_{y_1} \end{bmatrix} \tag{14b}$$

The equations RFDA associated with node 1 is the first part of equation (8). Using equations (9) and (10) to calculate $\widehat{\boldsymbol{M}}_1$ and $\widehat{\boldsymbol{f}}_1$, we get

$$\begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_1 \end{bmatrix} \begin{bmatrix} \ddot{r}_{x_1} \\ \ddot{r}_{y_1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \lambda_1 = \begin{bmatrix} f_{x_1} + f_{x_2} \\ f_{y_1} \end{bmatrix} \tag{15}$$

3.1.2 Coordinates of RFDA

We form a new coordinate by retaining the absolute coordinate of body 1 and replacing the absolute coordinate of body 2 by the joint variable r_{y_2}. The new coordinate $\bar{\boldsymbol{y}} = \begin{bmatrix} r_{x_1} & r_{y_1} & r_{y_2} \end{bmatrix}^T$ is related to the absolute coordinate $\boldsymbol{y}$, by the following relation.

$$\begin{bmatrix} \dot{r}_{x_1} \\ \dot{r}_{y_1} \\ \dot{r}_{x_2} \\ \dot{r}_{y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r}_{x_1} \\ \dot{r}_{y_1} \\ \dot{r}_{y_2} \end{bmatrix} \tag{16}$$

The constraint for the new coordinate is $\dot{r}_{y_1} = 0$. The equation of motion in terms of new coordinate is

$$\begin{bmatrix} m_1+m_2 & 0 & | & 0 \\ 0 & m_1 & | & 0 \\ - & - & - & - \\ 0 & 0 & | & m_2 \end{bmatrix} \begin{bmatrix} \ddot{r}_{x_1} \\ \ddot{r}_{y_1} \\ - \\ \ddot{r}_{y_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ - \\ 0 \end{bmatrix} \lambda_1 = \begin{bmatrix} f_{x_1}+f_{x_2} \\ f_{y_1} \\ - \\ f_{y_2} \end{bmatrix} \tag{17}$$

We notice that the mass matrix in the above equation of motion is block diagonal. Further, by comparing equation (17) with equation (15), we see that the equations corresponding to the first block is same as equations of RFDA. *Thus the coordinates $\bar{\boldsymbol{y}}$, defined in equation (16) is the coordinates of RFDA for the node.*

In the next subsection, we seek coordinates of RFDA for a two noded planar system with revolute joint.

3.2 Planar two rigid body system with revolute joint

Figure 3 shows two planar rigid bodies connected by a revolute joint. $x-y$ axes, with origin O, represent global reference frame. $\hat{x}_j - \hat{y}_j$ axes, with origin O_j represent local frame fixed to body j. $\boldsymbol{r}_j = \begin{bmatrix} x_j & y_j \end{bmatrix}^T$ is vector $\overrightarrow{OO_j}$, expressed in global coordinate. ϕ_j is angle from $\hat{x}_j$ to x. $\boldsymbol{s'}_j^{P_j} = \begin{bmatrix} s'^{P_j}_{jx} & s'^{P_j}_{jy} \end{bmatrix}^T$ represent vector $\overrightarrow{O_jP_j}$, expressed in local coordinate $\hat{x}_j - \hat{y}_j$. Similar conventions apply for body k also. We proceed on the same lines as previous subsection.

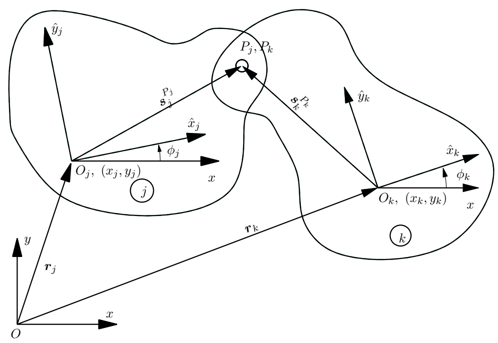


Figure 3: Two planar rigid bodies with a revolute joint at point $P_j = P_k$.

3.2.1 Absolute coordinate - equations of recursive algorithm

We take absolute coordinates for the system to be $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_j^T & \boldsymbol{y}_k^T \end{bmatrix}^T$, where $\boldsymbol{y}_j = \begin{bmatrix} x_j & y_j & \phi_j \end{bmatrix}^T$ and $\boldsymbol{y}_k = \begin{bmatrix} x_k & y_k & \phi_k \end{bmatrix}^T$.
The constraint equation is

$$\boldsymbol{r}_k + \boldsymbol{A}(\phi_k)\boldsymbol{s'}_k^{P_k} - \boldsymbol{r}_j - \boldsymbol{A}(\phi_j)\boldsymbol{s'}_j^{P_j} = \boldsymbol{0} \tag{18}$$

where $\quad \boldsymbol{A}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

The differential equations and constraints associated with each node are – $\{\boldsymbol{M}_j \ddot{\boldsymbol{y}}_j = \boldsymbol{f}_j - \boldsymbol{Q}_k^T \boldsymbol{\lambda}_k\}$, for node j, and $\{\boldsymbol{M}_k \ddot{\boldsymbol{y}}_k + \boldsymbol{G}_k^T \boldsymbol{\lambda}_k = \boldsymbol{f}_k, \quad \boldsymbol{G}_k \ddot{\boldsymbol{y}}_k = \boldsymbol{\gamma}_k - \boldsymbol{Q}_k \ddot{\boldsymbol{y}}_j\}$, for node k. $\boldsymbol{G}_k$, $\boldsymbol{Q}_k$ and $\boldsymbol{\gamma}_k$ could be evaluated using the constraint equation (18) ($\boldsymbol{Q}_k = \begin{bmatrix} -1 & 0 & \sin\phi_j {s'}_{j_x}^{P_j} + \cos\phi_j {s'}_{j_y}^{P_j} \\ 0 & -1 & -\cos\phi_j {s'}_{j_x}^{P_j} + \sin\phi_j {s'}_{j_y}^{P_j} \end{bmatrix}$, $\boldsymbol{G}_k = \begin{bmatrix} 1 & 0 & -\sin\phi_k {s'}_{k_x}^{P_k} - \cos\phi_k {s'}_{k_y}^{P_k} \\ 0 & 1 & \cos\phi_k {s'}_{k_x}^{P_k} - \sin\phi_k {s'}_{k_y}^{P_k} \end{bmatrix}$). Let, $\boldsymbol{M}_j = \begin{bmatrix} m_j & 0 & 0 \\ 0 & m_j & 0 \\ 0 & 0 & J_j \end{bmatrix}$, $\boldsymbol{M}_k = \begin{bmatrix} m_k & 0 & 0 \\ 0 & m_k & 0 \\ 0 & 0 & J_k \end{bmatrix}$, $\boldsymbol{f}_j = \begin{bmatrix} f_{j_x} & f_{j_y} & \tau_j \end{bmatrix}^T$ and $\boldsymbol{f}_k = \begin{bmatrix} f_{k_x} & f_{k_y} & \tau_k \end{bmatrix}^T$.

The equations of RFDA, associated with node 2 has the form

$$\widehat{\boldsymbol{M}}_j \ddot{\boldsymbol{y}}_j = \widehat{\boldsymbol{f}}_j \tag{19}$$

$\widehat{\boldsymbol{M}}_j$ and $\widehat{\boldsymbol{f}}_j$ are calculated using equations (9) and (10). The explicit expression for $\widehat{\boldsymbol{M}}_j(1,1)$ is given below. Rest of the elements and the elements of $\widehat{\boldsymbol{f}}_j$ could be easily obtained using any symbolic math software.

$$\begin{aligned}\widehat{\boldsymbol{M}}_j(1,1) = (&-2m_k^2 {s'}_{k_x}^{P_k} {s'}_{k_y}^{P_k} \sin(2\phi_k) + m_k^2 ({s'}_{k_x}^{P_k})^2 \cos(2\phi_k) + \\ &m_k^2 ({s'}_{k_x}^{P_k})^2 + 2m_j J_k + 2m_k J_k + 2m_j m_k ({s'}_{k_x}^{P_k})^2 + 2m_j m_k ({s'}_{k_y}^{P_k})^2 + \\ &m_k^2 ({s'}_{k_y}^{P_k})^2 - m_k^2 ({s'}_{k_y}^{P_k})^2 \cos(2\phi_k))/(2J_k + 2m_k ({s'}_{k_y}^{P_k})^2 + 2m_k ({s'}_{k_x}^{P_k})^2)\end{aligned}$$

3.2.2 Coordinates of RFDA

We look for a coordinates having the characteristic that the equation of motion of the planar system has a block diagonal mass matrix, with equation corresponding to one block same as equation (19). It is not easy to find such a coordinate. The trick of forming new coordinate by appending absolute coordinate of parent with the joint variable doesn't work here. For example, consider the coordinate $\bar{\boldsymbol{y}} = \begin{bmatrix} x_j & y_j & \phi_j & \theta_k \end{bmatrix}$ where $\theta_j = \phi_k - \phi_j$ is the joint angle. This coordinate is related to absolute coordinate by the relation

$$\begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{\phi}_j \\ \dot{x}_k \\ \dot{y}_k \\ \dot{\phi}_k \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} \boldsymbol{T_{22}} \\ 1 \end{matrix} & \begin{matrix} \boldsymbol{T_{23}} \\ 1 \end{matrix} \end{bmatrix} \begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{\phi}_j \\ \dot{\theta}_k \end{bmatrix} \tag{20}$$

where $\boldsymbol{T_{22}} = \boldsymbol{A}\left(\phi_j + \frac{\pi}{2}\right) \boldsymbol{s'}_j^{P_j} - \boldsymbol{A}\left(\phi_j + \theta_k + \frac{\pi}{2}\right) \boldsymbol{s'}_k^{P_k}$ and $\boldsymbol{T_{23}} = -\boldsymbol{A}\left(\phi_j + \theta_k + \frac{\pi}{2}\right) \boldsymbol{s'}_k^{P_k}$. The new coordinate is consistent with the k^{th} joint constraint and the equation of motion would be of the form $\bar{\boldsymbol{M}} \ddot{\bar{\boldsymbol{y}}} = \bar{\boldsymbol{f}}$. Explicit expression for $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{f}}$ could be obtained using first principles such as generalized d'Alembert's principle or using equation (64) . Some of the elements of $\bar{\boldsymbol{M}}$ are $\bar{\boldsymbol{M}}(1,1) = \bar{\boldsymbol{M}}(2,2) = m_j + m_k, \quad \bar{\boldsymbol{M}}(1,2) = \bar{\boldsymbol{M}}(2,1) = 0$.

It turns out that, $\bar{\boldsymbol{M}}$ is not block diagonal and equation (19) cannot be seen as a part of the equation, $\bar{\boldsymbol{M}} \ddot{\bar{\boldsymbol{y}}} = \bar{\boldsymbol{f}}$.

3.3 Motivation for new derivation

If one were to think equations of RFDA given in equation (8), as a part of equations of motion, then natural questions would be on the left out part of equations of motion and the coordinates associated with equations of motion. We have seen that these questions are not straight forward to answer even for simple two noded planar system with revolute joint. Addressing these questions for a general multibody system has been the motivation for the new derivation of this paper. Moreover answer to these questions gives insight into the RFDA, a cornerstone algorithm in multibody dynamics.

4 The new derivation of equations of RFDA

In this section, the derivation is explained for a two noded tree structure. In section 6, the derivation is extended to a general tree structure. The nodes of the tree are numbered as - k and j. k is considered as terminal node and j is its parent and the root node. There is joint k between nodes k and j, and joint j between root node j and global reference frame.

Equations of RFDA for terminal node k is nothing but equation of motion with respect to absolute coordinates of node k. Hence the absolute coordinates of terminal node itself is the coordinates of RFDA for the node. Following are the steps in the derivation of equations of RFDA for node j.

