Homogeneous coordinates, lines, screws and twists

In lecture 1 of module 2, a brief mention was made of homogeneous coordinates, lines in $\Re^3$, screws and twists to describe the general motion of a rigid body. Lines were used to represent joint axes and screws was used in connection with the properties ${}^A_B[T]$. In this write-up, we present a brief description of homogeneous coordinates, mathematical representation of lines, screws, and twists using Plücker coordinates and also present expressions for the angle and distance between two lines. We start with the concept of homogeneous coordinates.

Let (x, y) denote the Cartesian co-ordinates of a point in the Euclidean plane $\mathbf{E}^2$; then the *homogeneous* coordinates of the point are given by $(x, y, w) \in \mathbf{E}^3$ with $w \neq 0$. One of the key properties of homogeneous coordinates is that scaling does not matter, e,g., the coordinates (x, y, w) and $(\lambda x, \lambda y, \lambda w)$, where λ is a non-zero constant, represent the same point.

From elementary mathematics, we know that any point (x, y, z) on a line passing through two points, say ${}^A(x_0, y_0, z_0)^T$ and ${}^A(x_1, y_1, z_1)^T$, in $\{A\}$ satisfies

$$\frac{x - x_0}{x_0 - x_1} = \frac{y - y_0}{y_0 - y_1} = \frac{z - z_0}{z_0 - z_1} = c \tag{1}$$

where c is an arbitrary non-zero constant. In equation (1), if we consider a line through the origin, i.e., $(x_0, y_0, z_0) = \mathbf{0}$, we get $(x, y, z) = -c(x_1, y_1, z_1)$. If z is considered the same as w, then the equation of a line through the origin is equivalent to scaling. Hence, homogenous coordinates represent a point in $\mathbf{E}^2$ by a line through the origin in $\mathbf{E}^3$. Likewise, a line in $\mathbf{E}^2$ is a plane through the origin of $\mathbf{E}^3$.

To go from homogeneous coordinates to Cartesian coordinates, we simply extract from (x, y, w) the quantities $(x/w, y/w)$ and set w to 1. This implies that the Euclidean plane $\mathbf{E}^2$ with points (x, y) can be embedded as a $w = 1$ plane and the ordinary Euclidean point (with Cartesian coordinates (x, y)) can be thought of as a line through the origin intersecting the $w = 1$ plane.

In addition to the ordinary Euclidean points, it is possible to have homogeneous coordinates of the form $(x, y, 0)$. These are lines through the origin of $\mathbf{E}^3$ *parallel* to $w = 1$ plane. These are called *ideal points* or points at infinity which can be shown to form a line called the line at infinity. The set of lines through the origin of $\mathbf{E}^3$ defines the projective plane $\mathbf{P}^2$. The projective plane can be thought of as the Euclidean plane $\mathbf{E}^2$ to which we have added points at infinity. In this form, the projective plane has the interesting property of *duality* which states that in every axiom we can replace 'point' by 'line' and still make perfect sense without any exceptions.

For example, we can say 'two points determine a line' or 'two lines determine a point'. We can also state two parallel lines meet at infinity without any mathematical problem. The projective space $\mathbf{P}^2$ is one of the fundamental concepts in geometry and, as we will see in Module 3, allows us to 'correctly' count the number of solutions of non-linear equations.

The concept of a projective space is also useful in theoretical kinematics since the 4×1 homogeneous coordinates and the 4×4 transformation matrices of Lecture 1 can be put on a more formal footing. The 4×1 vector obtained by appending a '1' to $(x, y, z)^T$ are obtained from the homogeneous coordinates $(x, y, z, w)^T$ by setting $w = 1$. Similar to the discussion above we can also have points at infinity for $w = 0$. One difference between $\mathbf{P}^2$ earlier and $\mathbf{P}^3$ is that now the axioms and the notion of duality involve points, lines, and planes.

