Robotics: Advanced Concepts & Analysis

Module 4 - Kinematics of parallel robots

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OUTLINE

- Parallel manipulators: One or more loops $\rightarrow$ No first or last link.
- No natural choice of end-effector or output link $\rightarrow$ Output link *must be chosen*.
- Number of joints is more than the degree-of-freedom $\rightarrow$ several joints are *not* actuated.
- Un-actuated or *passive* joints can be multi- degree-of-freedom joints.
- Two main problems: Direct Kinematics and Inverse Kinematics.

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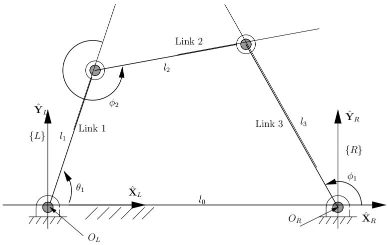
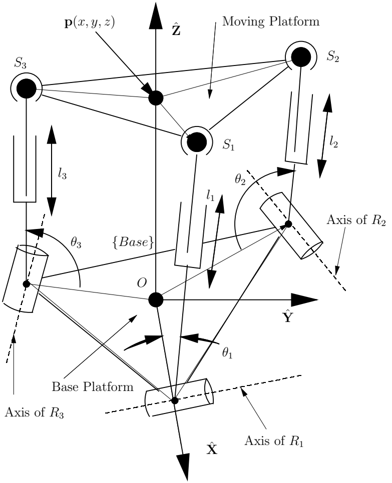
Introduction

Examples of Parallel Robots



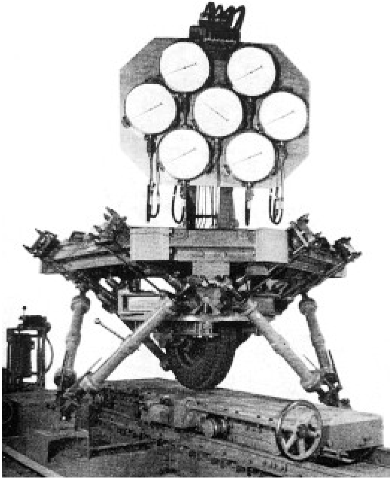
(a) Planar 4-bar Mechanism



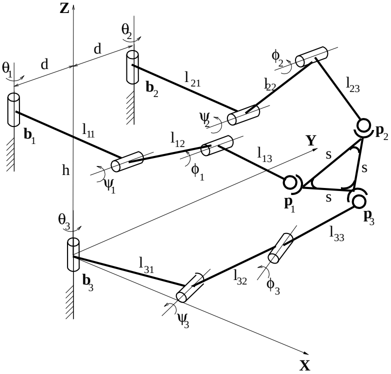
(b) Three- degree-of-freedom Manipulator

INTRODUCTION

EXAMPLES OF PARALLEL ROBOTS



(c) Original Stewart platform (1965)



(d) Model of a three-fingered hand

Figure 1: Example of Parallel Manipulators

- Grübler-Kutzbach's criterion

$$DOF = \lambda(N - J - 1) + \sum_{i=1}^{J} F_i \tag{1}$$

N – total number of links including the fixed link (or base),
J – total number of joints connecting *only* two links (if joint connects three links then it must be counted as two joints),
F_i – degrees of freedom at the i^{th} joint, and $\lambda = 6 forspatial$, 3 for planar manipulators and mechanisms.

- 4-bar mechanism – $N = 4$, $J = 4$, $\sum_{i=1}^{J} F_i = 1 + 1 + 1 + 1 = 4$, $\lambda = 3$ $\rightarrow$ $DOF = 1$
- 3-RPS manipulator – $N = 8$, $J = 9$, $\sum_{i=1}^{J} F_i = 6 \times 1 + 3 \times 3 = 15$, $\lambda = 6 \rightarrow$ $DOF = 3$.
- Three-fingered hand – $N = 11$, $J = 12$, $\sum_{i=1}^{J} F_i = 9 + 9 = 18$, $\lambda = 6$ $\rightarrow$ $DOF = 6$

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- Passive joints can be multi- degree-of-freedom joints.
  - In 3-RPS manipulator, three- degree-of-freedom spherical (S) joints are passive.
  - In a Stewart platform, the S and U joints are passive.
- Configuration space $\mathbf{q} = (\theta, \phi)$
  - θ are actuated joints & $\theta \in \Re^n$ $(n = DOF)$
  - ϕ is the set of passive joints & $\phi \in \Re^m$
- All *passive* joints $\notin \phi \Rightarrow (n+m) \leq J$

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Loop-closure Constraint Equations

- m passive joint variables $\rightarrow$ m *independent* equations required to solve for ϕ for given n actuated variable, θ_i, $i = 1, 2, ..., n$.
- General approach to derive m loop-closure constraint equations.
  1. 'Break' parallel manipulator into 2 or more serial manipulators
  2. Determine D-H parameters for serial chains and obtain position and orientation of the 'Break' for each chain
  3. Use joint constraint (see Module 2, Lecture 2) at the 'Break(s)' to re-join (close) the parallel manipulator.
- Trick is to 'break' such that
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- One loop – fixed frames $\{L\}$ and $\{R\}$, $\{R\}$ is translated by l_0 along the $X-$ axis.
- $\{1\}$, $\{2\}$, $\{3\}$, and $\{Tool\}$ are as shown. Note only $\hat{\mathbf{X}}$ shown for convenience.
- The sequence O_L-O_1-O_2-O_3-O_{Tool} can be thought of as a planar 3R manipulator

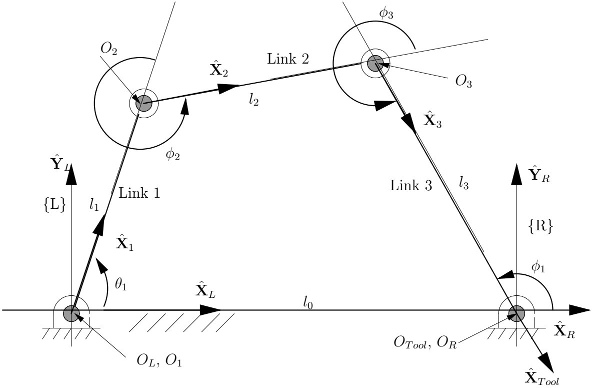

Figure 2: The four-bar mechanism

- D-H parameters of the planar 3R manipulator are

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	l_1	0	ϕ_2
3	0	l_2	0	ϕ_3

- From D-H table find ${}^{0}_{3}[T]$ (See Lecture 2, Module 2)
- For planar 3R and tool of length l_3, find ${}^{3}_{Tool}[T]$.
- ${}^{Tool}_{R}[T]$ is given

$$
{}^{Tool}_{R}[T] = \begin{pmatrix} -\cos\phi_1 & -\sin\phi_1 & 0 & 0 \\ \sin\phi_1 & -\cos\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

- The loop-closure equations for the four-bar mechanism is

$$ {}^{L}_{1}[T]\,{}^{1}_{2}[T]\,{}^{2}_{3}[T]\,{}^{3}_{Tool}[T]\,{}^{Tool}_{R}[T] = {}^{L}_{R}[T] $$

- Planar loop → only 3 independent equations

$$
\begin{aligned}
l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2) + l_3 \cos(\theta_1 + \phi_2 + \phi_3) &= l_0 \\
l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \phi_2) + l_3 \sin(\theta_1 + \phi_2 + \phi_3) &= 0 \\
\theta_1 + \phi_2 + \phi_3 + (\pi - \phi_1) = 4\pi & \qquad (2)
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- Loop-closure equations: *all* four joint variables present.
  - $\mathbf{q} = (\theta_1, \phi_1, \phi_2, \phi_3)$
  - The actuated joint $\theta = \theta_1$
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- In this approach $n = 1$, $m = 3$ and $J = 4$.

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Constraint equations – 4-bar example

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    1. Presence of multi- degree-of-freedom spherical (S) and Hooke (U) joints in a loop.
    2. Obtaining *independent* loops in the presence of several loops.
- Represent multi- degree-of-freedom joint by two or more one-degree-of-freedom joints and obtain an equivalent 4×4 transformation matrix.
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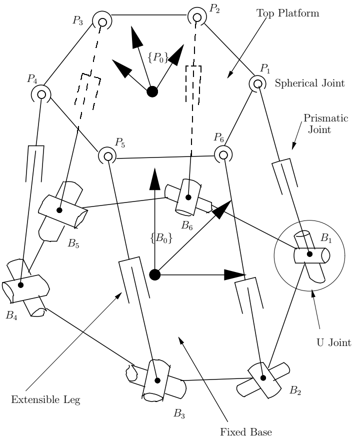
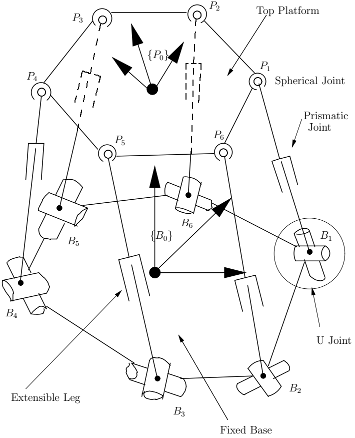


Figure 3: The Stewart-Gough platform

- Each leg is U-P-S chain, $\lambda = 6$, $N = 14$, $J = 18$, $\sum_{i=1}^{J} F_i = 36 \rightarrow DOF = 6$.
- 6 P joints actuated $\rightarrow$ 30 passive variables.
- Many loops – for example, 5 of the form $B_i - P_i - P_{i+1} - B_{i+1} - B_i, i = 1,..,5$ 4 of the form $B_i - P_i - P_{i+2} - B_{i+2} - B_i, i = 1,..,4$, and 3 of the form $B_i - P_i - P_{i+3} - B_{i+3} - B_i, i = 1,2,3$.
- Each of the 12 loops can have (potentially) 6 independent equations $\rightarrow$ Which 30 equations to choose?!