Step 1 : Coordinates having free and constrained partitions with block diagonal mass matrix -

Find coordinates $\tilde{\boldsymbol{y}}_k = \begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$, describing the rigid body k, and satisfying following properties.

Property 1: $\dot{\tilde{\boldsymbol{y}}}_{k_c}$ should be fully determined by $\dot{\boldsymbol{y}}_j$ and $\tilde{\boldsymbol{y}}_{k_f}$ should not be constrained in any way by parent coordinates. Equivalently, if the constraint equation due to joint k is represented in terms of $\boldsymbol{y}_j$ and $\tilde{\boldsymbol{y}}_k$ as

$$\boldsymbol{Q}_k\dot{\boldsymbol{y}}_j + \begin{bmatrix}\tilde{\boldsymbol{G}}_{k_c} & \tilde{\boldsymbol{G}}_{k_f}\end{bmatrix}\begin{bmatrix}\dot{\tilde{\boldsymbol{y}}}_{k_c}\\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} = \tilde{\boldsymbol{\nu}}_k \tag{21}$$

then $\tilde{\boldsymbol{G}}_{k_c}$ is nonsingular square matrix and $\tilde{\boldsymbol{G}}_{k_f}$ is zero matrix. As a result, the way $\dot{\boldsymbol{y}}_j$ determines $\dot{\tilde{\boldsymbol{y}}}_{k_c}$, is given by

$$\dot{\tilde{\boldsymbol{y}}}_{k_c} = \boldsymbol{S}_k\dot{\boldsymbol{y}}_j + \boldsymbol{a}_k \tag{22}$$

where $\boldsymbol{S}_k = -\tilde{\boldsymbol{G}}_{k_c}^{-1}\boldsymbol{Q}_k$ and $\boldsymbol{a}_k = \tilde{\boldsymbol{G}}_{k_c}^{-1}\tilde{\boldsymbol{\nu}}_k$.

If coordinates satisfy above property, then we say that it has free and constrained partitions.

Property 2: $\tilde{\boldsymbol{y}}_k$ describes the rigid body k and we can write equation of motion of body k in terms of $\tilde{\boldsymbol{y}}_k$. The mass matrix should be block diagonal corresponding to the partitions $\tilde{\boldsymbol{y}}_{k_c}$ and $\tilde{\boldsymbol{y}}_{k_f}$. In other words, the equation of motion in terms of $\tilde{\boldsymbol{y}}_k$ should be of the form

$$\begin{bmatrix} \tilde{\boldsymbol{M}}_{k_c} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{M}}_{k_f} \end{bmatrix} \begin{bmatrix} \ddot{\tilde{\boldsymbol{y}}}_{k_c} \\ \ddot{\tilde{\boldsymbol{y}}}_{k_f} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{G}}_{k_c}^T \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{\lambda}_k = \begin{bmatrix} \tilde{\boldsymbol{f}}_{k_c} \\ \tilde{\boldsymbol{f}}_{k_f} \end{bmatrix} \tag{23}$$

Example 1: In the Featherstone's example (see figure 2), the coordinates $\boldsymbol{y}_2$ describing body 2, satisfy all the above properties.
It has the partition $\boldsymbol{y}_2 = \begin{bmatrix} [r_{x2}] & [r_{y2}] \end{bmatrix}^T$, with the following features.
1) The constraint equation for joint 2 is of the form (see section 3.1.1) $\begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{r}_{x1} \\ \dot{r}_{y1} \end{bmatrix} + \begin{bmatrix} [1] & [0] \end{bmatrix} \begin{bmatrix} [\dot{r}_{x2}] \\ [\dot{r}_{y2}] \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$ and $\begin{bmatrix} \dot{r}_{x2} \end{bmatrix}$ is determined by the equation $\begin{bmatrix} \dot{r}_{x2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{r}_{x1} \\ \dot{r}_{y1} \end{bmatrix}$
2) The equation of motion for body 2, with respect to $\boldsymbol{y}_2$ is, $\begin{bmatrix} [m_2] & [0] \\ [0] & [m_2] \end{bmatrix} \begin{bmatrix} [\ddot{r}_{x2}] \\ [\ddot{r}_{y2}] \end{bmatrix} + \begin{bmatrix} [1] \\ [0] \end{bmatrix} \boldsymbol{\lambda}_2 = \begin{bmatrix} [f_{x2}] \\ [f_{y2}] \end{bmatrix}$. Clearly the mass matrix is block diagonal.
Example 2: Consider the planar system with revolute joint shown in figure 3. The usual coordinates for body k, $\boldsymbol{y}_k = \begin{bmatrix} x_k & y_k & \phi_k \end{bmatrix}^T$, does not have a partition that satisfy the property 1 of step 1, even though the mass matrix is diagonal (see $\boldsymbol{M}_k$ in the section 3.2.1).

Consider another set of coordinates $\bar{\boldsymbol{y}}_k = \begin{bmatrix} [x^{P_k} & y^{P_k}] & [\phi_k] \end{bmatrix}^T$, defined by the relation

$$\dot{\boldsymbol{y}}_k = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} s'^{P_k}_{k_x} \sin\phi_k + s'^{P_k}_{k_y} \cos\phi_k \\ -s'^{P_k}_{k_x} \cos\phi_k + s'^{P_k}_{k_y} \sin\phi_k \\ 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x}^{P_k} \\ \dot{y}^{P_k} \end{bmatrix} \\ [\dot{\phi}_k] \end{bmatrix} \tag{24}$$

From the definition, it should be clear that $\begin{bmatrix} \dot{x}^{P_k} & \dot{y}^{P_k} \end{bmatrix}^T$ is the velocity of pivot point P^k of body k (see figure 3).

The constraint equation in terms of $\bar{\boldsymbol{y}}_k$ is (using equation (62)),

$$\boldsymbol{Q}_k \begin{bmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{\phi}_j \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x}^{P_k} \\ \dot{y}^{P_k} \end{bmatrix} \\ \\ [\dot{\phi}_k] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $\boldsymbol{Q}_k$ is given in section 3.2.1. Clearly , $\bar{\boldsymbol{y}}_k$ satisfy the first property stated above .

The mass matrix for body k in terms of $\bar{\boldsymbol{y}}_k$ is (using equation (64))

$$\begin{bmatrix} \begin{bmatrix} m_k & 0 \\ 0 & m_k \end{bmatrix} & -m_k \boldsymbol{A}(\phi_k + \pi/2) \boldsymbol{s'}^{P_k}_k \\ \\ -(m_k \boldsymbol{A}(\phi_k + \pi/2) \boldsymbol{s'}^{P_k}_k)^T & J_k + m_k^2 (|\boldsymbol{s'}^{P_k}_k|^2) \end{bmatrix} \tag{25}$$

where $\boldsymbol{A}(\theta)$ is described in section 3.2.1. The mass matrix is not block diagonal and the property 2 is not satisfied.

Thus even in specific example as above, it is not straightforward to come up with coordinates satisfying both properties. In section 5 we deduce coordinates satisfying both the properties for a general system.

Step 2: Form new coordinates describing both nodes- Define new coordinates $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ as

$$\begin{bmatrix}\dot{\boldsymbol{y}}_j \\ \dot{\tilde{\boldsymbol{y}}}_{k_c} \\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{S}_k & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{y}}_j \\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} + \begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{a}_k \\ \boldsymbol{0}\end{bmatrix} \tag{26}$$

This coordinate describes the entire system consisting of two rigid bodies with a joint between them. We later see that this is the coordinates of RFDA for node j.

Step 3: Obtain equations of motion in terms of coordinates of step 2 - For writing the constraint and equation of motion in terms of above coordinates, we make use of equations (62) and (63), with coordinate transformation given by equation (26). This requires that we know the constraint equation and equation of motion in terms of $\begin{bmatrix}\dot{\boldsymbol{y}}_j^T & \dot{\tilde{\boldsymbol{y}}}_k^T\end{bmatrix}^T = \begin{bmatrix}\dot{\boldsymbol{y}}_j^T & \dot{\tilde{\boldsymbol{y}}}_{k_c}^T & \dot{\tilde{\boldsymbol{y}}}_{k_f}^T\end{bmatrix}^T$. These are as given below.

$$\boldsymbol{G}_j\dot{\boldsymbol{y}}_j = \boldsymbol{\nu}_j, \qquad \boldsymbol{Q}_k\dot{\boldsymbol{y}}_j + \tilde{\boldsymbol{G}}_{k_c}\dot{\tilde{\boldsymbol{y}}}_{k_c} = \tilde{\boldsymbol{\nu}}_k$$

$$\begin{bmatrix}\boldsymbol{M}_j & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{M}}_{k_c} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \tilde{\boldsymbol{M}}_{k_f}\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{y}}_j \\ \ddot{\tilde{\boldsymbol{y}}}_{k_c} \\ \ddot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} + \begin{bmatrix}\boldsymbol{G}_j^T \\ \boldsymbol{0} \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{\lambda}_j + \begin{bmatrix}\boldsymbol{Q}_k^T \\ \tilde{\boldsymbol{G}}_{k_c}^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{\lambda}_k = \begin{bmatrix}\boldsymbol{f}_k \\ \tilde{\boldsymbol{f}}_{k_c} \\ \tilde{\boldsymbol{f}}_{k_f}\end{bmatrix}$$

The above equations are consequence of equations (2), (6), (21) and (23).

After coordinate transformation through equation (26), the constraints (after removing redundant constraints) and equation of motion in terms of $\begin{bmatrix}\dot{\boldsymbol{y}}_j^T & \dot{\tilde{\boldsymbol{y}}}_{k_f}^T\end{bmatrix}^T$, becomes

$$\boldsymbol{G}_j\dot{\boldsymbol{y}}_j = \boldsymbol{\nu}_j \tag{27}$$

$$\begin{bmatrix}\boldsymbol{M}_j + \boldsymbol{S}_k^T\tilde{\boldsymbol{M}}_{k_c}\boldsymbol{S}_k & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{M}}_{k_f}\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{y}}_j \\ \ddot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} + \begin{bmatrix}\boldsymbol{G}_j^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{\lambda}_j = \begin{bmatrix}\boldsymbol{f}_d \\ \tilde{\boldsymbol{f}}_{k_f}\end{bmatrix} \tag{28}$$

where $\boldsymbol{f}_d = \boldsymbol{f}_j + \boldsymbol{S}_k^T\tilde{\boldsymbol{f}}_{k_c} - \boldsymbol{S}_k^T\tilde{\boldsymbol{M}}_{k_c}(\dot{\boldsymbol{a}}_k + \dot{\boldsymbol{S}}_k\dot{\boldsymbol{y}}_j)$.

Step 4: Recognize that equations of RFDA is part of the equations of motion - For the two noded tree structure, first of equations (8) for node j (i.e equations of RFDA), would be of form $\widehat{\boldsymbol{M}}_j\ddot{\boldsymbol{y}}_j + \boldsymbol{G}_j^T\boldsymbol{\lambda}_j = \widehat{\boldsymbol{f}}_j$. The rows of matrix equation (28) associated with $\boldsymbol{y}_j$, i.e,

$$\left(\boldsymbol{M}_j + \boldsymbol{S}_k^T\tilde{\boldsymbol{M}}_{k_c}\boldsymbol{S}_k\right)\ddot{\boldsymbol{y}}_j + \boldsymbol{G}_j^T\boldsymbol{\lambda}_j = \boldsymbol{f}_j + \boldsymbol{S}_k^T\tilde{\boldsymbol{f}}_{k_c} - \boldsymbol{S}_k^T\tilde{\boldsymbol{M}}_{k_c}(\dot{\boldsymbol{a}}_k + \dot{\boldsymbol{S}}_k\dot{\boldsymbol{y}}_j) \tag{29}$$

is essentially the equations of RFDA associated with the root node of two noded multibody system. Thus coordinates $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$, defined in step 2, is the coordinates of RFDA for node j. Further the only constraint on $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$, given in equation (27), corresponds to the second of equations (8).