In equation (1), instead of three equations, we can also represent the line as

$$\mathcal{L} =^A (x_0, y_0, z_0)^T + t\hat{\mathbf{Q}}_A \tag{2}$$

where t is an arbitrary constant and $\hat{\mathbf{Q}}_A$ is a *unit* vector from ${}^A(x_0, y_0, z_0)^T$ to ${}^A(x_1, y_1, z_1)^T$ in the coordinate system $\{A\}$. In addition, the point ${}^A(x_0, y_0, z_0)^T$ need not have three independent parameters. Since the line extends to infinity in both directions, along the line, we can choose the point ${}^A(x_0, y_0, z_0)^T$ as the point where the line intersects any of the three coordinate planes ($x = 0$ or $y = 0$ or $z = 0$). Hence, a line in $\Re^3$ can be described by four independent parameters.

We represent a line by a pair of vectors of the form $({}^A\mathbf{Q}\ ;\ {}^A\mathbf{Q}_0)$ where ${}^A\mathbf{Q}$ is the *direction* vector and ${}^A\mathbf{Q}_0$ is the *moment* vector given by

$${}^A\mathbf{Q}_0 =^A \mathbf{r} \times^A \mathbf{Q} \tag{3}$$

where ${}^A\mathbf{r}$ locates a point on the line. It can be shown that ${}^A\mathbf{Q}_0$ is independent of the chosen point on the line.

The vector pair $({}^A\mathbf{Q}\ ;\ {}^A\mathbf{Q}_0)$ are the six Plücker coordinates of a line in $\Re^3$. It may be noted that there are only four independent parameters since ${}^A\mathbf{Q} \cdot {}^A\mathbf{Q}_0 = 0$ and $c({}^A\mathbf{Q}\ ;\ {}^A\mathbf{Q}_0)$ $(c \in \Re^1 \neq 0)$ is the same line as $({}^A\mathbf{Q}\ ;\ {}^A\mathbf{Q}_0)$. Since, the Plücker coordinates of a line are unchanged by scaling, they are homogeneous coordinates. Hence, similar to choosing $w = 1$ in the case of points, as long as $|{}^A\mathbf{Q}| \neq 0$, we can represent lines in $\Re^3$ by a unit vector and its moment as

$$\begin{aligned} \hat{\mathbf{Q}}_A &= \frac{{}^A\mathbf{Q}}{|{}^A\mathbf{Q}|} \\ {}^A\hat{\mathbf{Q}}_0 &= {}^A\mathbf{r} \times \hat{\mathbf{Q}}_A \end{aligned} \tag{4}$$

Note that the vector pair $(\hat{\mathbf{Q}}_A \; ; \; {}^A\hat{\mathbf{Q}}_0)$ has four independent parameters.

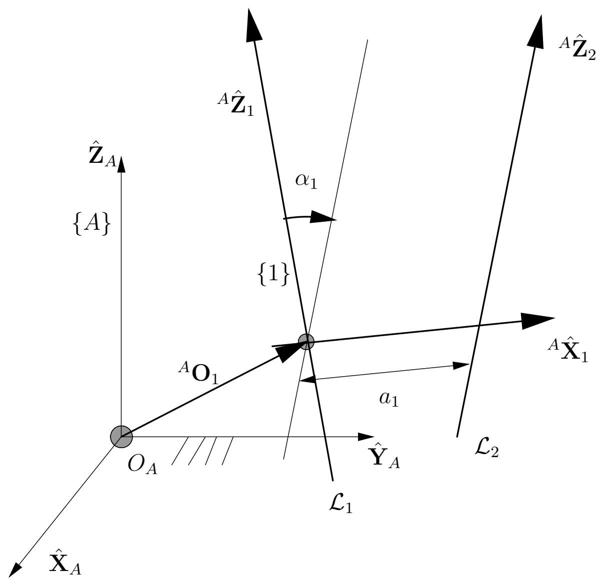

Figure 1: Line in $\Re^3$

The Denavit-Hartenberg parameters, discussed in Lecture 2, are based on the notion of *distance* and *angle* between lines in three-dimensional space. In the following, we present analytical expressions for the common perpendicular, angle and distance between two lines in $\Re^3$. These expressions can be used to compute numerical values of D-H parameters from a CAD model of a robot.