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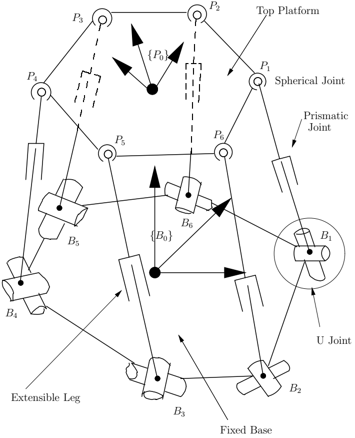

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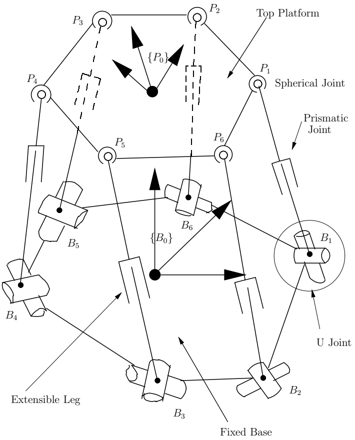

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LOOP-CLOSURE CONSTRAINT EQUATIONS

CONSTRAINT EQUATIONS – 4-BAR EXAMPLE REVISITED

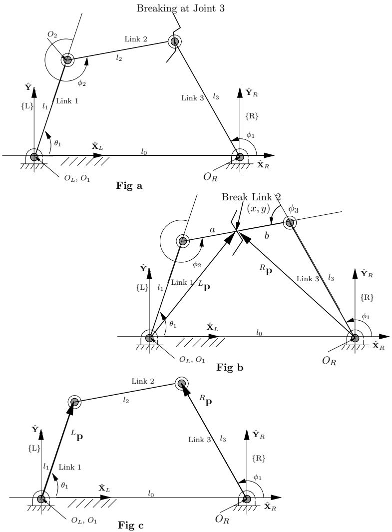

Figure 4: The four-bar mechanism 'broken ' in different ways

- Alternate way – 'break' loop at the third joint (see figure 4(a)).
  - One planar 2R manipulator + one planar 1R manipulator.
  - Easy to obtain the D-H tables for both (see Lecture 3, Module 2)
  - Easy to obtain ${}^{L}_{1}[T]$, ${}^{1}_{2}[T]$ & ${}^{R}_{1}[T]$.
  - Using l_2 and l_3, obtain ${}^{L}_{Tool}[T]$ and ${}^{R}_{Tool}[T]$.
  - From ${}^{L}_{Tool}[T]$ extract vector ${}^{L}\mathbf{p}$. The X and Y components are

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2), \quad y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \phi_2)$$

  - From ${}^{R}_{Tool}[T]$, extract vector ${}^{R}\mathbf{p}$ to get

$$x = l_3 \cos\phi_1, \quad y = l_3 \sin\phi_1$$

- Use constraint for a rotary (R) joint (see Lecture 2, Module 2)

$$\begin{aligned} x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2) &= l_0 + l_3 \cos\phi_1 \\ y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \phi_2) &= l_3 \sin\phi_1 \end{aligned} \tag{3}$$

  l_0 is the distance along the $X-$ axis between $\{L\}$ and $\{R\}$.
- In this case *only two constraint equation*: $\mathbf{q} = (\theta_1, \phi_1, \phi_2)$ – $n = 1$, $m = 2$ and $J = 3$

CONSTRAINT EQUATIONS – 4-BAR EXAMPLE REVISITED

- Alternate way – 'break' loop at the third joint (see figure 4(a)).
  - One planar 2R manipulator + one planar 1R manipulator.
  - Easy to obtain the D-H tables for both (see Lecture 3, Module 2)
  - Easy to obtain ${}^L_1[T]$, ${}^1_2[T]$ & ${}^R_1[T]$.
  - Using l_2 and l_3, obtain ${}^L_{Tool}[T]$ and ${}^R_{Tool}[T]$.
  - From ${}^L_{Tool}[T]$ extract vector ${}^L\mathbf{p}$. The X and Y components are

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LOOP-CLOSURE CONSTRAINT EQUATIONS

CONSTRAINT EQUATIONS – 4-BAR EXAMPLE REVISITED

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Loop-closure constraint equations

Constraint equations – 4-bar example revisited

- Another way to 'break' loop is to break the second link (see figure 4(b)).
- Two planar 2R manipulators
- Obtain the X and Y components of ${}^L\mathbf{p}$ as

$$x = l_1 \cos\theta_1 + a\ \cos(\theta_1 + \phi_2), \quad y = l_1 \sin\theta_1 + a\ \sin(\theta_1 + \phi_2)$$

- Likewise X and Y components of ${}^R\mathbf{p}$ are

$$x = l_3 \cos\phi_1 + b\ \cos(\phi_1 + \phi_3), \quad y = l_3 \sin\phi_1 + b\ \sin(\phi_1 + \phi_3)$$

  where $l_2 = a + b$ and the angle ϕ_3 is as shown in figure 4(b).
- Impose the constraint that the broken link is actually rigid

$$\begin{aligned} x = l_1 \cos\theta_1 + a\ \cos(\theta_1 + \phi_2) &= l_0 + l_3 \cos\phi_1 + b\ \cos(\phi_1 + \phi_3) \\ y = l_1 \sin\theta_1 + a\ \sin(\theta_1 + \phi_2) &= l_3 \sin\phi_1 + b\ \sin(\phi_1 + \phi_3) \\ \theta_1 + \phi_2 &= \phi_1 + \phi_3 + \pi \end{aligned} \qquad (4)$$

- Similar to equation (2) – $n = 1$, $m = 3$ and $J = 4$

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- Obtain ${}^{L}\mathbf{p}$ and ${}^{R}\mathbf{p}$ as

$$ {}^{L}\mathbf{p} = (l_1 \cos\theta_1, l_1 \sin\theta_1)^T, \quad {}^{R}\mathbf{p} = (l_3 \cos\phi_1, l_3 \sin\phi_1)^T $$

- Enforce the constraint of constant length l_2 to obtain

$$ \eta_1(\theta_1, \phi_1) = (l_1 \cos\theta_1 - l_0 - l_3 \cos\phi_1)^2 + (l_1 \sin\theta_1 - l_3 \sin\phi_1)^2 - l_2^2 = 0 \quad (5) $$

  This constraint is analogue of $S-S$ pair constraint (see Lecture 2, Module 2) for planar $R-R$ pair.
- Only *one* constraint equation[1] – $\mathbf{q} = (\theta_1, \phi_1)$, $n = m = 1$ & $J = 4$.

[1] In the four-bar kinematics this is the well known *Freudenstein's equation* (see Freudenstein 1954).

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Two Problems in Kinematics of Parallel Manipulators

- **Direct Kinematics Problem**: Two-part problem statement
  - **Step 1**: Given the geometry of the manipulator and the *actuated* joint variables, obtain *passive* joint variables.
  - **Step 2**: Obtain position and orientation of a *chosen* output link.
- *Much harder* than the direct kinematics problem for a serial manipulator.
- Leads to the notion of *mobility* and *assemble-ability* of a parallel manipulator or a closed-loop mechanism.
- **Inverse Kinematics Problem**:
  Given the geometry of the manipulator and the position and orientation of the *chosen* end-effector or output link, obtain the actuated *and* passive joint variables.
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Outline

- The link dimensions and other geometrical parameters are known.
- The values of the n *actuated* joints are known.
- First obtain m *passive* joint variables.
  - Obtain (minimal) m loop-closure constraint equations in m passive and n active joint variables.
  - Use elimination theory/Sylvester's dialytic method/Bézout's method (see Module 3, Lecture 4)
  - Solve set of m non-linear equations, if possible, in closed-form for the passive joint variables ϕ_i, $i = 1, .., m$
- Obtain position and orientation of *chosen output* link from known θ and ϕ – Recall no natural end-effector and hence have to be chosen!
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- Simplest possible closed-loop mechanism and studied extensively (see, for example Uicker et al., 2003).
- A good example to illustrate *all* steps in kinematics of parallel manipulators!!
- Simple loop-closure equations → All steps can be done manually!

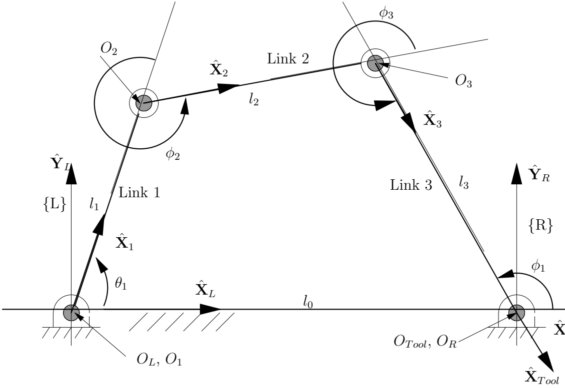

Figure 5: The four-bar mechanism - revisited

- From loop-closure equations (4) (see figure 4(b)),

$$x - l_0 = l_3 \cos\phi_1 - b\cos(\theta_1 + \phi_2), \quad y = l_3 \sin\phi_1 - b\sin(\theta_1 + \phi_2)$$

- Denote $\delta = \theta_1 + \phi_2$, squaring and adding

$$A_1 \cos\delta + B_1 \sin\delta + C_1 = 0 \tag{6}$$

where $A_1 = x - l_0,\ B_1 = y,\ C_1 = (1/2b)[(x - l_0)^2 + y^2 + b^2 - l_3^2]$

- From the first part of two equation (4)

$$x = l_1 \cos\theta_1 + a\cos(\theta_1 + \phi_2), \quad y = l_1 \sin\theta_1 + a\sin(\theta_1 + \phi_2)$$

- Squaring, adding, and after simplification gives

$$A_2 \cos\delta + B_2 \sin\delta + C_2 = 0 \tag{7}$$

where $A_2 = x,\ B_2 = y,\ C_2 = (1/2a)[l_1^2 - a^2 - x^2 - y^2]$

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- Convert equations (6) and (7) to a pair of quadratics by using the tangent half-angle substitutions (see Module 3, Lecture 4)
- Following Sylvester's dialytic elimination method (see Module 3, Lecture 4), $\det[SM] = 0$ gives

$$(A_1 B_2 - A_2 B_1)^2 = (A_1 C_2 - A_2 C_1)^2 + (B_1 C_2 - B_2 C_1)^2$$

and $\delta = -2\tan^{-1}\left(\frac{A_1 C_2 - A_2 C_1}{(B_1 C_2 - B_2 C_1) + (A_1 B_2 - A_2 B_1)}\right)$.

- $\det[SM] = 0$, after some simplification, gives

$$\begin{aligned} 4a^2 b^2 l_0{}^2 y^2 = {} & [b(x - l_0)(l_1^2 - a^2 - x^2 - y^2) - \\ & ax\{(x - l_0)^2 + y^2 + b^2 - l_3^2\}]^2 + \\ & y^2[b(l_1^2 - a^2 - x^2 - y^2) - a\{(x - l_0)^2 + y^2 + b^2 - l_3^2\}]^2 \quad (8) \end{aligned}$$

Equation (8) is known as the coupler curve[2]– a sixth-degree curve.

[2] The coupler curve is extensively studied in kinematics of mechanisms. For a more general form of the coupler curve and its interesting properties, see Chapter 6 of Hartenberg and Denavit (1964).

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- $\det[SM] = 0$, after some simplification, gives

$$\begin{aligned} 4a^2 b^2 l_0{}^2 y^2 = {} & [b(x - l_0)(l_1^2 - a^2 - x^2 - y^2) - \\ & ax\{(x - l_0)^2 + y^2 + b^2 - l_3^2\}]^2 + \\ & y^2[b(l_1^2 - a^2 - x^2 - y^2) - a\{(x - l_0)^2 + y^2 + b^2 - l_3^2\}]^2 \end{aligned} \quad (8)$$

Equation (8) is known as the coupler curve[2]– a sixth-degree curve.

[2]The coupler curve is extensively studied in kinematics of mechanisms. For a more general form of the coupler curve and its interesting properties, see Chapter 6 of Hartenberg and Denavit (1964).

- Convert equations (6) and (7) to a pair of quadratics by using the tangent half-angle substitutions (see Module 3, Lecture 4)
- Following Sylvester's dialytic elimination method (see Module 3, Lecture 4), $\det[SM] = 0$ gives

$$(A_1 B_2 - A_2 B_1)^2 = (A_1 C_2 - A_2 C_1)^2 + (B_1 C_2 - B_2 C_1)^2$$

and $\delta = -2\tan^{-1}\left(\frac{A_1 C_2 - A_2 C_1}{(B_1 C_2 - B_2 C_1) + (A_1 B_2 - A_2 B_1)}\right)$.