Illustration: We now illustrate the steps 2, 3 and 4 for Featherstone's example.
Step 2 Define a new coordinate $\begin{bmatrix}[r_{x_1} & r_{y_1}] & [y_{y_2}]\end{bmatrix}^T$ by the transformation

$$\begin{bmatrix}\begin{bmatrix}\dot{r}_{x_1}\\ \dot{r}_{y_1}\end{bmatrix}\\ \begin{bmatrix}\dot{r}_{x_2}\\ \dot{r}_{y_2}\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & \begin{bmatrix}0\\ 1\end{bmatrix}\end{bmatrix} \begin{bmatrix}\begin{bmatrix}\dot{r}_{x_1}\\ \dot{r}_{y_1}\end{bmatrix}\\ \begin{bmatrix}\dot{r}_{y_2}\end{bmatrix}\end{bmatrix}$$

Step 3 Equations of motion in terms of the above coordinate is

$$\begin{bmatrix}\begin{bmatrix}m_1+m_2 & 0\\ 0 & m_1\end{bmatrix} & \begin{bmatrix}0\\ 0\end{bmatrix}\\ \begin{bmatrix}0 & 0\end{bmatrix} & \begin{bmatrix}m_2\end{bmatrix}\end{bmatrix} \begin{bmatrix}\begin{bmatrix}\ddot{r}_{x_1}\\ \ddot{r}_{y_1}\end{bmatrix}\\ \begin{bmatrix}\ddot{r}_{y_2}\end{bmatrix}\end{bmatrix} + \begin{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ \begin{bmatrix}0\end{bmatrix}\end{bmatrix}\lambda_1 = \begin{bmatrix}\begin{bmatrix}f_{x_1}+f_{x_2}\\ f_{y_1}\end{bmatrix}\\ \begin{bmatrix}f_{y_2}\end{bmatrix}\end{bmatrix} \tag{30}$$

Step 4 Indeed the first row block of the above matrix equation is same as equations of RFDA obtained in equation (15).

Thus, given that coordinates of step 1 would be deduced in section 5, we have derived equations of RFDA in equation (29), based on finding coordinates of RFDA, defined in equation (26).

5 Finding coordinates of Step 1

In this section we rewrite the properties 1 and 2 of step 1 in section 4 as rigorous linear algebraic conditions and deduce the relation between $\tilde{\boldsymbol{y}}_k$ and $\boldsymbol{y}_k$ coordinates. This relation itself defines $\tilde{\boldsymbol{y}}_k$. We discuss two methods to deduce the relation.

5.1 Linear algebraic conditions for coordinates of step I

Let $\tilde{\boldsymbol{y}}_k$ be coordinates having the partition as $\begin{bmatrix}(\tilde{\boldsymbol{y}}_{k_c})^T & (\tilde{\boldsymbol{y}}_{k_f})^T\end{bmatrix}^T$. The coordinates be related to the existing coordinates by the following relation

$$\dot{\boldsymbol{y}}_k = [\tilde{\boldsymbol{E}}_k \quad \tilde{\boldsymbol{D}}_k]\begin{bmatrix}\dot{\tilde{\boldsymbol{y}}}_{k_c}\\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} \tag{31}$$

where $\begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k\end{bmatrix}$ is non-singular square matrix.

The constraint equation of the joint between body k and j when written in terms of $\begin{bmatrix}(\tilde{\boldsymbol{y}}_{k_c})^T & (\tilde{\boldsymbol{y}}_{k_f})^T\end{bmatrix}^T$, takes the following form (see equations (1) and (62)).

$$\boldsymbol{Q}_k\dot{\boldsymbol{y}}_j + [\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k \quad \boldsymbol{G}_k\tilde{\boldsymbol{D}}_k]\begin{bmatrix}\dot{\tilde{\boldsymbol{y}}}_{k_c}\\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} = \boldsymbol{\nu}_k \tag{32}$$

The mass matrix in terms of the coordinates $\begin{bmatrix}(\tilde{\boldsymbol{y}}_{k_c})^T & (\tilde{\boldsymbol{y}}_{k_f})^T\end{bmatrix}^T$ would take the following form (see appendix (B)

$$\begin{bmatrix}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{D}}_k \\ \\ \tilde{\boldsymbol{D}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{D}}_k\end{bmatrix} \tag{33}$$

Let the dimension of velocity space of body k when it is unconstrained be represented by p_k (same as the number of components of $\dot{\boldsymbol{y}}_k$). The dimension of row space of $\boldsymbol{G}_k$ be represented by p_{k_c}. If we require $\begin{bmatrix}(\tilde{\boldsymbol{y}}_{k_c})^T & (\tilde{\boldsymbol{y}}_{k_f})^T\end{bmatrix}^T$ to be constrained and free partitions (property 1 of step 1 in section 4), then the following condition should be satisfied.

Condition 1: Columns of $\tilde{\boldsymbol{D}}_k$ should be basis for the null space of $\boldsymbol{G}_k$. In other words, Columns of $\tilde{\boldsymbol{D}}_k$ should be basis for the orthogonal complement of column space of $\boldsymbol{G}_k^T$.

This condition would render the term $\boldsymbol{G}_k\tilde{\boldsymbol{D}}_k$ in equation (32) to be zero matrix. Hence $\dot{\boldsymbol{y}}_{k_f}$ is in no way constrained by the parent coordinates. This condition also ensures that the matrix $\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k$ is invertible square matrix and hence $\dot{\boldsymbol{y}}_{k_c}$ is fully determined by the parent coordinates. Proof to show that $\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k$ is invertible square matrix, is given below.

$\boldsymbol{G}_k$ is assumed to be full rank. So it has p_{k_c} rows. The null space of $\boldsymbol{G}_k$ has the dimension $p_k-p_{k_c}$. (see for example, [Strang, 1998].) From the above condition, the number of columns in $\tilde{\boldsymbol{D}}_k$ is $p_k - p_{k_c}$. $\begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k\end{bmatrix}$ is assumed to be non-singular square matrix. Hence number of columns in $\tilde{\boldsymbol{E}}_k$ is p_{k_c}. Thus the matrix $\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k$ is square.

To prove that $\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k$ is non-singular, it is enough to show that there is not a non-zero vector say $\boldsymbol{v}_1$, such that $\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k\boldsymbol{v}_1 = \boldsymbol{0}$. Suppose there is a $\boldsymbol{v}_1$ such that $\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k\boldsymbol{v}_1 = \boldsymbol{0}$, $\boldsymbol{v}_1 \neq \boldsymbol{0}$. The vector $\tilde{\boldsymbol{E}}_k\boldsymbol{v}_1$ is non-zero (because $\tilde{\boldsymbol{E}}_k$ is full rank matrix) and lies in the null space of $\boldsymbol{G}_k$. As per condition 1 above, columns of $\tilde{\boldsymbol{D}}_k$ forms the basis for the null space of $\boldsymbol{G}_k$. Hence there is a unique non-zero $\boldsymbol{v}_2$ such that

$$\tilde{\boldsymbol{E}}_k\boldsymbol{v}_1 = \tilde{\boldsymbol{D}}_k\boldsymbol{v}_2 \tag{34}$$

This means non-zero vector $\begin{bmatrix}\boldsymbol{v}_1^T & -\boldsymbol{v}_2^T\end{bmatrix}^T$ multiplied with non-singular matrix $\begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k\end{bmatrix}$ is zero. This is a contradiction. Hence there cannot be a non-zero $\boldsymbol{v}_1$ such that $\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k\boldsymbol{v}_1 = \boldsymbol{0}$.

Additionally if the mass matrix corresponding to the partition $\begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ is to be block diagonal (property 2 of step 1 of section 4) then the following condition should also be satisfied.

Condition 2: Column space of $\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k$ should lie in the orthogonal complement of column space of $\tilde{\boldsymbol{D}}_k$.

This condition implies $\tilde{\boldsymbol{D}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k = (\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{D}}_k)^T = \boldsymbol{0}$. Hence the mass matrix corresponding to the coordinates $\begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ would become block diagonal (see equation (33)).

As seen in the example of planar-revolute system, finding the transformation

$$\dot{\boldsymbol{y}}_k = \begin{bmatrix}\boldsymbol{E}_k & \boldsymbol{D}_k\end{bmatrix}\begin{bmatrix}\dot{\tilde{\boldsymbol{y}}}_{k_c} \\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix}, \quad (\begin{bmatrix}\boldsymbol{E}_k & \boldsymbol{D}_k\end{bmatrix} \text{ is non-singular}) \tag{35}$$

which satisfy first condition (i.e, columns of $\boldsymbol{D}_k$ being the basis for the null space of $\boldsymbol{G}_k$) is not hard. For the planar revolute system, we extract $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$ from equation (24), as

$$\boldsymbol{E}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{D}_k = \begin{bmatrix} s'^{P_k}_{k_x} s\phi_k + s'^{P_k}_{k_y} c\phi_k \\ -s'^{P_k}_{k_x} c\phi_k + s'^{P_k}_{k_y} s\phi_k \\ 1 \end{bmatrix} \tag{36}$$

In equation (60) of Appendix A, we give one general procedure of finding $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$ which satisfy condition 1. Most of the time, one could arrive at $\boldsymbol{E}$ and $\boldsymbol{D}$ by looking at the geometry of the joint. In the next two sections, given $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$, we show two different ways of finding $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$, which satisfy both condition 1 and condition 2.

5.2 Two approaches to find $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$.

5.2.1 Method 1

As discussed in equation (35), we can find a matrix $\boldsymbol{D}_k$ whose columns are the basis for the null-space of $\boldsymbol{G}_k$. As per condition 1, we require columns of $\tilde{\boldsymbol{D}}_k$ to be also basis for null-space of $\boldsymbol{G}_k$. We may very well take $\tilde{\boldsymbol{D}}_k$ to be $\boldsymbol{D}_k$ itself. More generally we can take $\tilde{\boldsymbol{D}}_k$ to be

$$\tilde{\boldsymbol{D}}_k = \boldsymbol{D}_k \boldsymbol{C}_f \tag{37}$$

where $\boldsymbol{C}_f$ is *any* non-singular square matrix of size $p_{k_f} \times p_{k_f}$.