We assume that the two lines $\mathcal{L}_1$ and $\mathcal{L}_2$, as shown in figure 1, are described by the vector pairs $({}^A\hat{\mathbf{Z}}_1 \; ; \; {}^A\hat{\mathbf{Z}}_{\mathbf{0}1})$ and $({}^A\hat{\mathbf{Z}}_2 \; ; \; {}^A\hat{\mathbf{Z}}_{\mathbf{0}2})$, respectively. The unit vector along the common perpendicular is given by

$$ {}^A\hat{\mathbf{X}}_1 = \frac{{}^A\hat{\mathbf{Z}}_1 \times^A \hat{\mathbf{Z}}_2}{|{}^A\hat{\mathbf{Z}}_1 \times^A \hat{\mathbf{Z}}_2|} \tag{5} $$

It may be noted that the unit vector ${}^A\hat{\mathbf{X}}_1$ is from $\mathcal{L}_1$ to $\mathcal{L}_2$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect, then the unit vector ${}^A\hat{\mathbf{X}}_1$ is normal to the plane formed by the two intersecting lines, and there are two choices for the direction of ${}^A\hat{\mathbf{X}}_1$. If the lines are parallel, the common perpendicular is not unique and the vector cross-product in equation (5) is zero. For this case any line perpendicular to $\mathcal{L}_1$ and $\mathcal{L}_2$ is a common perpendicular.

The angle between the two lines is

$$\alpha_1 = \cos^{-1}({}^A\hat{\mathbf{Z}}_1 \ \cdot \ {}^A\hat{\mathbf{Z}}_2), \qquad 0 \leq \alpha_1 \leq \pi \tag{6}$$

The angle α_1 can also be negative, $(-\pi \leq \alpha_1 \leq 0)$, and we can choose the correct sign by ensuring that the angle is measured from ${}^A\hat{\mathbf{Z}}_1$ to ${}^A\hat{\mathbf{Z}}_2$ about ${}^A\hat{\mathbf{X}}_1$ using the right-hand rule. If the two lines are parallel, the angle is 0 or π.
The shortest distance is along the common perpendicular and is given by

$$a_1 = \frac{{}^A\hat{\mathbf{Z}}_1 \ \cdot \ {}^A\hat{\mathbf{Z}}_{\mathbf{0}2} +^A \hat{\mathbf{Z}}_2 \ \cdot \ {}^A\hat{\mathbf{Z}}_{\mathbf{0}1}}{|{}^A\hat{\mathbf{Z}}_1 \times^A \hat{\mathbf{Z}}_2|} \tag{7}$$

If the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect, then a_1 is zero. If the lines are parallel, the length of any of the common perpendiculars is a_1.
The point of intersection of the common perpendicular line with $\mathcal{L}_1$ can be obtained by solving simultaneously the equations of line $\mathcal{L}_1$ and the plane formed by line $\mathcal{L}_2$ and the common perpendicular line. Denoting the point by the vector ${}^A\mathbf{O}_1$, we have

$${}^A\mathbf{O}_1 = \frac{({}^A\hat{\mathbf{X}}_1 \ \cdot \ {}^A\hat{\mathbf{Z}}_{\mathbf{0}2}){}^A\hat{\mathbf{Z}}_1 -^A \hat{\mathbf{Z}}_{\mathbf{0}1} \times ({}^A\hat{\mathbf{Z}}_2 \times^A \hat{\mathbf{X}}_1)}{|{}^A\hat{\mathbf{Z}}_1 \times^A \hat{\mathbf{Z}}_2|} \tag{8}$$