- $\det[SM] = 0$, after some simplification, gives

$$\begin{aligned} 4a^2 b^2 l_0{}^2 y^2 = {} & [b(x - l_0)(l_1^2 - a^2 - x^2 - y^2) - \\ & ax\{(x - l_0)^2 + y^2 + b^2 - l_3^2\}]^2 + \\ & y^2[b(l_1^2 - a^2 - x^2 - y^2) - a\{(x - l_0)^2 + y^2 + b^2 - l_3^2\}]^2 \end{aligned} \quad (8)$$

Equation (8) is known as the coupler curve[2]– a sixth-degree curve.

[2]The coupler curve is extensively studied in kinematics of mechanisms. For a more general form of the coupler curve and its interesting properties, see Chapter 6 of Hartenberg and Denavit (1964).

- The elimination procedure gives δ as a function of (x, y) and the link lengths.
- Since θ_1 is given,

$$\phi_2 = \delta - \theta_1 = -2\tan^{-1}\left(\frac{A_1 C_2 - A_2 C_1}{(B_1 C_2 - B_2 C_1) + (A_1 B_2 - A_2 B_1)}\right) - \theta_1 \quad (9)$$

- The angle ϕ_1 can be obtained from equation (5).

$$l_0^2 + l_1^2 + l_3^2 - l_2^2 = \cos\phi_1(2l_1 l_3 \cos\theta_1 - 2l_0 l_3) + \sin\phi_1(2l_1 l_3) \quad (10)$$

- Finally, ϕ_3 can be solved from the third equation in equation (4)

$$\phi_3 = \theta_1 + \phi_2 - \phi_1 - \pi \quad (11)$$

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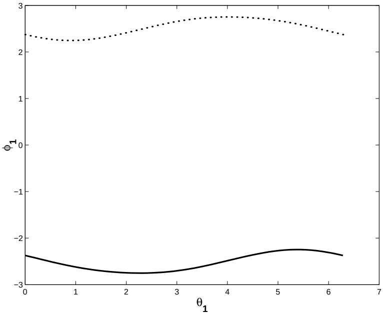
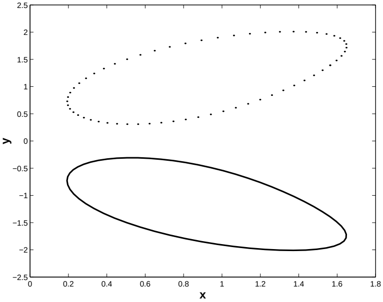
- Finally, ϕ_3 can be solved from the third equation in equation (4)

$$\phi_3 = \theta_1 + \phi_2 - \phi_1 - \pi \quad (11)$$

- $l_0 = 5.0$, $l_1 = 1.0$, $l_2 = 3.0$, and $l_3 = 4.0$ – the input link rotates fully (*Grashof's criteria*)
- Figure 6(a) shows plot of ϕ_1 vs θ_1 – both values are plotted.
- From ϕ_1 obtain ϕ_2 and ϕ_3 $\rightarrow$ Two coupler curves shown in figure 6(b)



(a) ϕ_1 vs θ_1 for 4-bar mechanism



(b) Coupler curves for 4-bar mechanism

Figure 6: Numerical example for a 4-bar

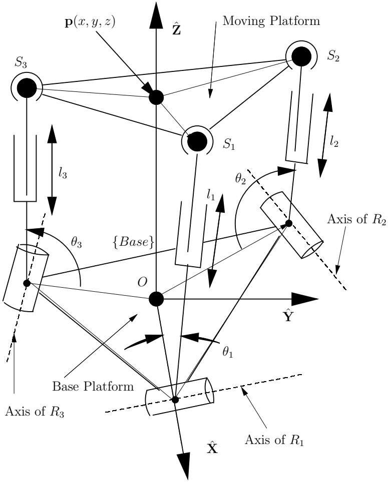


Figure 7: The 3-RPS parallel manipulator – revisited

D-H Table for a R-P-S leg (see Module 2, Lecture 2)

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	l_1	0

All legs are same.

$\theta_1,\ i = 1, 2, 3$ are passive variables.

$l_i,\ i = 1, 2, 3$ are actuated variables.

- Position vectors of three S joints (see Module 2, Lecture 2)

$$
\begin{aligned}
{}^{Base}\mathbf{S}_1 &= (b - l_1\cos\theta_1, 0, l_1\sin\theta_1)^T \\
{}^{Base}\mathbf{S}_2 &= (-\frac{b}{2} + \frac{1}{2}l_2\cos\theta_2, \frac{\sqrt{3}}{2}b - \frac{\sqrt{3}}{2}l_2\cos\theta_2, l_2\sin\theta_2)^T \\
{}^{Base}\mathbf{S}_3 &= (-\frac{b}{2} + \frac{1}{2}l_3\cos\theta_3, -\frac{\sqrt{3}}{2}b + \frac{\sqrt{3}}{2}l_3\cos\theta_3, l_3\sin\theta_3)^T \quad (12)
\end{aligned}
$$

Base an equilateral triangle circumscribed by circle of radius b.

- Impose $S - S$ pair constraint (see Module 2, Lecture 2)

$$
\begin{aligned}
\eta_1(l_1,\theta_1,l_2,\theta_2) &= ({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2)\cdot({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2) = k_{12}^2 \\
\eta_2(l_2,\theta_2,l_3,\theta_3) &= ({}^{Base}\mathbf{S}_2 - {}^{Base}\mathbf{S}_3)\cdot({}^{Base}\mathbf{S}_2 - {}^{Base}\mathbf{S}_3) = k_{23}^2 \\
\eta_3(l_3,\theta_3,l_1,\theta_1) &= ({}^{Base}\mathbf{S}_3 - {}^{Base}\mathbf{S}_1)\cdot({}^{Base}\mathbf{S}_3 - {}^{Base}\mathbf{S}_1) = k_{31}^2 \quad (13)
\end{aligned}
$$

- Spherical joint variables do not appear – due to $S - S$ pair equations!
- *Three* loop-closure equations in *three* passive variables – simplest!

- Position vectors of three S joints (see Module 2, Lecture 2)

$$
\begin{aligned}
{}^{Base}\mathbf{S}_1 &= (b - l_1\cos\theta_1, 0, l_1\sin\theta_1)^T \\
{}^{Base}\mathbf{S}_2 &= (-\frac{b}{2} + \frac{1}{2}l_2\cos\theta_2, \frac{\sqrt{3}}{2}b - \frac{\sqrt{3}}{2}l_2\cos\theta_2, l_2\sin\theta_2)^T \\
{}^{Base}\mathbf{S}_3 &= (-\frac{b}{2} + \frac{1}{2}l_3\cos\theta_3, -\frac{\sqrt{3}}{2}b + \frac{\sqrt{3}}{2}l_3\cos\theta_3, l_3\sin\theta_3)^T \quad (12)
\end{aligned}
$$

Base an equilateral triangle circumscribed by circle of radius b.

- Impose $S - S$ pair constraint (see Module 2, Lecture 2)

$$
\begin{aligned}
\eta_1(l_1, \theta_1, l_2, \theta_2) &= ({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2)\cdot({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2) = k_{12}^2 \\
\eta_2(l_2, \theta_2, l_3, \theta_3) &= ({}^{Base}\mathbf{S}_2 - {}^{Base}\mathbf{S}_3)\cdot({}^{Base}\mathbf{S}_2 - {}^{Base}\mathbf{S}_3) = k_{23}^2 \\
\eta_3(l_3, \theta_3, l_1, \theta_1) &= ({}^{Base}\mathbf{S}_3 - {}^{Base}\mathbf{S}_1)\cdot({}^{Base}\mathbf{S}_3 - {}^{Base}\mathbf{S}_1) = k_{31}^2 \quad (13)
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$$

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  Base an equilateral triangle circumscribed by circle of radius b.
- Impose $S-S$ pair constraint (see Module 2, Lecture 2)

$$
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\eta_1(l_1,\theta_1,l_2,\theta_2) &= ({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2)\cdot({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2) = k_{12}^2 \\
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\end{aligned}
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$$

- Spherical joint variables do not appear – due to $S-S$ pair equations!
- *Three* loop-closure equations in *three* passive variables – simplest!

- Position vectors of three S joints (see Module 2, Lecture 2)

$$\begin{aligned} {}^{Base}\mathbf{S}_1 &= (b - l_1\cos\theta_1, 0, l_1\sin\theta_1)^T \\ {}^{Base}\mathbf{S}_2 &= (-\frac{b}{2} + \frac{1}{2}l_2\cos\theta_2, \frac{\sqrt{3}}{2}b - \frac{\sqrt{3}}{2}l_2\cos\theta_2, l_2\sin\theta_2)^T \\ {}^{Base}\mathbf{S}_3 &= (-\frac{b}{2} + \frac{1}{2}l_3\cos\theta_3, -\frac{\sqrt{3}}{2}b + \frac{\sqrt{3}}{2}l_3\cos\theta_3, l_3\sin\theta_3)^T \quad (12) \end{aligned}$$

  Base an equilateral triangle circumscribed by circle of radius b.

- Impose $S-S$ pair constraint (see Module 2, Lecture 2)

$$\begin{aligned} \eta_1(l_1,\theta_1,l_2,\theta_2) &= ({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2)\cdot({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2) = k_{12}^2 \\ \eta_2(l_2,\theta_2,l_3,\theta_3) &= ({}^{Base}\mathbf{S}_2 - {}^{Base}\mathbf{S}_3)\cdot({}^{Base}\mathbf{S}_2 - {}^{Base}\mathbf{S}_3) = k_{23}^2 \\ \eta_3(l_3,\theta_3,l_1,\theta_1) &= ({}^{Base}\mathbf{S}_3 - {}^{Base}\mathbf{S}_1)\cdot({}^{Base}\mathbf{S}_3 - {}^{Base}\mathbf{S}_1) = k_{31}^2 \quad (13) \end{aligned}$$

- Spherical joint variables do not appear – due to $S-S$ pair equations!
- *Three* loop-closure equations in *three* passive variables – simplest!

- Assume $b = 1$ and $k_{12} = k_{23} = k_{31} = \sqrt{3}a$.
- Eliminate using Sylvester's dialytic method (see Module 3, Lecture 4), θ_1 from $\eta_1(\cdot) = 0$ and $\eta_3(\cdot) = 0$

$$\eta_4(l_1, l_2, l_3, \theta_2, \theta_3) = (A_1 C_2 - A_2 C_1)^2 + (B_1 C_2 - B_2 C_1)^2 - (A_1 B_2 - A_2 B_1)^2 = 0$$

where

$$\begin{aligned} C_1 &= 3 - 3a^2 + l_1^2 + l_2^2 - 3l_2 c_2, \; A_1 = l_1 l_2 c_2 - 3l_1, \; B_1 = -2l_1 l_2 s_2 \\ C_2 &= 3 - 3a^2 + l_1^2 + l_3^2 - 3l_3 c_3, \; A_2 = l_1 l_3 c_3 - 3l_1, \; B_2 = -2l_1 l_3 s_3 \end{aligned}$$

- Eliminate θ_2 from $\eta_4(\cdot) = 0$ and $\eta_2(\cdot) = 0$, with $x_3 = \tan(\theta_3/2)$.

$$q_8(x_3^2)^8 + q_7(x_3^2)^7 + ... + q_1(x_3^2) + q_0 = 0 \tag{14}$$

An eight degree polynomial in x_3^2.