We will now find $\tilde{\boldsymbol{E}}_k$. Condition 2 requires columns of $\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k$ to lie in the orthogonal complement of column space of $\tilde{\boldsymbol{D}}_k$. This orthogonal complement has dimension p_{k_c}. However $\tilde{\boldsymbol{E}}_k$ is full column rank with p_{k_c} columns and $\boldsymbol{M}_k$ is non-singular matrix. Hence columns of $\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k$ have to be basis for the orthogonal complement of the column space of $\tilde{\boldsymbol{D}}_k$. From the condition 1 (and the $\boldsymbol{G}_k$ is full row rank), columns of $\boldsymbol{G}_k^T$ is the basis for orthogonal complement of column space of $\tilde{\boldsymbol{D}}_k$. So we may very well take $\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k = \boldsymbol{G}_k^T$, or with more generality $\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k = \boldsymbol{G}_k^T\boldsymbol{C}_c$ where $\boldsymbol{C}_c$ is any $p_{k_c} \times p_{k_c}$ non-singular matrix. Hence

$$\tilde{\boldsymbol{E}}_k = \boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T\boldsymbol{C}_c \tag{38}$$

5.2.2 Method 2

In the example shown in figure 3, we defined coordinates $\begin{bmatrix}(\bar{\boldsymbol{y}}_{k_c})^T & (\bar{\boldsymbol{y}}_{k_f})^T\end{bmatrix}^T$ in equation (24), such that $(\bar{\boldsymbol{y}}_{k_c})$ is determined by $\boldsymbol{y}_j$ and $(\bar{\boldsymbol{y}}_{k_f})$ is unconstrained by $\boldsymbol{y}_j$. Figures 4(a), (4(b)) and (4(c)) shows the displacement of rigid body k due to small changes in each of the components of these coordinates. For the same system, consider another set of coordinates defined by

$$\dot{\boldsymbol{y}}_k = \begin{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha\boldsymbol{t}_3 & \begin{bmatrix}0\\1\\0\end{bmatrix} + \beta\boldsymbol{t}_3 & \boldsymbol{t}_3 \end{bmatrix} \begin{bmatrix} \dot{\check{\boldsymbol{y}}}_{k_{c_1}} \\ \dot{\check{\boldsymbol{y}}}_{k_{c_2}} \\ \dot{\check{\boldsymbol{y}}}_{k_{f_1}} \end{bmatrix}, \text{ where} \tag{39}$$

$$\boldsymbol{t}_3 = \left(s'^{P_k}_{k_x}\sin\phi_k + s'^{P_k}_{k_y}\cos\phi_k, \quad -s'^{P_k}_{k_x}\cos\phi_k + s'^{P_k}_{k_y}\sin\phi_k, \quad 1\right)^T$$

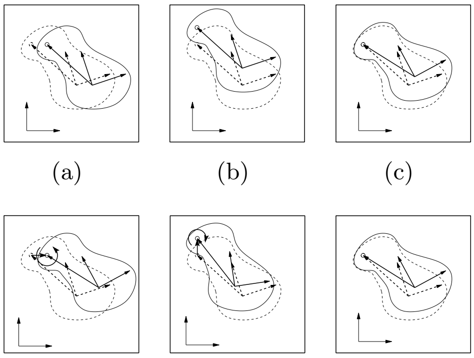


Figure 4: Visualization of various coordinates for body k – (a), (b), (c) indicates small changes in x^{Pk}, y^{Pk}, ϕ_k respectively as a part of coordinate $\bar{\boldsymbol{y}}_k$ (see equation (24)); (d), (e), (f) indicates small change in components of $\breve{y}_{k_c}$ (see equation (39)) .

The change in the system due to infinitesimal change in $\begin{bmatrix}\breve{\boldsymbol{y}}_{k_c}^T & \breve{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ is shown in figures 4(d), 4(e) and 4(f). From these figures, it may be noted that $\delta\breve{y}_{k_{c_1}}$ (or δx^{P_j}) is still determined by $\boldsymbol{\delta y}_j$. The point P^k has small displacement in x-direction if and only if there is small change in $\breve{y}_{k_{c_1}}$. Similar arguments holds for $\breve{y}_{k_{c_2}}$. From these figures it is also clear that $\breve{y}_{k_{f_1}}$ is still no way constrained by the parent body. (It could however compensate for the extra rotation due to $\boldsymbol{\delta y}_{k_c}$.) Thus $\begin{bmatrix}\breve{\boldsymbol{y}}_{k_c}^T & \breve{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ defined in equation (39) also has the partition into constrained and free parts.

The generalization of above concept is as follows:

If $\begin{bmatrix}\bar{\boldsymbol{y}}_{k_c}^T & \bar{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ is a coordinates for body k, defined by

$$\dot{\boldsymbol{y}}_k = \begin{bmatrix}\boldsymbol{E}_k & \boldsymbol{D}_k\end{bmatrix}\begin{bmatrix}\dot{\bar{\boldsymbol{y}}}_{k_c} \\ \dot{\bar{\boldsymbol{y}}}_{k_f}\end{bmatrix} \qquad \begin{bmatrix}\boldsymbol{E}_k & \boldsymbol{D}_k\end{bmatrix} \text{ is non-singular} \tag{40}$$

such that $\bar{\boldsymbol{y}}_{k_c}$ and $\bar{\boldsymbol{y}}_{k_f}$ are the constrained and free partitions of $\bar{\boldsymbol{y}}_k$, [5] then another coordinates say $\breve{\boldsymbol{y}}_k = \begin{bmatrix}\breve{\boldsymbol{y}}_{k_c}^T & \breve{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ defined by

$$\dot{\boldsymbol{y}}_k = \begin{bmatrix}\boldsymbol{E}_k + \boldsymbol{D}_k\boldsymbol{A} & \boldsymbol{D}_k\end{bmatrix}\begin{bmatrix}\dot{\breve{\boldsymbol{y}}}_{k_c} \\ \dot{\breve{\boldsymbol{y}}}_{k_f}\end{bmatrix}, \quad \boldsymbol{A} \text{ any compatible matrix.} \tag{41}$$

also has the constrained and free partitions ($\breve{\boldsymbol{y}}_{k_c}$ and $\breve{\boldsymbol{y}}_{k_c}$).

To prove the above generalization, we should show that $\begin{bmatrix}\boldsymbol{E}_k + \boldsymbol{D}_k\boldsymbol{A} & \boldsymbol{D}_k\end{bmatrix}$ is non-singular and columns of $\boldsymbol{D}_k$ is the basis for the null-space of $\boldsymbol{G}_k$ (See condition 1 in section 5.1).

Proof : $\begin{bmatrix}\boldsymbol{E}_k & \boldsymbol{D}_k\end{bmatrix}$ is a full column rank square matrix. For a column of a matrix, if we add linear combination of other columns of the matrix, then the column rank is unchanged.

[5]Appendix A shows how to find such an $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$.

Hence $\begin{bmatrix}\boldsymbol{E}_k + \boldsymbol{D}_k\boldsymbol{A} & \boldsymbol{D}_k\end{bmatrix}$ is full column rank square matrix or non-singular matrix. Since $\begin{bmatrix}\bar{\boldsymbol{y}}_{k_c}^T & \bar{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ defined in equation (40) has partition into constrained and free part, $\boldsymbol{D}_k$ satisfy condition 1 given in section 5.1, i.e, columns of $\boldsymbol{D}_k$ are the basis for the null-space of $\boldsymbol{G}_k$.

We now show that we can find a special matrix $\tilde{\boldsymbol{A}}_k$ such that $\tilde{\boldsymbol{E}}_k = \boldsymbol{E}_k + \boldsymbol{D}_k\tilde{\boldsymbol{A}}_k$ and $\tilde{\boldsymbol{D}}_k = \boldsymbol{D}_k$, while satisfying condition 1, also satisfy condition 2. From condition 2 we have

$$\tilde{\boldsymbol{D}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k = \boldsymbol{D}_k^T\boldsymbol{M}_k(\boldsymbol{E}_k + \boldsymbol{D}_k\tilde{\boldsymbol{A}}_k) = \boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{E}_k + (\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)\tilde{\boldsymbol{A}}_k = \boldsymbol{0}$$

$\boldsymbol{D}_k$ is full column rank and $\boldsymbol{M}_k$ is positive definite. Hence $\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k$ is also positive definite and $(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}$ exists. So we can write

$$\tilde{\boldsymbol{A}}_k = -(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{E}_k \tag{42}$$

$$\tilde{\boldsymbol{E}}_k = \boldsymbol{E}_k + \boldsymbol{D}_k\tilde{\boldsymbol{A}}_k = \boldsymbol{E}_k - \boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{E}_k \tag{43}$$

$$\tilde{\boldsymbol{D}}_k = \boldsymbol{D}_k \tag{44}$$

Thus we have obtained $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$ satisfying both conditions 1 and 2. The expressions are in terms of $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$ (see equation (35)).

To summarize, if we know coordinates $\bar{\boldsymbol{y}}_k = \begin{bmatrix}\bar{\boldsymbol{y}}_{k_c}^T & \bar{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$, satisfying property 1 of step 1 and its relation $\dot{\boldsymbol{y}}_k = \begin{bmatrix}\boldsymbol{E}_k & \boldsymbol{D}_k\end{bmatrix}\begin{bmatrix}\dot{\bar{\boldsymbol{y}}}_{k_c} \\ \dot{\bar{\boldsymbol{y}}}_{k_f}\end{bmatrix}$ with $\boldsymbol{y}_k$, then the coordinates $\tilde{\boldsymbol{y}}_k = \begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ satisfying both properties of step 1 in section 4 is defined by $\dot{\boldsymbol{y}}_k = \begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k\end{bmatrix}\begin{bmatrix}\dot{\tilde{\boldsymbol{y}}}_{k_c} \\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix}$, where $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$ are as in equation (38) and (37) or (43) and (44). If the constraint equation corresponding to joint k and the joint variables are known, then Appendix A gives the procedure to find $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$.

5.3 Simplification of equations of RFDA to standard form

The coordinates $\tilde{\boldsymbol{y}}_k$ discussed in step 1, is now defined as $\dot{\boldsymbol{y}}_k = \begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k\end{bmatrix}\begin{bmatrix}\dot{\tilde{\boldsymbol{y}}}_{k_c} \\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix}$, where $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$, is given by equations (38, 37) or (43, 44). $\tilde{\boldsymbol{G}}_{k_c}$, $\tilde{\boldsymbol{\nu}}_k$, $\boldsymbol{S}_k$, $\boldsymbol{a}_k$, $\tilde{\boldsymbol{M}}_{k_c}$, $\tilde{\boldsymbol{M}}_{k_f}$, $\tilde{\boldsymbol{f}}_{k_c}$ and $\tilde{\boldsymbol{f}}_{k_f}$, in equations (21), (22) and (23), assume the following forms : $\tilde{\boldsymbol{G}}_{k_c} = (\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)$, $\tilde{\boldsymbol{\nu}}_k = \boldsymbol{\nu}_k$, $\boldsymbol{S}_k = \left(-(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-1}\boldsymbol{Q}_k\right)$, $\boldsymbol{a}_k = (\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-1}\boldsymbol{\nu}_k$, $\tilde{\boldsymbol{M}}_{k_c} = \tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k$, $\tilde{\boldsymbol{M}}_{k_f} = \tilde{\boldsymbol{D}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{D}}_k$, $\tilde{\boldsymbol{f}}_{k_c} = \tilde{\boldsymbol{E}}_k^T(\boldsymbol{f}_k - \boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c} + \dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}))$, $\tilde{\boldsymbol{f}}_{k_f} = \tilde{\boldsymbol{D}}_k^T(\boldsymbol{f}_k - \boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c} + \dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}))$.

Substitution of above terms in equation (28) and further simplification by using the expressions for $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$ in equations (38) and (37), we get

$$\begin{bmatrix}\boldsymbol{M}_j + \boldsymbol{Q}_k^T(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-1}\boldsymbol{Q}_k & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C}_f^T\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k\boldsymbol{C}_f\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{y}}_j \\ \ddot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} + \begin{bmatrix}\boldsymbol{G}_j^T \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{\lambda}_j = \begin{bmatrix}\boldsymbol{f}_j + \boldsymbol{Q}_k^T(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-1}(\gamma_k - \boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{f}_k) \\ \boldsymbol{C}_f^T\boldsymbol{D}_k^T(\boldsymbol{f}_k - \boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c} + \dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}))\end{bmatrix} \tag{45}$$

In equation (66) of Appendix C.2, we show that $\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c}+\dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}+\tilde{\boldsymbol{E}}_k(\dot{\boldsymbol{S}}_k\dot{\boldsymbol{y}}_j+\dot{\boldsymbol{a}}_k))=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-1}\boldsymbol{\gamma}$. Rest of the simplifications in the equation (45) is straightforward.