The moment vector ${}^A\hat{\mathbf{X}}_{\mathbf{0}1}$ can be obtained by noting that the distances between $\mathcal{L}_1$, $\mathcal{L}_2$ and the common perpendicular are zero, and ${}^A\hat{\mathbf{X}}_{\mathbf{0}1} \cdot {}^A\hat{\mathbf{X}}_1 = 0$. The vector ${}^A\hat{\mathbf{X}}_{\mathbf{0}1}$ is given by

$${}^A\mathbf{X}_{\mathbf{0}1} = [\{\cos\alpha_1 \ {}^A\hat{\mathbf{Z}}_{\mathbf{0}2} - {}^A\hat{\mathbf{Z}}_{\mathbf{0}1}\} \cdot {}^A\hat{\mathbf{X}}_1]{}^A\hat{\mathbf{Z}}_1 + [\{\cos\alpha_1 \ {}^A\hat{\mathbf{Z}}_{\mathbf{0}1} - {}^A\hat{\mathbf{Z}}_{\mathbf{0}2}\} \cdot {}^A\hat{\mathbf{X}}_1]{}^A\hat{\mathbf{Z}}_2 \tag{9}$$

The lines $({}^A\hat{\mathbf{X}}_1 \ ; \ {}^A\hat{\mathbf{X}}_{\mathbf{0}1})$, $({}^A\hat{\mathbf{Z}}_1 \ ; \ {}^A\hat{\mathbf{Z}}_{\mathbf{0}}1)$ and their point of intersection, ${}^A\mathbf{O}_1$, completely determine the coordinate system $\{1\}$ with respect to the fixed coordinate system $\{A\}$.

A screw $\mathcal{S}$ with respect to $\{A\}$ can be specified by a line and a pitch denoted by h. The screw coordinates denoted by $({}^A\mathbf{S} \ ; \ {}^A\mathbf{S}_0)$ are defined as

$$\begin{aligned} {}^A\mathbf{S} &= {}^A\mathbf{Q} \\ {}^A\mathbf{S}_0 &= {}^A\mathbf{Q}_0 + h{}^A\mathbf{Q} \end{aligned} \tag{10}$$

The pitch can be obtained from a given $({}^A\mathbf{S} \ ; \ {}^A\mathbf{S}_0)$ by

$$h = \frac{{}^A\mathbf{S} \ \cdot \ {}^A\mathbf{S}_0}{{}^A\mathbf{S} \ \cdot \ {}^A\mathbf{S}} \tag{11}$$

Since a line has four independent parameters, a screw has five independent parameters. If the pitch h is zero, the screw coordinates are the same as the line coordinates. A screw has infinite pitch if ${}^{A}\mathbf{S} = \mathbf{0}$. A screw is an element of $\mathbf{P}^5$.

A twist is a six-dimensional entity which completely describes the motion of a rigid body in $\Re^3$. It can be thought of as a screw with a magnitude. Denoting a twist by $\mathcal{V}$, we can write

$$\mathcal{V} = c\left(\frac{{}^{A}\mathbf{S}}{|{}^{A}\mathbf{S}|}\ ;\ \frac{{}^{A}\mathbf{S}_0}{|{}^{A}\mathbf{S}|}\right), \quad c \in \Re^1 \tag{12}$$

In terms of (normalized) line coordinates, we can write

$$\mathcal{V} = c\left(\hat{\mathbf{Q}}_A\ ;\ {}^{A}\hat{\mathbf{Q}}_0 + h\hat{\mathbf{Q}}_A\right), \quad c \in \Re^1 \tag{13}$$

The six independent parameters are the four in normalized line coordinates $(\hat{\mathbf{Q}}_A\ ;\ {}^{A}\hat{\mathbf{Q}}_0)$, the pitch h, and the magnitude c. A twist of zero pitch is pure rotation and is of the form $\theta(\hat{\mathbf{Q}}_A\ ;\ {}^{A}\hat{\mathbf{Q}}_0)$ where θ is the amount of rotation. A twist of infinite pitch is a pure translation and is of the form $(\mathbf{0}; d\hat{\mathbf{Q}}_A)$, where d is the amount of translation.