- Assume $b = 1$ and $k_{12} = k_{23} = k_{31} = \sqrt{3}a$.
- Eliminate using Sylvester's dialytic method (see Module 3, Lecture 4), θ_1 from $\eta_1(\cdot) = 0$ and $\eta_3(\cdot) = 0$

$$\eta_4(l_1, l_2, l_3, \theta_2, \theta_3) =$$
$$(A_1 C_2 - A_2 C_1)^2 + (B_1 C_2 - B_2 C_1)^2 - (A_1 B_2 - A_2 B_1)^2 = 0$$

where

$$\begin{aligned} C_1 &= 3 - 3a^2 + l_1^2 + l_2^2 - 3l_2 c_2, \; A_1 = l_1 l_2 c_2 - 3l_1, \; B_1 = -2l_1 l_2 s_2 \\ C_2 &= 3 - 3a^2 + l_1^2 + l_3^2 - 3l_3 c_3, \; A_2 = l_1 l_3 c_3 - 3l_1, \; B_2 = -2l_1 l_3 s_3 \end{aligned}$$

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- Eliminate using Sylvester's dialytic method (see Module 3, Lecture 4), θ_1 from $\eta_1(\cdot) = 0$ and $\eta_3(\cdot) = 0$

$$\eta_4(l_1, l_2, l_3, \theta_2, \theta_3) = (A_1C_2 - A_2C_1)^2 + (B_1C_2 - B_2C_1)^2 - (A_1B_2 - A_2B_1)^2 = 0$$

where

$$\begin{aligned} C_1 &= 3 - 3a^2 + l_1^2 + l_2^2 - 3l_2c_2, \ A_1 = l_1l_2c_2 - 3l_1, \ B_1 = -2l_1l_2s_2 \\ C_2 &= 3 - 3a^2 + l_1^2 + l_3^2 - 3l_3c_3, \ A_2 = l_1l_3c_3 - 3l_1, \ B_2 = -2l_1l_3s_3 \end{aligned}$$

- Eliminate θ_2 from $\eta_4(\cdot) = 0$ and $\eta_2(\cdot) = 0$, with $x_3 = \tan(\theta_3/2)$.

$$q_8(x_3^2)^8 + q_7(x_3^2)^7 + \ldots + q_1(x_3^2) + q_0 = 0 \qquad (14)$$

An eight degree polynomial in x_3^2.

- Expressions for q_i obtained using symbolic algebra software, *MAPLE*®, are very large. Two smaller ones are given

$$q_8 = (p_0 a^4 + p_1 a^3 + p_2 a^2 + p_3 a + p_4)^2 (p_0 a^4 - p_1 a^3 + p_2 a^2 - p_3 a + p_4)^2$$
$$q_0 = (r_0 a^4 + r_1 a^3 + r_2 a^2 + r_3 a + r_4)^2 (r_0 a^4 - r_1 a^3 + r_2 a^2 - r_3 a + r_4)^2$$

where $r_0 = p_0 = -9$, $r_1 = 12(l_3 - 3)$, $p_1 = 12(l_3 + 3)$,
$r_2 = 3(l_1^2 + l_2^2 - l_3(l_3 - 10) - 15)$, $p_2 = 3(l_1^2 + l_2^2 - l_3(l_3 + 10) - 15)$,
$r_3 = -2(l_3 - 3)(l_1^2 + l_2^2 + l_3^2 - 3)$, $p_3 = -2(l_3 + 3)(l_1^2 + l_2^2 + l_3^2 - 3)$,
$r_4 = l_3^4 - 8l_3^3 + 3l_2^2 + 18l_3^2 - 2l_3(l_2^2 + 6) - l_1^2(l_2^2 + 2l_3 - 3)$, and
$p_4 = l_3^4 + 8l_3^3 + 3l_2^2 + 18l_3^2 + 2l_3(l_2^2 + 6) + l_1^2(l_2^2 + 2l_3 - 3)$.

- 8 possible values of θ_3 for given a and actuated variables $(l_1, l_2, l_3)^T$.
- Once θ_3 is obtained, θ_2 obtained from $\eta_2(\cdot) = 0$ and θ_1 from $\eta_3(\cdot) = 0$.

- Expressions for q_i obtained using symbolic algebra software, *MAPLE*®, are very large. Two smaller ones are given

$$
\begin{aligned}
q_8 &= (p_0 a^4 + p_1 a^3 + p_2 a^2 + p_3 a + p_4)^2 (p_0 a^4 - p_1 a^3 + p_2 a^2 - p_3 a + p_4)^2 \\
q_0 &= (r_0 a^4 + r_1 a^3 + r_2 a^2 + r_3 a + r_4)^2 (r_0 a^4 - r_1 a^3 + r_2 a^2 - r_3 a + r_4)^2
\end{aligned}
$$

  where $r_0 = p_0 = -9$, $r_1 = 12(l_3 - 3)$, $p_1 = 12(l_3 + 3)$,
  $r_2 = 3(l_1^2 + l_2^2 - l_3(l_3 - 10) - 15)$, $p_2 = 3(l_1^2 + l_2^2 - l_3(l_3 + 10) - 15)$,
  $r_3 = -2(l_3 - 3)(l_1^2 + l_2^2 + l_3^2 - 3)$, $p_3 = -2(l_3 + 3)(l_1^2 + l_2^2 + l_3^2 - 3)$,
  $r_4 = l_3^4 - 8l_3^3 + 3l_2^2 + 18l_3^2 - 2l_3(l_2^2 + 6) - l_1^2(l_2^2 + 2l_3 - 3)$, and
  $p_4 = l_3^4 + 8l_3^3 + 3l_2^2 + 18l_3^2 + 2l_3(l_2^2 + 6) + l_1^2(l_2^2 + 2l_3 - 3)$.
- 8 possible values of θ_3 for given a and actuated variables $(l_1, l_2, l_3)^T$.
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- Expressions for q_i obtained using symbolic algebra software, *MAPLE*®, are very large. Two smaller ones are given

$$\begin{aligned} q_8 &= (p_0 a^4 + p_1 a^3 + p_2 a^2 + p_3 a + p_4)^2 (p_0 a^4 - p_1 a^3 + p_2 a^2 - p_3 a + p_4)^2 \\ q_0 &= (r_0 a^4 + r_1 a^3 + r_2 a^2 + r_3 a + r_4)^2 (r_0 a^4 - r_1 a^3 + r_2 a^2 - r_3 a + r_4)^2 \end{aligned}$$

  where $r_0 = p_0 = -9$, $r_1 = 12(l_3 - 3)$, $p_1 = 12(l_3 + 3)$, $r_2 = 3(l_1^2 + l_2^2 - l_3(l_3 - 10) - 15)$, $p_2 = 3(l_1^2 + l_2^2 - l_3(l_3 + 10) - 15)$, $r_3 = -2(l_3 - 3)(l_1^2 + l_2^2 + l_3^2 - 3)$, $p_3 = -2(l_3 + 3)(l_1^2 + l_2^2 + l_3^2 - 3)$, $r_4 = l_3^4 - 8l_3^3 + 3l_2^2 + 18l_3^2 - 2l_3(l_2^2 + 6) - l_1^2(l_2^2 + 2l_3 - 3)$, and $p_4 = l_3^4 + 8l_3^3 + 3l_2^2 + 18l_3^2 + 2l_3(l_2^2 + 6) + l_1^2(l_2^2 + 2l_3 - 3)$.
- 8 possible values of θ_3 for given a and actuated variables $(l_1, l_2, l_3)^T$.
- Once θ_3 is obtained, θ_2 obtained from $\eta_2(\cdot) = 0$ and θ_1 from $\eta_3(\cdot) = 0$.

- A *natural output link* is the moving platform.
- Position and orientation of the moving platform:
  - Centroid of moving platform,

$$
{}^{Base}\mathbf{p} = \frac{1}{3}({}^{Base}\mathbf{S}_1 + {}^{Base}\mathbf{S}_2 + {}^{Base}\mathbf{S}_3) \tag{15}
$$

  - Orientation of moving platform or ${}^{Base}_{Top}[R]$ is

$$
{}^{Base}_{Top}[R] = \left[\begin{array}{ccc} \frac{{}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2}{|{}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2|} & \hat{\mathbf{Y}} & \frac{({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2) \times ({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_3)}{|({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_2) \times ({}^{Base}\mathbf{S}_1 - {}^{Base}\mathbf{S}_3)|} \end{array} \right] \tag{16}
$$

  where $\hat{\mathbf{Y}}$ is obtained from the cross-product of the third and first columns.
- Once $l_i, \theta_i \;\; i = 1, 2, 3$ are known ${}^{Base}\mathbf{p}$ and ${}^{Base}_{Top}[R]$ can be found.
- Key step was the elimination of passive variables and obtaining a single equation in one passive variable!

- A *natural output link* is the moving platform.
- Position and orientation of the moving platform:
  - Centroid of moving platform,

$$^{Base}\mathbf{p} = \frac{1}{3}(^{Base}\mathbf{S}_1 + ^{Base}\mathbf{S}_2 + ^{Base}\mathbf{S}_3) \tag{15}$$

  - Orientation of moving platform or $^{Base}_{Top}[R]$ is

$$^{Base}_{Top}[R] = \left[\frac{^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2}{|^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2|} \quad \hat{\mathbf{Y}} \quad \frac{(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2)\times(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_3)}{|(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2)\times(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_3)|} \right] \tag{16}$$

  where $\hat{\mathbf{Y}}$ is obtained from the cross-product of the third and first columns.

- Once $l_i, \theta_i \;\; i = 1,2,3$ are known $^{Base}\mathbf{p}$ and $^{Base}_{Top}[R]$ can be found.
- Key step was the elimination of passive variables and obtaining a single equation in one passive variable!