In equations (43) and (44), we had obtained alternate expression for $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$, in terms of $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$ of equation (35). In Appendix A, one way to obtain $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$ is given (see equation (60)), along with its relation with $\boldsymbol{B}_k$, $\boldsymbol{H}_k$ and $\boldsymbol{c}_k$ (see equation (61). Using equations (43), (44), (60) and (61), we get yet another simplification as -

$$\begin{bmatrix}\boldsymbol{M}_j+\boldsymbol{B}_k^T(\boldsymbol{I}-\boldsymbol{M}_k\boldsymbol{H}_k(\boldsymbol{H}_k^T\boldsymbol{M}_k\boldsymbol{H}_k)^{-1}\boldsymbol{H}_k^T)\boldsymbol{M}_k\boldsymbol{B}_k & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{y}}_j\\ \ddot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix}$$
$$+\begin{bmatrix}\boldsymbol{G}_j^T\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{\lambda}_j=\begin{bmatrix}\boldsymbol{f}_j+\boldsymbol{B}_k^T(\boldsymbol{I}-\boldsymbol{M}_k\boldsymbol{H}_k(\boldsymbol{H}_k^T\boldsymbol{M}_k\boldsymbol{H}_k)^{-1}\boldsymbol{H}_k^T)(\boldsymbol{f}_k-\boldsymbol{M}_k\boldsymbol{d}_k)\\ \boldsymbol{D}_k^T(\boldsymbol{f}_k-\boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c}+\dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} \quad (46)$$

where $\boldsymbol{d}_k=\dot{\boldsymbol{B}}_k\dot{\boldsymbol{y}}_j+\dot{\boldsymbol{H}}_k\dot{\boldsymbol{q}}_k+\dot{\boldsymbol{c}}_k$, with $\boldsymbol{B}_k$, $\boldsymbol{H}_k$ and $\boldsymbol{c}_k$ as given in equation (61). Details of simplification of equation (46) is presented in Appendix C.3.

Note that if j and k (k treated as terminal node) are the only two bodies of the multibody system, then equation (8), with equations (9) and (10) used for $\widehat{\boldsymbol{M}}_j$ and $\widehat{\boldsymbol{f}}_j$, is same as $\boldsymbol{y}_j$ part of the equation (45). Similarly, equation (8), with equations (11) and (12) used for $\widehat{\boldsymbol{M}}_j$ and $\widehat{\boldsymbol{f}}_j$, is same as $\boldsymbol{y}_j$ part of the equation (46). We recall that equations (9) and (10) are presented in reference [Lubich et al, 1992], and equations (11) and (12) are presented in [Lubich et al, 1992], [Bae and Haug, 1987] and [Featherstone, 1983]. Thus we have reduced the equations of RFDA obtained by our method into standard form found in literature.

5.4 Visualization of coordinates for planar revolute joint example

5.4.1 Coordinates of step 1

For the planar revolute joint system, using equation (36) and $\boldsymbol{M}_k$ given in section 3.2.1, $\tilde{\boldsymbol{A}}_k$ of equation (42), evaluates to

$$\begin{bmatrix}-\frac{m_k(\sin(\phi_k)s_{k_x}^{\prime P_k}+\cos(\phi_k)s_{k_y}^{\prime P_k})}{J_k+m_k((s_{k_x}^{\prime P_k})^2+(s_{k_y}^{\prime P_k})^2)} & \frac{m_k(\cos(\phi_k)s_{k_x}^{\prime P_k}-\sin(\phi_k)s_{k_y}^{\prime P_k})}{J_k+m_k((s_{k_x}^{\prime P_k})^2+(s_{k_y}^{\prime P_k})^2)}\end{bmatrix} \quad (47)$$

Let the two elements in the above matrix be represented by $\tilde{\alpha}$ and $\tilde{\beta}$.

The coordinates $\tilde{\boldsymbol{y}}_k$ of step 1, is same as the coordinates $\breve{\boldsymbol{y}}_k$, defined in equation (39), except that α and β are replaced by $\tilde{\alpha}$ and $\tilde{\beta}$. It turns out that the equation (after replacing by $\tilde{\alpha}$ and $\tilde{\beta}$) is non-integrable. Hence $\tilde{\boldsymbol{y}}_k$ is a pseudo-coordinate for the system. We can visualize it by examining the changes that body k undergoes, due to small changes in the components of $\tilde{\boldsymbol{y}}_k$. The figures (4(d)), (4(e)) and (4(f)), hold good here also. The rotation in figure (4(d)), is $\tilde{\alpha}$ times the translation along x-axis. The rotation in figure (4(e)), is $\tilde{\beta}$ times the translation along y-axis.

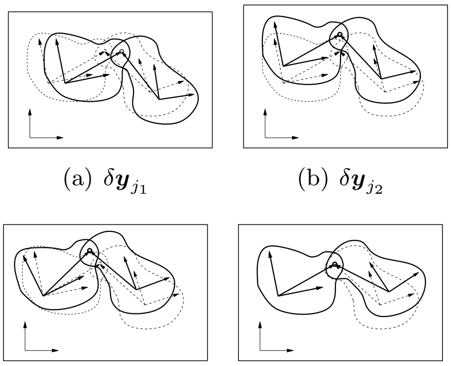
5.4.2 Coordinates of step 2 – coordinates of RFDA

Equation (31), gives the relation between $\boldsymbol{y}_k$ and $\begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$. Equation (26), gives the relation between $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ and $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$. These two relations could be composed to obtain the relation between $\begin{bmatrix}\boldsymbol{y}_j^T & \boldsymbol{y}_k^T\end{bmatrix}^T$ and $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$. For the planar revolute joint example, such a relation assumes following form –

$$\begin{bmatrix}\dot{\boldsymbol{y}}_j \\ \dot{\boldsymbol{y}}_k\end{bmatrix} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0} \\ -\left(\boldsymbol{E}_k + \boldsymbol{D}_k\begin{bmatrix}\tilde{\alpha} & \tilde{\beta}\end{bmatrix}\right)\boldsymbol{Q}_k & \boldsymbol{D}_k\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{y}}_j \\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\end{bmatrix} \tag{48}$$

where $\begin{bmatrix}\tilde{\alpha} & \tilde{\beta}\end{bmatrix} = \tilde{\boldsymbol{A}}$ is given in equation (47), $\boldsymbol{E}_k$ and $\boldsymbol{D}_k$ in equation (36) and $\boldsymbol{Q}_k$ in section 3.2.1.

To get a feel for the coordinates of RFDA, figures 5(a), 5(b), 5(c) and 5(d), shows (highly exaggerated) displacement of the system due to infinitesimal changes in each of the components of $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$. Also to see how different these coordinates are from standard coordinates, in figures 6(a), 6(b), 6(c) and 6(d), we have shown the exaggerated displacement of the system due to infinitesimal change in each of the components of $\bar{\boldsymbol{y}}$ defined in equation (20).



(a) $\delta\boldsymbol{y}_{j_1}$

(b) $\delta\boldsymbol{y}_{j_2}$

(c) $\delta\boldsymbol{y}_{j_3}$

(d) $\delta\tilde{\boldsymbol{y}}_{k_{f_1}}$

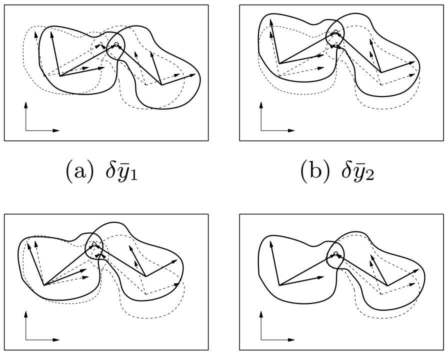
Figure 5: Changes in system due to small changes in the coordinate $\begin{bmatrix}\boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{k_f}^T\end{bmatrix}^T$ defined in equation (48).

The comparison between the two coordinates is summarized as follows.
If, $[\delta\bar{\boldsymbol{y}}_1 \quad \delta\bar{\boldsymbol{y}}_2 \quad \delta\bar{\boldsymbol{y}}_3 \quad \delta\bar{\boldsymbol{y}}_4]^T = \begin{bmatrix}\delta\boldsymbol{y}_{j_1} & \delta\boldsymbol{y}_{j_2} & \delta\boldsymbol{y}_{j_3} & \delta\tilde{\boldsymbol{y}}_{k_{f_1}}\end{bmatrix}^T$, then

1. displacement of body j is same in the corresponding figures for the two coordinates.

2. The corresponding figures differ only in the rotation that the body k undergoes about the point P^k. The rotation of body k in figures 5(a), 5(b), 5(c) and 5(d), are $\tilde{\boldsymbol{\alpha}}\delta\boldsymbol{y}_{j_1}$, $\tilde{\boldsymbol{\beta}}\delta\boldsymbol{y}_{j_2}$, $\left(\left(-\boldsymbol{s'}^{P_k}_{k_x}s\phi_k - \boldsymbol{s'}^{P_k}_{k_y}c\phi_k\right)\tilde{\alpha} + \left(\boldsymbol{s'}^{P_k}_{k_x}c\phi_k - \boldsymbol{s'}^{P_k}_{k_y}s\phi_k\right)\tilde{\beta}\right)\delta\boldsymbol{y}_{j_3}$, and $\delta\tilde{\boldsymbol{y}}_{k_{f_1}}$, respectively ($s\phi_k$ represent $\sin\phi_k$ and $c\phi_k$ represent $\cos\phi_k$). The rotations in figures 6(a), 6(b), 6(c) and 6(d), are 0, 0, $\delta\bar{\boldsymbol{y}}_3$ and $\delta\bar{\boldsymbol{y}}_4$, respectively.

In this section we defined the coordinates of Step 1 by the relation given in equation (31) and gave explicit expressions for $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$ using two methods. First method lead to equations (37) and (38), while the second method resulted in equations (43) and (44).



(a) $\delta\bar{y}_1$ (b) $\delta\bar{y}_2$

(c) $\delta\bar{y}_3$ (d) $\delta\bar{y}_4$

Figure 6: Changes in system due to small changes in the coordinate $\bar{\boldsymbol{y}}$, defined in equation (20).

Kinematic observation was used to motivate second method. We further simplified equations of RFDA to the form generally found in literature. We illustrated the $\tilde{\boldsymbol{y}}_k$ coordinates of step 1, and the coordinates of RFDA $[\boldsymbol{y}_j^T \ \tilde{\boldsymbol{y}}_{k_f}^T]^T$, using planar revolute joint example.

6 Generalization to all nodes

Consider any non-terminal node, j, of the branched multibody system. Suppose that for every child, k, of node j, there are coordinates $\begin{bmatrix}\boldsymbol{y}_k^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$, having the following properties

1. $\begin{bmatrix}\boldsymbol{y}_k^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$ describes the system consisting of node k and all its descendants. In other words, the relation between $\begin{bmatrix}\boldsymbol{y}_k^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$ and absolute coordinates of k and its descendants is of the form

$$\begin{bmatrix} \dot{\boldsymbol{y}}_k \\ \begin{bmatrix} \dot{\boldsymbol{y}}_{h(k,1)} \\ \vdots \\ \dot{\boldsymbol{y}}_{h(k,d_k)} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{P}_k & \boldsymbol{R}_k \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{y}}_k \\ \dot{\boldsymbol{\xi}}_k \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\varpi}_k \end{bmatrix} \quad \forall k : j = \mathcal{P}(k)$$

where $h(i,p)$ is the p^{th} descendant of node i, with the descendants are arranged in some order. [6] ($\mathcal{P}(i)$ denote parent index of i, and $k : j = \mathcal{P}(k)$ denote 'k is such that, j is its parent'.) An example for above equation is equation (48), where there is a single descendant node.