- A *natural output link* is the moving platform.
- Position and orientation of the moving platform:
  - Centroid of moving platform,

$$^{Base}\mathbf{p} = \frac{1}{3}(^{Base}\mathbf{S}_1 + ^{Base}\mathbf{S}_2 + ^{Base}\mathbf{S}_3) \tag{15}$$

  - Orientation of moving platform or $^{Base}_{Top}[R]$ is

$$^{Base}_{Top}[R] = \left[\begin{array}{ccc} \frac{^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2}{|^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2|} & \hat{\mathbf{Y}} & \frac{(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2)\times(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_3)}{|(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_2)\times(^{Base}\mathbf{S}_1 - ^{Base}\mathbf{S}_3)|} \end{array} \right] \tag{16}$$

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- Polynomial in equation (14) is eight degree in $(\tan\theta_3/2)^2$
- Not possible to obtain closed-form expressions for θ_1, θ_2, and θ_3.
- Numerical solution using *Matlab*®
  - For $a = 1/2$, and for $l_1 = 2/3$, $l_2 = 3/5$ and $l_3 = 3/4$
  - Two sets values $\theta_3 = \pm 0.8111, \ \pm 0.8028$ radians.
  - For the positive values of θ_3, $\theta_2 = 0.4809, \ 0.2851$ radians and $\theta_1 = 0.7471, \ 0.7593$ radians respectively.
  - For the set $(0.7471, 0.4809, 0.8111)$, ${}^{Base}\mathbf{p} = (0.0117, -0.0044, 0.4248)^T$, and
  - The rotation matrix ${}^{Base}_{Top}[R]$ is given by

$$
{}^{Base}_{Top}[R] = \begin{pmatrix} 0.8602 & 0.5069 & -0.0564 \\ -0.4681 & 0.8285 & 0.3074 \\ 0.2026 & -0.2380 & 0.9499 \end{pmatrix}
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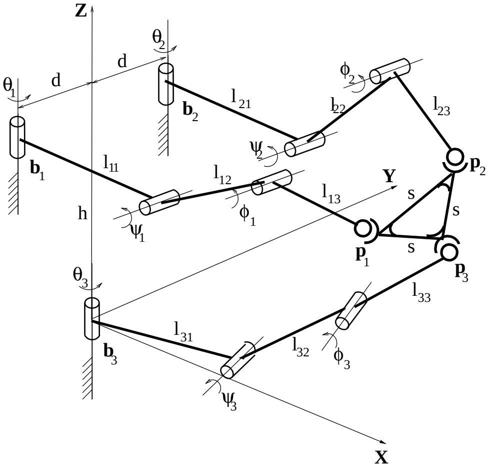

Figure 8: The 3-RRRS parallel manipulator – revisited

- D-H parameters for R-R-R-S chain (see Module 2, Lecture 2)

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	l_{11}	0	ψ_1
3	0	l_{12}	0	ϕ_1

- D-H parameters for three fingers with respect to $\{F_i\}$, $i = 1,2,3$ identical.
- $6DOF$ parallel manipulator $\rightarrow$ Only 6 out of 12 θ_i, ψ_i, ϕ_i are actuated.

- Position vector of spherical joint i

$$ {}^{F_i}\mathbf{p}_i = \begin{pmatrix} \cos\theta_i(l_{i1} + l_{i2}\cos\psi_i + l_{i3}\cos(\psi_i + \phi_i)) \\ \sin\theta_i(l_{i1} + l_{i2}\cos\psi_i + l_{i3}\cos(\psi_i + \phi_i)) \\ l_{i2}\sin\psi_i + l_{i3}\sin(\psi_i + \phi_i) \end{pmatrix} $$

- With respect to $\{Base\}$, the locations of $\{F_i\}$, $i = 1, 2, 3$, are known and constant
  ${}^{Base}\mathbf{b}_1 = (0, -d, h)^T$ ${}^{Base}\mathbf{b}_2 = (0, d, h)^T$ ${}^{Base}\mathbf{b}_3 = (0, 0, 0)^T$
- Orientation of $\{F_i\}$, $i = 1, 2, 3$, with respect to $\{Base\}$ are also known - $\{F_1\}$ and $\{F_2\}$ are parallel to $\{Base\}$ and $\{F_3\}$ is rotated by γ about the $\hat{\mathbf{Y}}$.
- The transformation matrices ${}^{Base}_{p_i}[T]$ is ${}^{Base}_{F_1}[T]^0_1[T]^1_2[T]^2_3[T]^3_{p_1}[T]$ – last transformation includes l_{13}.

- Position vector of spherical joint i

$$^{F_i}\mathbf{p}_i = \begin{pmatrix} \cos\theta_i(l_{i1} + l_{i2}\cos\psi_i + l_{i3}\cos(\psi_i + \phi_i)) \\ \sin\theta_i(l_{i1} + l_{i2}\cos\psi_i + l_{i3}\cos(\psi_i + \phi_i)) \\ l_{i2}\sin\psi_i + l_{i3}\sin(\psi_i + \phi_i) \end{pmatrix}$$

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$^{Base}\mathbf{b}_1 = (0, -d, h)^T$ $^{Base}\mathbf{b}_2 = (0, d, h)^T$ $^{Base}\mathbf{b}_3 = (0, 0, 0)^T$

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- Extract the position vector ${}^{Base}\mathbf{p}_1$ from the last column of ${}^{Base}_{F_1}[T]$

  ${}^{Base}\mathbf{p}_1 = {}^{Base}\mathbf{b}_1 + {}^{F_1}\mathbf{p}_1 =$

$$\begin{pmatrix} \cos\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ -d + \sin\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ h + l_{12}\sin\psi_1 + l_{13}\sin(\psi_1+\phi_1) \end{pmatrix}$$

- Similarly for second leg

$${}^{Base}\mathbf{p}_2 = \begin{pmatrix} \cos\theta_2(l_{21} + l_{22}\cos\psi_2 + l_{23}\cos(\psi_2+\phi_2)) \\ d + \sin\theta_2(l_{21} + l_{22}\cos\psi_2 + l_{23}\cos(\psi_2+\phi_2)) \\ h + l_{22}\sin\psi_2 + l_{23}\sin(\psi_2+\phi_2) \end{pmatrix}$$

- For third leg

$${}^{Base}\mathbf{p}_3 = [R(\hat{\mathbf{Y}},\gamma)] \begin{pmatrix} \cos\theta_3(l_{31} + l_{32}\cos\psi_3 + l_{33}\cos(\psi_3+\phi_3)) \\ \sin\theta_3(l_{31} + l_{32}\cos\psi_3 + l_{33}\cos(\psi_3+\phi_3)) \\ l_{32}\sin\psi_3 + l_{33}\sin(\psi_3+\phi_3) \end{pmatrix}$$

- Extract the position vector ${}^{Base}\mathbf{p}_1$ from the last column of ${}^{Base}_{F_1}[T]$

$$ {}^{Base}\mathbf{p}_1 = {}^{Base}\mathbf{b}_1 + {}^{F_1}\mathbf{p}_1 = \begin{pmatrix} \cos\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ -d + \sin\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ h + l_{12}\sin\psi_1 + l_{13}\sin(\psi_1+\phi_1) \end{pmatrix} $$

- Similarly for second leg

$$ {}^{Base}\mathbf{p}_2 = \begin{pmatrix} \cos\theta_2(l_{21} + l_{22}\cos\psi_2 + l_{23}\cos(\psi_2+\phi_2)) \\ d + \sin\theta_2(l_{21} + l_{22}\cos\psi_2 + l_{23}\cos(\psi_2+\phi_2)) \\ h + l_{22}\sin\psi_2 + l_{23}\sin(\psi_2+\phi_2) \end{pmatrix} $$

- For third leg

$$ {}^{Base}\mathbf{p}_3 = [R(\hat{\mathbf{Y}},\gamma)] \begin{pmatrix} \cos\theta_3(l_{31} + l_{32}\cos\psi_3 + l_{33}\cos(\psi_3+\phi_3)) \\ \sin\theta_3(l_{31} + l_{32}\cos\psi_3 + l_{33}\cos(\psi_3+\phi_3)) \\ l_{32}\sin\psi_3 + l_{33}\sin(\psi_3+\phi_3) \end{pmatrix} $$

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$${}^{Base}\mathbf{p}_1 = {}^{Base}\mathbf{b}_1 + {}^{F_1}\mathbf{p}_1 =$$
$$\begin{pmatrix} \cos\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ -d + \sin\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ h + l_{12}\sin\psi_1 + l_{13}\sin(\psi_1+\phi_1) \end{pmatrix}$$
- Similarly for second leg
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- Use $S - S$ pair constraint to get 3 loop-closure equations.

$$
\begin{aligned}
\eta_1(\theta_1, \psi_1, \phi_1, \theta_2, \psi_2, \phi_2) &= |{}^{Base}\mathbf{p}_1 - {}^{Base}\mathbf{p}_2|^2 = k_{12}^2 \\
\eta_2(\theta_2, \psi_2, \phi_2, \theta_3, \psi_3, \phi_3) &= |{}^{Base}\mathbf{p}_2 - {}^{Base}\mathbf{p}_3|^2 = k_{23}^2 \\
\eta_3(\theta_3, \psi_3, \phi_3, \theta_1, \psi_1, \phi_1) &= |{}^{Base}\mathbf{p}_3 - {}^{Base}\mathbf{p}_1|^2 = k_{31}^2
\end{aligned}
\tag{17}
$$

where k_{12}, k_{23} and k_{31} are constants.

- Actuated: θ_1, ψ_1, θ_2, ψ_2, θ_3, and ψ_3 & Passive: ϕ_1, ϕ_2, and ϕ_3.
- Obtain explicit expressions for the passive variables using elimination.
- Eliminate ϕ_1 from first and third equation (17)$\rightarrow$ $\eta_4(\phi_2, \phi_3, \cdot, \cdot) = 0$
- Eliminate ϕ_2 from $\eta_4(\phi_2, \phi_3, \cdot, \cdot) = 0$ and second equation (17) $\rightarrow$ Single equation in ϕ_3.
- Final equation is 16$^{\text{th}}$ degree polynomial in $\tan(\phi_3/2)$ – Obtained using symbolic algebra software *MAPLE*®.
- Expressions for the coefficients of the polynomial very long! – Numerical example shown next.

- Use $S - S$ pair constraint to get 3 loop-closure equations.

$$\begin{aligned}
\eta_1(\theta_1, \psi_1, \phi_1, \theta_2, \psi_2, \phi_2) &= |{}^{Base}\mathbf{p}_1 - {}^{Base}\mathbf{p}_2|^2 = k_{12}^2 \\
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- Use $S-S$ pair constraint to get 3 loop-closure equations.

$$
\begin{aligned}
\eta_1(\theta_1,\psi_1,\phi_1,\theta_2,\psi_2,\phi_2) &= |{}^{Base}\mathbf{p}_1 - {}^{Base}\mathbf{p}_2|^2 = k_{12}^2 \\
\eta_2(\theta_2,\psi_2,\phi_2,\theta_3,\psi_3,\phi_3) &= |{}^{Base}\mathbf{p}_2 - {}^{Base}\mathbf{p}_3|^2 = k_{23}^2 \\
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Direct Kinematics of Parallel Manipulators

A six- degree-of-freedom parallel manipulator – Numerical Results

- Assume $d = 1/2$, $h = \sqrt{3}/2$, $l_{i1} = 1$, $l_{i2} = 1/2$, $l_{i3} = 1/4$ $(i = 1, 2, 3)$, $\gamma = \pi/4$ and $k_{12} = k_{23} = k_{13} = \sqrt{3}/2$.
- For the actuated joint variables, choose $\theta_1 = 0.1$, $\psi_1 = -1.0$, $\theta_2 = 0.1$, $\psi_2 = -1.2$, $\theta_3 = 0.3$, and $\psi_3 = 1.0$ in radians.
- The sixteenth degree polynomial is obtained as

$$\begin{aligned}
0.00012t_3^{16} &- 0.00182t_3^{15} + 0.01376t_3^{14} - 0.05230t_3^{13} + 0.13148t_3^{12} \\
&- 0.24391t_3^{11} + 0.35247t_3^{10} - 0.40965t_3^{9} + 0.38696t_3^{8} \\
&- 0.29811t_3^{7} + 0.18502t_3^{6} - 0.09104t_3^{5} + 0.03433t_3^{4} \\
&- 0.00968t_3^{3} + 0.00201t_3^{2} - 0.00037t_3 + 0.00006 = 0
\end{aligned}$$

  where $t_3 = \tan(\phi_3/2)$.
- Numerical solution gives two real values of ϕ_3 as $(0.8831, 1.8239)$ radians.
- Corresponding values of ϕ_1 and ϕ_2 are $(0.3679, 0.1146)$ radians and $(1.4548, 1.0448)$ radians, respectively.