2. Any value for $\begin{bmatrix}\dot{\boldsymbol{y}}_k^T & \dot{\boldsymbol{\xi}}_k^T\end{bmatrix}^T$ is consistent with all the joint constraints among body k and its descendants. The only constraint on the coordinates $\begin{bmatrix}\dot{\boldsymbol{y}}_k^T & \dot{\boldsymbol{\xi}}_k^T\end{bmatrix}^T$, is due to joint between body k and body j, given by

$$\boldsymbol{Q}_k\dot{\boldsymbol{y}}_j + \boldsymbol{G}_k\dot{\boldsymbol{y}}_k + [\boldsymbol{0}]\dot{\boldsymbol{\xi}}_k = \boldsymbol{\nu}_k \qquad \forall k : j = \mathcal{P}(k) \tag{49}$$

[6] Descendants of node i can always be arranged in a sequence. For example, the descendants of node 2 in figure 1 can be arranged as - 4, 5, 7, 10, 11, 12. For this order, $h(2,3) = 7$.

3. The equation of motion in terms of $\begin{bmatrix}\boldsymbol{y}_k^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$, for the system consisting of body k and all its descendants, is of the form

$$\begin{bmatrix}\widehat{\boldsymbol{M}}_k & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Lambda}_k\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{y}}_k\\ \ddot{\boldsymbol{\xi}}_k\end{bmatrix}+\begin{bmatrix}\boldsymbol{G}_k^T\boldsymbol{\lambda}_k\\ \boldsymbol{0}\end{bmatrix}=\begin{bmatrix}\widehat{\boldsymbol{f}}_k\\ \boldsymbol{\eta}_k\end{bmatrix}\quad \forall k: j=\mathcal{P}(k) \tag{50}$$

then we claim that we can find coordinates $\begin{bmatrix}\boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T\end{bmatrix}^T$ which has same properties listed above, i.e 1) $\begin{bmatrix}\boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T\end{bmatrix}^T$ describes body j and all its descendants, 2) the coordinates are consistent with all the joint constraints between node j and its descendants, and 3) mass matrix with respect to the coordinates is block diagonal.

Proof: Find $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$ (with $\begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k\end{bmatrix}$ being non-singular) that satisfy condition 1 and condition 2 of section 5.1, with $\boldsymbol{M}_k$ replaced by $\widehat{\boldsymbol{M}}_k$. The only property of $\boldsymbol{M}_k$ used in deriving $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$ in subsection 5.2 was positive definiteness of $\boldsymbol{M}_k$. $\widehat{\boldsymbol{M}}_k$ is diagonal block of the positive definite matrix given in equation (50). Hence $\widehat{\boldsymbol{M}}_k$ is also positive definite. Thus we can use theory in subsection 5.2 to find $\tilde{\boldsymbol{E}}_k$ and $\tilde{\boldsymbol{D}}_k$, even after the replacement.

Define new coordinates for the system consisting of node k and its descendants, as

$$\begin{bmatrix}\dot{\boldsymbol{y}}_k\\ \dot{\boldsymbol{\xi}}_k\end{bmatrix}=\begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\dot{\tilde{\boldsymbol{y}}}_{k_c}\\ \dot{\tilde{\boldsymbol{y}}}_{k_f}\\ \dot{\boldsymbol{\xi}}_k\end{bmatrix}\quad \forall \quad k: j=\mathcal{P}(k) \tag{51}$$

Since $\begin{bmatrix}\tilde{\boldsymbol{E}}_k & \tilde{\boldsymbol{D}}_k\end{bmatrix}$ is non-singular, the above transformation as a whole is non-singular.

Under the transformation from $\begin{bmatrix}\boldsymbol{y}_k^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$ coordinates to $\begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$, the only constraint equation involving $\begin{bmatrix}\boldsymbol{y}_k^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$, i.e, equation (49), becomes (see equation (62)

$$\boldsymbol{Q}_k\dot{\boldsymbol{y}}_j+\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c}+[\boldsymbol{0}]\,\dot{\tilde{\boldsymbol{y}}}_{k_f}+[\boldsymbol{0}]\,\dot{\boldsymbol{\xi}}_k=\boldsymbol{\nu}_k,\quad \forall k: j=\mathcal{P}(k) \tag{52}$$

(from condition 1 of section 5.1, $\boldsymbol{G}_k\tilde{\boldsymbol{D}}_k=\boldsymbol{0}$.) This is the only constraint equation involving $\begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$. Hence, both $\dot{\tilde{\boldsymbol{y}}}_{k_f}$ and $\dot{\boldsymbol{\xi}}_k$ are kinematically unconstrained. $\dot{\tilde{\boldsymbol{y}}}_{k_c}$ is determined by $\dot{\boldsymbol{y}}_j$ by the following equation.

$$\dot{\tilde{\boldsymbol{y}}}_{k_c}=\boldsymbol{S}_k\dot{\boldsymbol{y}}_j+\boldsymbol{a}_k\quad \forall k: j=\mathcal{P}(k) \tag{53}$$

where $\boldsymbol{S}_k=-\left(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k\right)^{-1}\boldsymbol{Q}_k$, and $\boldsymbol{a}_k=\left(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k\right)^{-1}\boldsymbol{\nu}_k$. The above equation follows from equation (52). The equation of motion of the system consisting of node k and its descendants, written in terms of coordinates $\begin{bmatrix}\tilde{\boldsymbol{y}}_{k_c}^T & \tilde{\boldsymbol{y}}_{k_f}^T & \boldsymbol{\xi}_k^T\end{bmatrix}^T$, becomes (see equation (64)

$$\begin{bmatrix}\tilde{\boldsymbol{M}}_{k_c} & \boldsymbol{0}\\ \boldsymbol{0} & \tilde{\boldsymbol{M}}_{k_f}\end{bmatrix}\begin{bmatrix}\ddot{\tilde{\boldsymbol{y}}}_{k_c}\\ \ddot{\tilde{\boldsymbol{\xi}}}_k\end{bmatrix}=\begin{bmatrix}\tilde{\boldsymbol{f}}_{k_c}\\ \tilde{\boldsymbol{\eta}}_k\end{bmatrix}+\begin{bmatrix}\left(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k\right)^T\boldsymbol{\lambda}_k\\ \boldsymbol{0}\end{bmatrix}\quad \forall k: j=\mathcal{P}(k) \tag{54}$$

where $\boldsymbol{M}_{k_c}=\tilde{\boldsymbol{E}}_k^T\widehat{\boldsymbol{M}}_k\tilde{\boldsymbol{E}}_k$, $\tilde{\boldsymbol{M}}_{k_f}=\begin{bmatrix}\tilde{\boldsymbol{D}}_k^T\widehat{\boldsymbol{M}}_k\tilde{\boldsymbol{D}}_k & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Lambda}_k\end{bmatrix}$, $\tilde{\boldsymbol{\xi}}_k=\begin{bmatrix}\tilde{\boldsymbol{y}}_{k_f}\\ \boldsymbol{\xi}_k\end{bmatrix}$,

$\tilde{\boldsymbol{f}}_{k_c}=\tilde{\boldsymbol{E}}_k^T\left(\widehat{\boldsymbol{f}}_k-\widehat{\boldsymbol{M}}_k\left(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c}+\dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}\right)\right)$, $\tilde{\boldsymbol{\eta}}_k=\begin{bmatrix}\tilde{\boldsymbol{D}}_k^T\left(\widehat{\boldsymbol{f}}_k-\widehat{\boldsymbol{M}}_k\left(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c}+\dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}\right)\right)\\ \boldsymbol{\eta}_k\end{bmatrix}$.

Let $g(i,p)$ denote p^{th} child of node i, with the children of node i arranged in some order. Also let h_i represent the number of children of node i. Then, the coordinates

$$\begin{bmatrix} \boldsymbol{y}_j^T & \tilde{\boldsymbol{y}}_{g(j,1)_c}^T & \tilde{\boldsymbol{\xi}}_{g(j,1)}^T & \cdots & \tilde{\boldsymbol{y}}_{g(j,h_j)_c}^T & \tilde{\boldsymbol{\xi}}_{g(j,h_j)}^T \end{bmatrix}^T$$

describes node j and all its descendants. The constraints on these coordinates are equation (52), and the constraint due to joint between body j and its parent, i.e $\boldsymbol{Q}_j\dot{\boldsymbol{y}}_{\mathcal{P}(j)}+\boldsymbol{G}_j\dot{\boldsymbol{y}}_j = \boldsymbol{\nu}_j$.

Using the relation (53), we obtain a smaller coordinates $\begin{bmatrix} \boldsymbol{y}_j^T & \tilde{\boldsymbol{\xi}}_{g(j,1)}^T & \cdots & \tilde{\boldsymbol{\xi}}_{g(j,h_j)}^T \end{bmatrix}^T$ by the following transformation.

$$\begin{bmatrix} \dot{\boldsymbol{y}}_j \\ \dot{\tilde{\boldsymbol{y}}}_{g(j,1)_c} \\ \dot{\tilde{\boldsymbol{\xi}}}_{g(j,1)} \\ \vdots \\ \dot{\tilde{\boldsymbol{y}}}_{g(j,h_j)_c} \\ \dot{\tilde{\boldsymbol{\xi}}}_{g(j,h_j)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{S}_{g(j,1)} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{S}_{g(j,h_j)} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{y}}_j \\ \dot{\tilde{\boldsymbol{\xi}}}_{g(j,1)} \\ \vdots \\ \dot{\tilde{\boldsymbol{\xi}}}_{g(j,h_j)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{a}_{g(j,1)} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{a}_{g(j,h_j)} \\ \boldsymbol{0} \end{bmatrix} \tag{55}$$

From the expression of $\boldsymbol{S}_k$ and $\boldsymbol{a}_k$ given in equation (53), it is easy to see that constraint equations (52) become redundant under the new coordinates defined above. Let $\boldsymbol{\xi}_j = \begin{bmatrix} \tilde{\boldsymbol{\xi}}_{g(j,1)}^T & \cdots & \tilde{\boldsymbol{\xi}}_{g(j,h_j)}^T \end{bmatrix}^T$. Then the only constraint equation on $\begin{bmatrix} \boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T \end{bmatrix}^T$ coordinates is due to joint between j and its parent can be written as

$$\boldsymbol{Q}_j\dot{\boldsymbol{y}}_{\mathcal{P}(j)} + \boldsymbol{G}_j\dot{\boldsymbol{y}}_j + \boldsymbol{0}\dot{\boldsymbol{\xi}}_j = \boldsymbol{\nu}_j \tag{56}$$

Analogous to equation (28) the equation of motion in terms of $\begin{bmatrix} \boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T \end{bmatrix}^T$ is obtained below.

$$\begin{bmatrix} \boldsymbol{M}_j + \sum\limits_{k:\mathcal{P}(k)=j} \boldsymbol{S}_k^T \tilde{\boldsymbol{M}}_{k_c} \boldsymbol{S}_k & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Lambda}_j \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{y}}_j \\ \ddot{\boldsymbol{\xi}}_j \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_j^T \boldsymbol{\lambda}_j \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{d_j} \\ \boldsymbol{\eta}_j \end{bmatrix} \tag{57}$$

where $\boldsymbol{\Lambda}_j = \begin{bmatrix} \tilde{\boldsymbol{M}}_{g(j,1)_f} & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \tilde{\boldsymbol{M}}_{g(j,h_j)_f} \end{bmatrix}$, $\boldsymbol{\eta}_j = \begin{bmatrix} \tilde{\boldsymbol{\eta}}_{g(j,1)} \\ \vdots \\ \tilde{\boldsymbol{\eta}}_{g(j,h_j)} \end{bmatrix}$, and $\boldsymbol{f}_{d_j} = \boldsymbol{f}_j + \sum\limits_{k:\mathcal{P}(k)=j} \left(\boldsymbol{S}_k^T \tilde{\boldsymbol{M}}_{k_c} \boldsymbol{b}_k \right)$.

Thus associated with node j we have found the coordinates $\begin{bmatrix} \boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T \end{bmatrix}^T$ having the required properties and the proof is complete.

If k is a terminal node, then the coordinates y_k trivially satisfies all the properties mentioned in the claim at the beginning of this section. Now, from the result obtained above, we can recursively obtain $\begin{bmatrix} \boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T \end{bmatrix}^T$ coordinates for all nodes of tree structure.

Thus associated with each node of the tree structure, we can obtain $\begin{bmatrix} \boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T \end{bmatrix}^T$ coordinates which has the constraint equation as in equation (56) and equation of motion as in equation (57). The rows of matrix equation (57) associated with $\boldsymbol{y}_j$ is the first part of the equation (8). The constraint equation (56) is the second part of the equation. So we have obtained equations of RFDA, based on finding coordinates of RFDA, $\begin{bmatrix} \boldsymbol{y}_j^T & \boldsymbol{\xi}_j^T \end{bmatrix}^T$, for every node the tree.

Analogous to simplification from equation (28) to equation (45) or equation (46), we can show the rows of matrix equation (57) associated with $\boldsymbol{y}_j$ simplifies to $\widehat{\boldsymbol{M}}_j + \boldsymbol{G}_j^T \boldsymbol{\lambda}_j = \widehat{\boldsymbol{f}}_j$, with $\widehat{\boldsymbol{M}}_j$ and $\widehat{f}_j$ as given in equation (9) and (10) or (11) and (12).

7 Conclusions

In this paper, we derive equations of RFDA using a new method. The method has two parts, 1) finding coordinates of RFDA, and 2) writing equations of motion in terms of it and extracting relevant portion of it as equations of RFDA. In section 4, the method has been described for a simple two noded tree structure, in 4 steps and the non-trivial coordinates of step 1 has been worked out in section 5. Steps 1 and 2 constitute the procedure to find the coordinates of RFDA. Steps 3 and 4 are about writing down equation of motion and extracting relevant portion of it.

The crux of the paper lies in section 5.2, where coordinates required for step 1 is defined. Two different methods of finding the coordinates has been explained. We use linear algebraic arguments, motivated by kinematic intuition, to get the coordinates. The originality of the paper lies here.

We simplified the relevant portion of its equation of motion, in terms of coordinates of RFDA, to standard form in equations (45) and (46). In section 6, our approach based on coordinates of RFDA was extended to general tree structure. Different nodes have different coordinates of RFDA associated with them, which describes the rigid bodies of the node and all its descendants. The coordinates of RFDA get defined recursively, as in equation (55) and equations of motion with respect to them are as in equation (57).

This derivation conclusively shows that equations of RFDA are actually part of equations of motion. More importantly this derivation gives coordinates associated with equation of motion, as well as left out part of equations of motion. These are significant insights into RFDA, a important algorithm in multibody dynamics.

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A Coordinate of child body in terms coordinate of parent and joint variables

Consider body j and k with a joint k between them. The constraint is represented as $\boldsymbol{Q}_k\dot{\boldsymbol{y}}_j + \boldsymbol{G}_k\dot{\boldsymbol{y}}_k = \boldsymbol{\gamma}_k$. If there are p_{k_c} rows in the constraint equation and p_k represent the degree of freedom of unconstrained body k, then one can associate joint variable $\boldsymbol{q}_k$ of dimension $(p_k - p_{k_c})$. For example if k^{th} joint is revolute joint then $\boldsymbol{q}_k = [\theta_k]$, where θ_k is joint rotation angle. If $\boldsymbol{q}_k$ is constrained as a function of time, say, $\dot{\boldsymbol{q}} = \boldsymbol{g}(t)$, then such a constraint is called driving constraint. Driving constraints have the form [7]

$$\bar{\boldsymbol{Q}}_{k_d}\dot{\boldsymbol{y}}_j + \bar{\boldsymbol{G}}_{k_d}\dot{\boldsymbol{y}}_k = \bar{\boldsymbol{\nu}}_{k_d} + \boldsymbol{\mathfrak{H}}_{k_d}\dot{\boldsymbol{q}}_k \tag{58}$$

([Haug, 1989] catalogs driving constraint and corresponding Jacobians for standard joints.) We consider only cases where $\boldsymbol{\mathfrak{H}}_{k_d}$ is invertible.

For example, consider planar revolute joint example given in figure 3. The constraint equation is $\boldsymbol{Q}_k\dot{\boldsymbol{y}}_j + \boldsymbol{G}_k\dot{\boldsymbol{y}}_k = \boldsymbol{\nu}_k$, where $\dot{\boldsymbol{y}}_j$, $\dot{\boldsymbol{y}}_k$, $\boldsymbol{Q}_k$ and $\boldsymbol{G}_k$ are given in section 3.2.1, and $\boldsymbol{\nu}_k = [0 \quad 0]^T$. The joint driving constraint can be written as $\phi_k - \phi_j = \theta_k(t)$, where θ_k is

[7] If the driving constraint is holonomic, then the constraint can be written as $\bar{\boldsymbol{\Phi}}(\boldsymbol{y}_j, \boldsymbol{y}_k, \boldsymbol{q}_k(t), t) = \mathbf{0}$. Its differentiation gives

$$\frac{\partial\bar{\boldsymbol{\Phi}}}{\partial\boldsymbol{y}_j}\dot{\boldsymbol{y}}_j + \frac{\partial\bar{\boldsymbol{\Phi}}}{\partial\boldsymbol{y}_k}\dot{\boldsymbol{y}}_k = -\frac{\partial\bar{\boldsymbol{\Phi}}}{\partial t} - \frac{\partial\bar{\boldsymbol{\Phi}}}{\partial\boldsymbol{q}_k}\dot{\boldsymbol{q}}_k$$

The above equation is the motivation for equation (58).

the joint angle. For this driving constraint, $\bar{\boldsymbol{Q}}_{k_d} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$, $\bar{\boldsymbol{G}}_{k_d} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $\bar{\boldsymbol{\nu}}_{k_d} = [0]$ and $\boldsymbol{\mathfrak{H}}_{k_d} = [1]$.

Assume that driving constraints are present in the system so that we can write constraint equation as

$$\begin{bmatrix} \boldsymbol{Q}_k \\ \boldsymbol{Q}_{k_d} \end{bmatrix} \dot{\boldsymbol{y}}_j + \begin{bmatrix} \boldsymbol{G}_k \\ \boldsymbol{G}_{k_d} \end{bmatrix} \dot{\boldsymbol{y}}_k = \begin{bmatrix} \boldsymbol{\nu}_k \\ \boldsymbol{\nu}_{k_d} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \dot{\boldsymbol{q}}_k \end{bmatrix} \tag{59}$$

where $\boldsymbol{Q}_{k_d} = \boldsymbol{\mathfrak{H}}_{k_d}^{-1} \bar{\boldsymbol{Q}}_{k_d}$, $\boldsymbol{G}_{k_d} = \boldsymbol{\mathfrak{H}}_{k_d}^{-1} \bar{\boldsymbol{G}}_{k_d}$, and $\boldsymbol{\nu}_{k_d} = \boldsymbol{\mathfrak{H}}_{k_d}^{-1} \bar{\boldsymbol{\nu}}_{k_d}$. We restrict ourselves to the cases where $\begin{bmatrix} \boldsymbol{G}_k \\ \boldsymbol{G}_{k_d} \end{bmatrix}$ is invertible. Let $\begin{bmatrix} \boldsymbol{G}_k \\ \boldsymbol{G}_{k_d} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{E}_k & \boldsymbol{D}_k \end{bmatrix}$ so that,

$$\begin{bmatrix} \boldsymbol{G}_k \\ \boldsymbol{G}_{k_d} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_k & \boldsymbol{D}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_k \boldsymbol{E}_k & \boldsymbol{G}_k \boldsymbol{D}_k \\ \boldsymbol{G}_{k_d} \boldsymbol{E}_k & \boldsymbol{G}_{k_d} \boldsymbol{D}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{p_{k_c} \times p_{k_c}} & \boldsymbol{0}_{p_{k_c} \times p_{k_f}} \\ \boldsymbol{0}_{p_{k_f} \times p_{k_c}} & \boldsymbol{I}_{p_{k_f} \times p_{k_f}} \end{bmatrix} \tag{60}$$

Multiplying equation (59) with $\begin{bmatrix} \boldsymbol{G}_k \\ \boldsymbol{G}_{k_d} \end{bmatrix}^{-1}$ and using equation (60), we get

$$\dot{\boldsymbol{y}}_k = \boldsymbol{B}_k \dot{\boldsymbol{y}}_j + \boldsymbol{H}_k \dot{\boldsymbol{q}}_k + \boldsymbol{c}_k \tag{61}$$

where $\boldsymbol{B}_k = -(\boldsymbol{E}_k \boldsymbol{Q}_k + \boldsymbol{D}_k \boldsymbol{Q}_{k_d})$, $\boldsymbol{H}_k = \boldsymbol{D}_k$ and $\boldsymbol{c}_k = (\boldsymbol{E}_k \boldsymbol{\nu}_k + \boldsymbol{D}_k \boldsymbol{\nu}_{k_d})$. Equation (61) essentially gives coordinate of body k in terms of coordinate of parent body j and joint coordinate.

B Equations of motion and change of coordinates

Let $\boldsymbol{u}$ be coordinates describing a rigid multibody system. There could be constraints on $\dot{\boldsymbol{u}}$. We consider only those constraints that could be expressed in the form $\boldsymbol{\Psi} \dot{\boldsymbol{u}} = \boldsymbol{\nu}$ with $\boldsymbol{\Psi}$ having full row rank. Application of generalized d'Alembert's principle (see for example, [Meirovitch, 1970]) to the multibody system leads to equation of motion of form, $\boldsymbol{M}_{\boldsymbol{u}} \ddot{\boldsymbol{u}} = \boldsymbol{f}_{\boldsymbol{u}} - \boldsymbol{\Psi}^T \boldsymbol{\lambda}$. $\boldsymbol{M}_u$ is function of $\boldsymbol{u}$ and t. $\boldsymbol{f}_u$, in general could be function of $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, t and even $\boldsymbol{\lambda}$. However, in this paper we restrict ourselves to cases where $\boldsymbol{f}_u$ is function of $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$ and t only.

Consider new coordinates $\boldsymbol{v}$, having the relation with $\boldsymbol{u}$ as, $\dot{\boldsymbol{u}} = \boldsymbol{T} \dot{\boldsymbol{v}} + \boldsymbol{e}$, where $\boldsymbol{T}$ is full column rank. The constraint equation in terms of $\boldsymbol{v}$ would be

$$\big(\boldsymbol{\Psi} \boldsymbol{T}\big) \dot{\boldsymbol{v}} = \boldsymbol{\nu} - \boldsymbol{\Psi} \boldsymbol{e} \tag{62}$$

When $\boldsymbol{T}$ is non-square (*rows* > *columns*), $\boldsymbol{T}$ and $\boldsymbol{e}$ used to define $\boldsymbol{v}$ cannot be arbitrary. The sufficient conditions on $\boldsymbol{T}$ and e are 1) the equation (62) should be consistent, 2) if $n_{\boldsymbol{u}}$ and $n_{\boldsymbol{v}}$ represent number of components of coordinates $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, then $n_{\boldsymbol{u}} - n_{\boldsymbol{v}}$ equations in (62) should be redundant. The constraint equation in terms of $\dot{\boldsymbol{v}}$ is obtained after removing redundant equations from (62).