- The position vector of centroid, computed as in the 3-RPS example, using the first set of θ_i, ψ_i, ϕ_i is

$$^{Base}\mathbf{p} = \frac{1}{3}(^{Base}\mathbf{p}_1 + ^{Base}\mathbf{p}_2 + ^{Base}\mathbf{p}_3) = (1.3768, 0.2624, 0.1401)^T$$

- The rotation matrix $^{Base}_{Object}[R]$, computed similar to the 3-RPS example, is

$$^{Base}_{Object}[R] = \begin{pmatrix} 0.0306 & 0.2099 & -0.9773 \\ -0.9811 & 0.1806 & 0.0695 \\ 0.1910 & -0.9609 & 0.2004 \end{pmatrix}$$

OUTLINE

- Concept of workspace in serial manipulators $\rightarrow$ All $(x, y, z; [R])$ such that *real* solutions for the inverse kinematics exists.
- In parallel manipulators two concepts: mobility and workspace.
  - Workspace dependent on the choice of output link.
  - Mobility: range of possible motion of the actuated joints in a parallel manipulator.
  - Mobility is more important in parallel manipulators!
- Mobility is determined by geometry/linkage dimensions $\rightarrow$ Loop-closure constraint equations.
- Mobility is related to the *ability to assemble* a parallel manipulator at a configuration.

- Concept of workspace in serial manipulators $\rightarrow$ All $(x, y, z; [R])$ such that *real* solutions for the inverse kinematics exists.
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  - Workspace dependent on the choice of output link.
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MOBILITY OF PARALLEL MANIPULATORS

REVIEW

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Mobility of Parallel Manipulators

Review (contd.)

- Mobility: All values of actuated variables such that *real* value(s) of passive variables exists $\rightarrow$ determined by *direct kinematics*.
- No *real* value of passive variable $\Rightarrow$ Cannot be *assembled*.
- Mobility $\rightarrow$ Obtain conditions for existence of real solutions for the polynomial in one passive variable obtained after *elimination*.
- Very few parallel manipulators where the polynomial degree is 4 or less.
- In most cases mobility determined numerically using search.
- In 4-bar mechanism, mobility can be obtained in closed-form.

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- Loop-closure constraint equation of a 4-bar

$$\eta_1(\theta_1, \phi_1) = (l_1 \cos\theta_1 - l_0 - l_3 \cos\phi_1)^2 + (l_1 \sin\theta_1 - l_3 \sin\phi_1)^2 - l_2^2 = 0$$

- On simplification η_1 becomes

$$P \cos\phi_1 + Q \sin\phi_1 + R = 0 \tag{18}$$

where P, Q, and R are given by

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l_0, l_1, l_2, and l_3 are the link lengths (see figure 2), and c_1, s_1 are the sine and cosine of the actuated joint angle θ_1, respectively.

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Mobility of Parallel Manipulators

Mobility of 4-bar mechanism - Contd.

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- The condition $c_1 > 1$ and $c_1 < -1$ leads to

$$(l_0 - l_1)^2 > (l_3 - l_2)^2 \tag{21}$$

and

$$(l_0 + l_1) < (l_3 + l_2) \tag{22}$$

- Two additional conditions from $c_1 > 1$, $c_1 < -1$ lead to $l_3 + l_2 + l_1 < l_0$ and $l_0 + l_1 + l_2 < l_3 \rightarrow$ violates triangle inequality.
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- Equations (22) and (24) imply that l_0, l_2 and l_3 are all larger than l_1.
- Equations (22) and (24)$\rightarrow$ $l + s < p + q$, s, l are the shortest and largest links, and p, p are intermediate links.
- Likewise, for $l_1 > l_0$

$$\begin{aligned} l_1 + l_2 &> l_0 + l_3 \\ l_1 + l_3 &> l_0 + l_2 \end{aligned} \tag{25}$$

and again l_0 is the shortest link.
- Concisely represent equations (22) and (25) as $l + s < p + q$ – *Grashof's Criterion* for 4-bar linkage.

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- Concisely represent equations (22) and (25) as $l + s < p + q$ – *Grashof's Criterion* for 4-bar linkage.

- Three- degree-of-freedom parallel manipulator – polynomial is eight degree in x_3^2.
- $a = 0.5$ and $(l_1, l_2, l_3) \in [0.5,\ 1.5]$.
- Points marked as '$*$' – no *real and positive* values of x_3^2.
- Finer search → More accurate the mobility region.

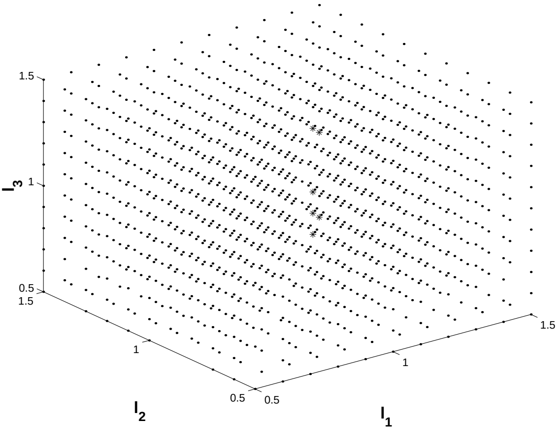

Figure 9: Values of (l_1, l_2, l_3) for imaginary θ_3 (marked by $*$)

Outline

- Problem statement: given
  - geometry and link parameters,
  - position and orientation of a *chosen output link* with respect to a fixed frame,

  Find the joint (actuated and passive) joint variables.
- Simpler than the direct kinematics problem since no need to worry about the multiple loops or the loop-closure constraint equations.
- Key idea is to ‘break’ the mechanism into serial chains and obtain the joint angles of each chain in ‘parallel’.
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- Worst case: Solution of inverse kinematics of a general 6R serial manipulator (See Module 3, Lecture 4).

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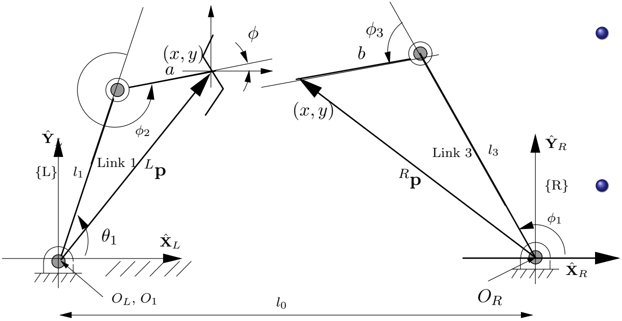

Figure 10: Inverse kinematics of a four-bar mechanism

- Coupler is the *chosen output link*.
- Given the position of a point ${}^L\mathbf{p}$ and the rotation matrix ${}^L_2[R]$ of the coupler link.
- Planar case $\rightarrow x, y$ coordinates and the orientation angle ϕ given.
- Lengths l_0, l_1, $l_2 = a + b$, a, b and l_3 are known.

- We have

$$x = l_1 \cos\theta_1 + a\cos(\theta_1 + \phi_2), \quad y = l_1 \sin\theta_1 + a\sin(\theta_1 + \phi_2)$$

where x and y are known.

- The angle ϕ denoting the orientation of link 2 is given by

$$\phi = \theta_1 + \phi_2 - 2\pi$$

- Solve for θ_1 and ϕ_2 as

$$\theta_1 = \mathrm{Atan2}(y - a\sin\phi,\ x - a\cos\phi), \quad \phi_2 = \phi - \theta_1$$

- In a similar manner, considering the equations

$$\begin{aligned} x &= l_0 + l_3\cos\phi_1 + b\cos(\phi_1 + \phi_3),\ y = l_3 \sin\phi_1 + b\sin(\phi_1 + \phi_3) \\ \phi &= \phi_1 + \phi_3 - \pi \end{aligned}$$

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solve for ϕ_1 and ϕ_3.

- ϕ obtained as $\theta_1 + \phi_2 - 2\pi$ and as $\phi_1 + \phi_3 - \pi$ must be same.
- The four-bar mechanism is a one- degree-of-freedom mechanism and only one of (x, y, ϕ) can be independent.
  - x and y are related through the sixth-degree coupler curve (see equation (8))
  - ϕ must satisfy

$$x \cos\phi + y \sin\phi = (1/2a)(x^2 + y^2 - a^2 - l_1^2)$$

- The constraints on the given position and orientation of the chosen output link, x, y, ϕ, are analogous to the case of the inverse kinematics of serial manipulators when $n < 6$ (see Module 3, Lecture 3).
- The inverse kinematics of a four-bar mechanism *can be solved when the given position and orientation is consistent.*

- ϕ obtained as $\theta_1 + \phi_2 - 2\pi$ and as $\phi_1 + \phi_3 - \pi$ must be same.
- The four-bar mechanism is a one- degree-of-freedom mechanism and only one of (x, y, ϕ) can be independent.
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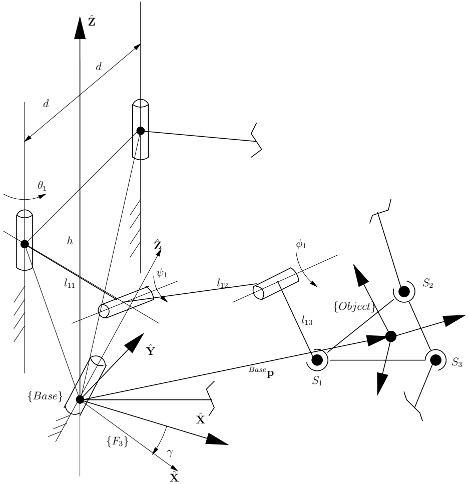

Figure 11: Inverse kinematics of six-degree-of-freedom parallel manipulator

- Figure shows one 'finger' RRRS chain.
- Given the position and orientation of the 'gripped' object with respect to $\{Base\}$.
- Obtain the rotations at the nine joints in the three 'fingers'.

- Vector ${}^{Base}\mathbf{p}$ locates the centroid of the gripped object.
- ${}^{Base}_{Object}[R]$is also available.
- In $\{Object\}$, the location of S_1, ${}^{Object}\mathbf{S}_1$, is known. Hence, $(x,y,z)^T = {}^{Base}\mathbf{S}_1 = {}^{Base}_{Object}[R]\,{}^{Object}\mathbf{S}_1 + {}^{Base}\mathbf{p}_{Object}$ is known.
- From above

$$(x,y,z)^T = \begin{pmatrix} \cos\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ -d + \sin\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ h + l_{12}\sin\psi_1 + l_{13}\sin(\psi_1+\phi_1) \end{pmatrix} \quad (26)$$

- Equation (26) can be solved for θ_1, ψ_1 and ϕ_1 using elimination (see Module 3, Lecture 4) from known $(x,y,z)^T$.

- Vector ${}^{Base}\mathbf{p}$ locates the centroid of the gripped object.
- ${}^{Base}_{Object}[R]$is also available.
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- From above

$$(x,y,z)^T = \begin{pmatrix} \cos\theta_1(l_{11}+l_{12}\cos\psi_1+l_{13}\cos(\psi_1+\phi_1)) \\ -d+\sin\theta_1(l_{11}+l_{12}\cos\psi_1+l_{13}\cos(\psi_1+\phi_1)) \\ h+l_{12}\sin\psi_1+l_{13}\sin(\psi_1+\phi_1) \end{pmatrix} \quad (26)$$

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- From above

$$(x,y,z)^T = \begin{pmatrix} \cos\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ -d + \sin\theta_1(l_{11} + l_{12}\cos\psi_1 + l_{13}\cos(\psi_1+\phi_1)) \\ h + l_{12}\sin\psi_1 + l_{13}\sin(\psi_1+\phi_1) \end{pmatrix} \quad (26)$$

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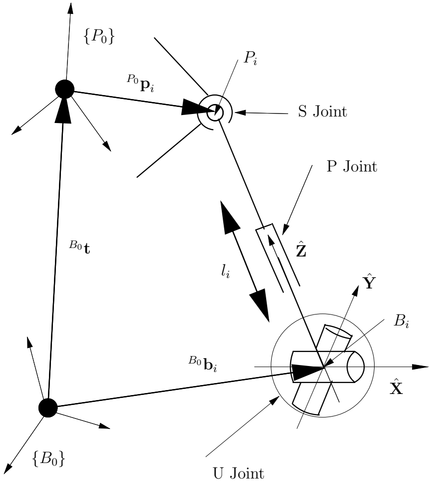

Figure 12: A leg of a Stewart platform

- From figure 12, an arbitrary platform point P_i can be written in $\{B_0\}$ as

$$^{B_0}\mathbf{p}_i = {}^{B_0}_{P_0}[R]\,{}^{P_0}\mathbf{p}_i + {}^{B_0}\mathbf{t} \qquad (27)$$

- The ${}^{P_0}\mathbf{p}_i$ is a known constant vector in $\{P_0\}$.
- The location of the base connection points ${}^{B_0}\mathbf{b}_i$ are known.

- From known ${}^{B_0}_{P_0}[R]$ and translation vector ${}^{B_0}\mathbf{t}$, obtain ${}^{B_0}\mathbf{p}_1$

$$\begin{aligned}[R(\hat{\mathbf{Z}},\gamma_i)]^T((x,y,z)^T - {}^{B_0}\mathbf{b}_1) &= [R(\hat{\mathbf{Y}},\phi_i)][R(\hat{\mathbf{X}},\psi_i)](0,0,l_i)^T \\ &= l_1 \begin{pmatrix} \sin\phi_1 \cos\psi_1 \\ -\sin\psi_1 \\ \cos\phi_1 \cos\psi_1 \end{pmatrix} \qquad (28)\end{aligned}$$

where ${}^{B_0}\mathbf{p}_1$ is denoted by $(x,y,z)^T$.

- Three non-linear equations in $l_1,\ \psi_1,\ \phi_1 \rightarrow$ solution

$$\begin{aligned} l_1 &= \pm\sqrt{[(x,y,z)^T - {}^{B_0}\mathbf{b}_1]^2} \\ \psi_1 &= \text{Atan2}(-Y, \pm\sqrt{X^2+Z^2}) \qquad (29) \\ \phi_1 &= \text{Atan2}(X/\cos\psi_1, Z/\cos\psi_1) \end{aligned}$$

where X,Y,Z are the components of $[R(\hat{\mathbf{Z}},\gamma_i)]^T((x,y,z)^T - {}^{B_0}\mathbf{b}_1)$.

- Perform for each leg to obtain $l_i,\ \psi_i$ and ϕ_i for $i = 1,...,6$.

- From known ${}^{B_0}_{P_0}[R]$ and translation vector ${}^{B_0}\mathbf{t}$, obtain ${}^{B_0}\mathbf{p}_1$

$$
\begin{aligned}
[R(\hat{\mathbf{Z}},\gamma_i)]^T((x,y,z)^T - {}^{B_0}\mathbf{b}_1) &= [R(\hat{\mathbf{Y}},\phi_i)][R(\hat{\mathbf{X}},\psi_i)](0,0,l_i)^T \\
&= l_1 \begin{pmatrix} \sin\phi_1 \cos\psi_1 \\ -\sin\psi_1 \\ \cos\phi_1 \cos\psi_1 \end{pmatrix}
\end{aligned}
\tag{28}
$$

where ${}^{B_0}\mathbf{p}_1$ is denoted by $(x,y,z)^T$.

- Three non-linear equations in $l_1,\ \psi_1,\ \phi_1 \rightarrow$ solution

$$
\begin{aligned}
l_1 &= \pm\sqrt{[(x,y,z)^T - {}^{B_0}\mathbf{b}_1]^2} \\
\psi_1 &= \mathrm{Atan2}(-Y, \pm\sqrt{X^2+Z^2}) \\
\phi_1 &= \mathrm{Atan2}(X/\cos\psi_1, Z/\cos\psi_1)
\end{aligned}
\tag{29}
$$

where X,Y,Z are the components of $[R(\hat{\mathbf{Z}},\gamma_i)]^T((x,y,z)^T - {}^{B_0}\mathbf{b}_1)$.

- Perform for each leg to obtain $l_i,\ \psi_i$ and ϕ_i for $i=1,\ldots,6$.

Inverse kinematics of Stewart platform (contd.)

- From known ${}^{B_0}_{P_0}[R]$ and translation vector ${}^{B_0}\mathbf{t}$, obtain ${}^{B_0}\mathbf{p}_1$

$$\begin{aligned} [R(\hat{\mathbf{Z}},\gamma_i)]^T((x,y,z)^T - {}^{B_0}\mathbf{b}_1) &= [R(\hat{\mathbf{Y}},\phi_i)][R(\hat{\mathbf{X}},\psi_i)](0,0,l_i)^T \\ &= l_1 \begin{pmatrix} \sin\phi_1 \cos\psi_1 \\ -\sin\psi_1 \\ \cos\phi_1 \cos\psi_1 \end{pmatrix} \end{aligned} \quad (28)$$

where ${}^{B_0}\mathbf{p}_1$ is denoted by $(x,y,z)^T$.

- Three non-linear equations in $l_1,\ \psi_1,\ \phi_1 \rightarrow$ solution

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where X, Y, Z are the components of $[R(\hat{\mathbf{Z}},\gamma_i)]^T((x,y,z)^T - {}^{B_0}\mathbf{b}_1)$.

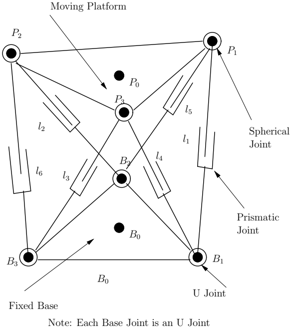
- Perform for each leg to obtain l_i, ψ_i and ϕ_i for $i = 1, \ldots, 6$.

Outline

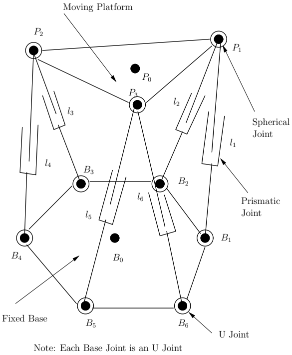
Stewart Platform Manipulators

Review

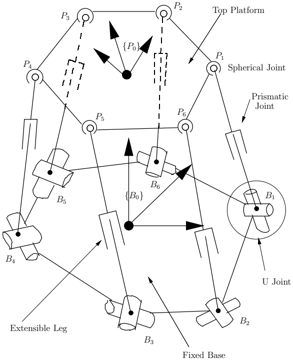
- Gough-Stewart platform – six- degree-of-freedom parallel manipulator.
- Extensively used in flight simulators, machine tools, force-torque sensors, orienting device etc. (see Merlet, 2001).



(a) 3–3 Stewart Platform



(b) 6–3 Stewart Platform



(c) 6–6 Stewart Platform

Figure 13: Three configurations of Stewart platform manipulator

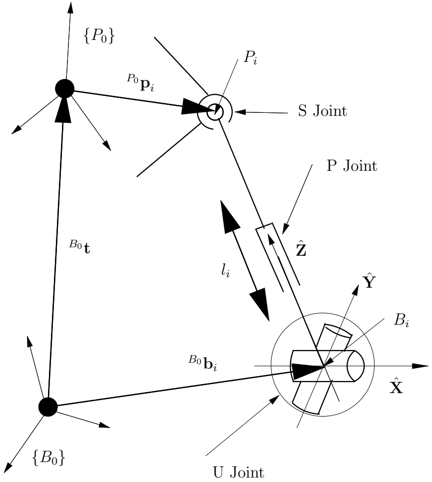


Figure 14: A leg of a Stewart platform -revisited

- Hooke ('U') joint modeled as 2 intersecting R joint $\rightarrow$ Each leg R-R-P-S chain.
- Hooke joint equivalent to successive Euler rotations (see Module 2, Lecture 2) ϕ_i about $\hat{\mathbf{Y}}_i$ and ψ_i about $\hat{\mathbf{X}}_i$.

- The vector ${}^{B_0}\mathbf{p}_i$ locating the spherical joint can be written as

$$
\begin{aligned}
{}^{B_0}\mathbf{p}_i &= {}^{B_0}\mathbf{b}_i + [R(\hat{\mathbf{Z}},\gamma_i)][R(\hat{\mathbf{Y}},\phi_i)][R(\hat{\mathbf{X}},\psi_i)](0,0,l_i)^T \\
&= {}^{B_0}\mathbf{b}_i + l_i \begin{pmatrix} \cos\gamma_i \sin\phi_i \cos\psi_i + \sin\gamma_i \sin\psi_i \\ \sin\gamma_i \sin\phi_i \cos\psi_i - \cos\gamma_i \sin\psi_i \\ \cos\phi_i \cos\psi_i \end{pmatrix}
\end{aligned} \tag{30}
$$

  - Constant vector ${}^{B_0}\mathbf{b}_i$ locates the origin O_i $\{i\}$ at the Hooke joint i,
  - Constant angle γ_i determines the orientation of $\{i\}$ with respect to $\{B_0\}$, and
  - l_i is the translation of the prismatic (P) joint in leg i.
- ${}^{B_0}\mathbf{p}_i$ is a function of two passive joint variables, ϕ_i and ψ_i, and the actuated joint variable l_i.

- The vector ${}^{B_0}\mathbf{p}_i$ locating the spherical joint can be written as

$$
\begin{aligned}
{}^{B_0}\mathbf{p}_i &= {}^{B_0}\mathbf{b}_i + [R(\hat{\mathbf{Z}},\gamma_i)][R(\hat{\mathbf{Y}},\phi_i)][R(\hat{\mathbf{X}},\psi_i)](0,0,l_i)^T \\
&= {}^{B_0}\mathbf{b}_i + l_i \begin{pmatrix} \cos\gamma_i \sin\phi_i \cos\psi_i + \sin\gamma_i \sin\psi_i \\ \sin\gamma_i \sin\phi_i \cos\psi_i - \cos\gamma_i \sin\psi_i \\ \cos\phi_i \cos\psi_i \end{pmatrix}
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- ${}^{B_0}\mathbf{p}_i$ is a function of two passive joint variables, ϕ_i and ψ_i, and the actuated joint variable l_i.