The equation of motion in terms of $\boldsymbol{v}$ would be of the form

$$\boldsymbol{M}_{\boldsymbol{v}} \ddot{\boldsymbol{v}} = \boldsymbol{f}_{\boldsymbol{v}} - (\boldsymbol{T}^T \boldsymbol{\Psi}^T) \boldsymbol{\lambda}, \qquad \text{where} \tag{63}$$

$$\boldsymbol{M}_{\boldsymbol{v}} = \boldsymbol{T}^T \boldsymbol{M}_{\boldsymbol{u}} \boldsymbol{T}, \text{ and } \boldsymbol{f}_{\boldsymbol{v}} = \boldsymbol{T}^T \bigg(\boldsymbol{f}_{\boldsymbol{u}} - \boldsymbol{M}_u (\dot{\boldsymbol{e}} - \dot{\boldsymbol{T}} \dot{\boldsymbol{v}}) \bigg) \tag{64}$$

C Simplifications

C.1 An useful relation

$$\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c}+\dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}+\tilde{\boldsymbol{E}}_k(\dot{\boldsymbol{S}}_k\dot{\boldsymbol{y}}_j+\dot{\boldsymbol{a}}_k))$$
$$=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\ddot{\boldsymbol{y}}_k-\tilde{\boldsymbol{E}}_k\ddot{\tilde{\boldsymbol{y}}}_{k_c}-\tilde{\boldsymbol{D}}_k\ddot{\tilde{\boldsymbol{y}}}_{k_f}+\tilde{\boldsymbol{E}}_k(\ddot{\tilde{\boldsymbol{y}}}_{k_c}-\boldsymbol{S}_k\ddot{\boldsymbol{y}}_j))$$
(see eqns (31) and (22))
$$=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\ddot{\boldsymbol{y}}_k+\tilde{\boldsymbol{E}}_k(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-1}\boldsymbol{Q}_k\ddot{\boldsymbol{y}}_j)-$$
$$\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}(\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{D}}_k)\ddot{\tilde{\boldsymbol{y}}}_{k_f}\text{(see } \boldsymbol{S}_k \text{ given in section 5.3)}$$
$$=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\ddot{\boldsymbol{y}}_k+\tilde{\boldsymbol{E}}_k(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-1}\boldsymbol{Q}_k\ddot{\boldsymbol{y}}_j) \tag{65}$$
(from condition 1 in section 5.1, $\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{D}}_k=\boldsymbol{0}$.)

C.2 Simplifications related to equation (45)

$$\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_c}+\dot{\tilde{\boldsymbol{D}}}_k\dot{\tilde{\boldsymbol{y}}}_{k_f}+\tilde{\boldsymbol{E}}_k(\dot{\boldsymbol{S}}_k\dot{\boldsymbol{y}}_j+\dot{\boldsymbol{a}}_k))$$
$$=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\ddot{\boldsymbol{y}}_k+\tilde{\boldsymbol{E}}_k(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-1}\boldsymbol{Q}_k\ddot{\boldsymbol{y}}_j)\quad\text{(see eqn (65))}$$
$$=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-T}\boldsymbol{C_c}^{-T}\boldsymbol{C}_c^T\boldsymbol{G}_k\boldsymbol{M}_k^{-T}\boldsymbol{M}_k(\ddot{\boldsymbol{y}}_k+$$
$$\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T\boldsymbol{C}_c\boldsymbol{C}_c^{-1}(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-1}\boldsymbol{Q}_k\ddot{\boldsymbol{y}}_j)\quad\text{(see eqn(38))}$$
$\boldsymbol{C}_c$ is invertible. $(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)$ and $\boldsymbol{M}_K$ are positive definite.
$$=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-1}(\boldsymbol{G}_k\ddot{\boldsymbol{y}}_k+(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-1}\boldsymbol{Q}_k\ddot{\boldsymbol{y}}_j)$$
$$=\boldsymbol{Q}_k^T(\boldsymbol{G}_k\boldsymbol{M}_k^{-1}\boldsymbol{G}_k^T)^{-1}(\boldsymbol{\gamma})\quad\text{(from equation (3))} \tag{66}$$

C.3 Simplifications related to equation (46)

$$\text{Let}\quad \tilde{\boldsymbol{B}}_k=\tilde{\boldsymbol{E}}_k\boldsymbol{Q}_k \tag{67}$$
$$=\boldsymbol{E}_k\boldsymbol{Q}_k-\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{E}_k\boldsymbol{Q}_k\quad\text{(see eqn (43))}$$
$$=-\boldsymbol{B}_k+\boldsymbol{D}_k\boldsymbol{Q}_{k_d}+\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{B}_k-$$
$$\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)\boldsymbol{Q}_{k_d}$$
(because, from eqn (61), we have $\boldsymbol{B}_k=-(\boldsymbol{E}_k\boldsymbol{Q}_k+\boldsymbol{D}_k\boldsymbol{Q}_{k_d})$)
$$\tilde{\boldsymbol{B}}_k=-\boldsymbol{B}_k+\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{B}_k \tag{68}$$
$$\tilde{\boldsymbol{B}}_k=(-\boldsymbol{I}+\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T\boldsymbol{M}_k)\boldsymbol{B}_k \tag{69}$$

It may be noted that

$$\boldsymbol{D}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{B}}_k=-\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{B}_k+\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k\boldsymbol{M}_k\boldsymbol{B}_k=\boldsymbol{0} \tag{70}$$

Also, using equations (43) and (60), we get

$$\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k=\boldsymbol{G}_k(\boldsymbol{E}_k+\boldsymbol{D}_k\tilde{\boldsymbol{A}}_k)=\boldsymbol{G}_k\boldsymbol{E}_k+(\boldsymbol{G}_k\boldsymbol{D}_k)\tilde{\boldsymbol{A}}_k=\boldsymbol{I}+\boldsymbol{0}=\boldsymbol{I} \tag{71}$$

C.3.1 Mass matrix related term

$$
\begin{aligned}
& \boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-1}\boldsymbol{Q}_k \\
& = \boldsymbol{Q}_k^T\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k\boldsymbol{Q}_k = \tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{B}}_k \quad \text{(see equations (71) and (67))} \\
& = \tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k(-\boldsymbol{B}_k + \boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k\boldsymbol{M}_k\boldsymbol{B}_k) \quad \text{(see eqn(68))} \\
& = -\tilde{\boldsymbol{B}}_k\boldsymbol{M}_k\boldsymbol{B}_k + \boldsymbol{0} \quad \text{(see eqn(70))} \qquad (72) \\
& = \boldsymbol{B}_k^T(\boldsymbol{I} - \boldsymbol{M}_k\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T)\boldsymbol{M}_k\boldsymbol{B}_k \quad \text{(see eqn(69))} \\
& = \boldsymbol{B}_k^T(\boldsymbol{I} - \boldsymbol{M}_k\boldsymbol{H}_k(\boldsymbol{H}_k^T\boldsymbol{M}_k\boldsymbol{H}_k)^{-1}\boldsymbol{H}_k^T)\boldsymbol{M}_k\boldsymbol{B}_k \text{(from eqn(61), } \boldsymbol{D}_k = \boldsymbol{H}_k\text{)} \qquad (73)
\end{aligned}
$$

C.3.2 Force vector related term

$$
\begin{aligned}
& -\boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{f}_k \\
& = -\boldsymbol{Q}_k^T\tilde{\boldsymbol{E}}_k\boldsymbol{f}_k \quad \text{(see eqn(71))} \\
& = -\tilde{\boldsymbol{B}}_k^T\boldsymbol{f}_k = \boldsymbol{B}_k^T(\boldsymbol{I} - \boldsymbol{M}_k\boldsymbol{D}_k(\boldsymbol{D}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)^{-1}\boldsymbol{D}_k^T)\boldsymbol{f}_k \text{(see eqns(67),(69))} \\
& = \boldsymbol{B}_k^T(\boldsymbol{I} - \boldsymbol{M}_k\boldsymbol{H}_k(\boldsymbol{H}_k^T\boldsymbol{M}_k\boldsymbol{H}_k)^{-1}\boldsymbol{H}_k^T)\boldsymbol{f}_k \quad \text{(from eqn(61), } \boldsymbol{D}_k = \boldsymbol{H}_k\text{)} \qquad (74)
\end{aligned}
$$

C.3.3 Velocity related term

$$
\begin{aligned}
& \boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\dot{\tilde{\boldsymbol{E}}}_k\dot{\boldsymbol{y}}_{k_c} + \dot{\tilde{\boldsymbol{D}}}_k\dot{\boldsymbol{y}}_{k_f} + \tilde{\boldsymbol{E}}_k(\dot{\boldsymbol{S}}_k\dot{\boldsymbol{y}}_j + \dot{\boldsymbol{a}}_k)) \\
& = \boldsymbol{Q}_k^T(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-T}\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k(\ddot{\boldsymbol{y}}_k + \tilde{\boldsymbol{E}}_k(\boldsymbol{G}_k\tilde{\boldsymbol{E}}_k)^{-1}\boldsymbol{Q}_k\ddot{\boldsymbol{y}}_j) \quad \text{(see eqn(65))} \\
& = \boldsymbol{Q}_k^T\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\ddot{\boldsymbol{y}}_k + (\boldsymbol{Q}_k^T\tilde{\boldsymbol{E}}_k^T\boldsymbol{M}_k\tilde{\boldsymbol{E}}_k\boldsymbol{Q}_k)\ddot{\boldsymbol{y}}_j \quad \text{(from eqn(71))} \\
& = \tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k\ddot{\boldsymbol{y}}_k - \tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k\boldsymbol{B}_k\ddot{\boldsymbol{y}}_j \quad \text{(see eqns (67) and (72))} \\
& = \tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k\ddot{\boldsymbol{y}}_k - \tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k\boldsymbol{B}_k\ddot{\boldsymbol{y}}_j - (\tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k\boldsymbol{D}_k)\ddot{\boldsymbol{q}}_j \quad (\tilde{\boldsymbol{B}}_k\boldsymbol{M}_k\boldsymbol{D}_k = \boldsymbol{0}\text{, eq(70))} \\
& = -\tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k(-\ddot{\boldsymbol{y}}_k + \boldsymbol{B}_k\ddot{\boldsymbol{y}}_j + \boldsymbol{D}_k\ddot{\boldsymbol{q}}_k) \\
& = -\tilde{\boldsymbol{B}}_k^T\boldsymbol{M}_k(-(\dot{\boldsymbol{B}}_k\dot{\boldsymbol{y}}_j + \dot{\boldsymbol{D}}_k\dot{\boldsymbol{q}}_k + \dot{\boldsymbol{c}}_k)) \\
& \text{(from eqn(61), } \ddot{\boldsymbol{y}}_k = \boldsymbol{B}_k\ddot{\boldsymbol{y}}_j + \boldsymbol{D}_k\ddot{\boldsymbol{q}}_k + \dot{\boldsymbol{B}}_k\dot{\boldsymbol{y}}_j + \dot{\boldsymbol{D}}_k\dot{\boldsymbol{q}}_k + \dot{\boldsymbol{c}}_k\text{)} \\
& = \boldsymbol{B}_k^T(\boldsymbol{I} - \boldsymbol{M}_k\boldsymbol{H}_k(\boldsymbol{H}_k^T\boldsymbol{M}_k\boldsymbol{H}_k)^{-1}\boldsymbol{H}_k^T)\boldsymbol{M}_k(-\boldsymbol{d}_k), \quad \text{(see eqn(69))}
\end{aligned}
$$

where $\boldsymbol{d}_k = (\dot{\boldsymbol{B}}_k\dot{\boldsymbol{y}}_j + \dot{\boldsymbol{D}}_k\dot{\boldsymbol{q}}_k + \dot{\boldsymbol{c}}_k)$.