- 6 legs are $B_1 - P_1$, $B_1 - P_3$, $B_2 - P_1$, $B_2 - P_2$, $B_3 - P_2$ and $B_3 - P_3$ (see figure 13(a)).
- 6 actuated and 12 passive variables $\rightarrow$ 12 constraint equations needed.
- Three constraints: Distances between P_1, P_2 and P_3 are constant (similar to 3-RPS).
- Point P_1 reached in *two ways*: 3 vector equations or 9 scalar equations.

$$\begin{aligned}
{}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1 P_1} &= {}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2 P_1} \\
{}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2 P_2} &= {}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3 P_2} \\
{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3 P_3} &= {}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1 P_3}
\end{aligned}$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination (Nanua, Waldron, Murthy, 1990).

- 6 legs are $B_1 - P_1$, $B_1 - P_3$, $B_2 - P_1$, $B_2 - P_2$, $B_3 - P_2$ and $B_3 - P_3$ (see figure 13(a)).
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{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3P_3} &= {}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1P_3}
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- 6 legs are $B_1 - P_1$, $B_1 - P_3$, $B_2 - P_1$, $B_2 - P_2$, $B_3 - P_2$ and $B_3 - P_3$ (see figure 13(a)).
- 6 actuated and 12 passive variables $\rightarrow$ 12 constraint equations needed.
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{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3 P_3} &= {}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1 P_3}
\end{aligned}
$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination (Nanua, Waldron, Murthy, 1990).

Stewart Platform Manipulators

Direct kinematics of 6–3 configuration

- Direct kinematics similar to 3–3 configurations (see figure 13(b))
- 6 legs are $B_1 - P_1$, $B_2 - P_1$, $B_3 - P_2$, $B_4 - P_2$, $B_5 - P_3$ and $B_6 - P_3$.
- 6 actuated and 12 passive variables $\rightarrow$ 12 constraint equations needed.
- Three constraints: Distances between P_1, P_2 and P_3 are constant (similar to 3-RPS).
- P_1, P_2 and P_3 reached in two ways $\rightarrow$ 9 scalar equations

$$
\begin{aligned}
{}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1 P_1} &= {}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2 P_1} \\
{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3 P_2} &= {}^{B_0}\mathbf{b}_4 + \overrightarrow{B_4 P_2} \\
{}^{B_0}\mathbf{b}_5 + \overrightarrow{B_5 P_3} &= {}^{B_0}\mathbf{b}_6 + \overrightarrow{B_6 P_3}
\end{aligned}
$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination.

- Direct kinematics similar to 3–3 configurations (see figure 13(b))
- 6 legs are $B_1 - P_1$, $B_2 - P_1$, $B_3 - P_2$, $B_4 - P_2$, $B_5 - P_3$ and $B_6 - P_3$.
- 6 actuated and 12 passive variables $\rightarrow$ 12 constraint equations needed.
- Three constraints: Distances between P_1, P_2 and P_3 are constant (similar to 3-RPS).
- P_1, P_2 and P_3 reached in two ways $\rightarrow$ 9 scalar equations

$$
\begin{aligned}
{}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1 P_1} &= {}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2 P_1} \\
{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3 P_2} &= {}^{B_0}\mathbf{b}_4 + \overrightarrow{B_4 P_2} \\
{}^{B_0}\mathbf{b}_5 + \overrightarrow{B_5 P_3} &= {}^{B_0}\mathbf{b}_6 + \overrightarrow{B_6 P_3}
\end{aligned}
$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination.

- Direct kinematics similar to 3–3 configurations (see figure 13(b))
- 6 legs are $B_1 - P_1$, $B_2 - P_1$, $B_3 - P_2$, $B_4 - P_2$, $B_5 - P_3$ and $B_6 - P_3$.
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$$\begin{aligned}
{}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1P_1} &= {}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2P_1} \\
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{}^{B_0}\mathbf{b}_5 + \overrightarrow{B_5P_3} &= {}^{B_0}\mathbf{b}_6 + \overrightarrow{B_6P_3}
\end{aligned}$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination.

- Direct kinematics similar to 3–3 configurations (see figure 13(b))
- 6 legs are $B_1 - P_1$, $B_2 - P_1$, $B_3 - P_2$, $B_4 - P_2$, $B_5 - P_3$ and $B_6 - P_3$.
- 6 actuated and 12 passive variables $\rightarrow$ 12 constraint equations needed.
- Three constraints: Distances between P_1, P_2 and P_3 are constant (similar to 3-RPS).
- P_1, P_2 and P_3 reached in two ways $\rightarrow$ 9 scalar equations

$$
\begin{aligned}
{}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1 P_1} &= {}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2 P_1} \\
{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3 P_2} &= {}^{B_0}\mathbf{b}_4 + \overrightarrow{B_4 P_2} \\
{}^{B_0}\mathbf{b}_5 + \overrightarrow{B_5 P_3} &= {}^{B_0}\mathbf{b}_6 + \overrightarrow{B_6 P_3}
\end{aligned}
$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination.

- Direct kinematics similar to 3–3 configurations (see figure 13(b))
- 6 legs are $B_1 - P_1$, $B_2 - P_1$, $B_3 - P_2$, $B_4 - P_2$, $B_5 - P_3$ and $B_6 - P_3$.
- 6 actuated and 12 passive variables $\rightarrow$ 12 constraint equations needed.
- Three constraints: Distances between P_1, P_2 and P_3 are constant (similar to 3-RPS).
- P_1, P_2 and P_3 reached in two ways $\rightarrow$ 9 scalar equations

$$
\begin{aligned}
{}^{B_0}\mathbf{b}_1 + \overrightarrow{B_1P_1} &= {}^{B_0}\mathbf{b}_2 + \overrightarrow{B_2P_1} \\
{}^{B_0}\mathbf{b}_3 + \overrightarrow{B_3P_2} &= {}^{B_0}\mathbf{b}_4 + \overrightarrow{B_4P_2} \\
{}^{B_0}\mathbf{b}_5 + \overrightarrow{B_5P_3} &= {}^{B_0}\mathbf{b}_6 + \overrightarrow{B_6P_3}
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\end{aligned}
$$

- 16^{th} degree polynomial in tangent half-angle obtained after elimination.

- 6 distinct points in the fixed base and moving platform (see figure 13(c))
- Hooke joint modeled as 2 intersecting rotary (R) joint $\rightarrow$ 6 actuated and 12 passive variables $\rightarrow$ Need 12 constraint equations!.
- ${}^{B_0}\mathbf{p}_i$ revisited

$$\begin{aligned} {}^{B_0}\mathbf{p}_i &= {}^{B_0}\mathbf{b}_i + [R(\hat{\mathbf{Z}},\gamma_i)][R(\hat{\mathbf{Y}},\phi_i)][R(\hat{\mathbf{X}},\psi_i)](0,0,l_i)^T \\ &= {}^{B_0}\mathbf{b}_i + l_i \begin{pmatrix} \cos\gamma_i \sin\phi_i \cos\psi_i + \sin\gamma_i \sin\psi_i \\ \sin\gamma_i \sin\phi_i \cos\psi_i - \cos\gamma_i \sin\psi_i \\ \cos\phi_i \cos\psi_i \end{pmatrix} \end{aligned} \quad (31)$$

- 6 constraint equations from $S-S$ pair constraints (see Module 2, Lecture 2)

- 6 distinct points in the fixed base and moving platform (see figure 13(c))
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$$
\begin{aligned}
{}^{B_0}\mathbf{p}_i &= {}^{B_0}\mathbf{b}_i + [R(\hat{\mathbf{Z}},\gamma_i)][R(\hat{\mathbf{Y}},\phi_i)][R(\hat{\mathbf{X}},\psi_i)](0,0,l_i)^T \\
&= {}^{B_0}\mathbf{b}_i + l_i \begin{pmatrix} \cos\gamma_i \sin\phi_i \cos\psi_i + \sin\gamma_i \sin\psi_i \\ \sin\gamma_i \sin\phi_i \cos\psi_i - \cos\gamma_i \sin\psi_i \\ \cos\phi_i \cos\psi_i \end{pmatrix}
\end{aligned} \tag{31}
$$

- 6 constraint equations from $S-S$ pair constraints (see Module 2, Lecture 2)

- 6 distinct points in the fixed base and moving platform (see figure 13(c))
- Hooke joint modeled as 2 intersecting rotary (R) joint $\rightarrow$ 6 actuated and 12 passive variables $\rightarrow$ Need 12 constraint equations!.
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- 6 constraint equations from $S-S$ pair constraints (see Module 2, Lecture 2)

- 6 $S-S$ pair constraints

$$
\begin{aligned}
\eta_1(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_2|^2 - d_{12}^2 = 0 \\
\eta_2(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_2 - {}^{B_0}\mathbf{p}_3|^2 - d_{23}^2 = 0 \\
\eta_3(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_3 - {}^{B_0}\mathbf{p}_4|^2 - d_{34}^2 = 0 \\
\eta_4(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_4 - {}^{B_0}\mathbf{p}_5|^2 - d_{45}^2 = 0 \\
\eta_5(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_5 - {}^{B_0}\mathbf{p}_6|^2 - d_{56}^2 = 0 \\
\eta_6(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_6 - {}^{B_0}\mathbf{p}_1|^2 - d_{61}^2 = 0
\end{aligned}
\tag{32}
$$

- Need another 6 constraint equations.

- 6 $S - S$ pair constraints

$$\begin{aligned}
\eta_1(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_2|^2 - d_{12}^2 = 0 \\
\eta_2(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_2 - {}^{B_0}\mathbf{p}_3|^2 - d_{23}^2 = 0 \\
\eta_3(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_3 - {}^{B_0}\mathbf{p}_4|^2 - d_{34}^2 = 0 \\
\eta_4(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_4 - {}^{B_0}\mathbf{p}_5|^2 - d_{45}^2 = 0 \\
\eta_5(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_5 - {}^{B_0}\mathbf{p}_6|^2 - d_{56}^2 = 0 \\
\eta_6(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_6 - {}^{B_0}\mathbf{p}_1|^2 - d_{61}^2 = 0
\end{aligned} \tag{32}$$

- Need another 6 constraint equations.

- All six points $P_i,\ i = 1,...,6$ must lie on a plane

$$
\begin{aligned}
\eta_7(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_3|^2 - d_{13}^2 = 0 \\
\eta_8(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4|^2 - d_{14}^2 = 0 \\
\eta_9(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_5|^2 - d_{15}^2 = 0 \\
\eta_{10}(\mathbf{q}) &= ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_3) \times ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4) \cdot ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_2) = 0 \\
\eta_{11}(\mathbf{q}) &= ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4) \times ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_5) \cdot ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_3) = 0 \\
\eta_{12}(\mathbf{q}) &= ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_5) \times ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_6) \cdot ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4) = 0
\end{aligned}
\tag{33}
$$

- d_{ij} is the known distance between the spherical joints S_i and S_j on the top platform.

- All six points $P_i, \ i = 1, ..., 6$ must lie on a plane

$$
\begin{aligned}
\eta_7(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_3|^2 - d_{13}^2 = 0 \\
\eta_8(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4|^2 - d_{14}^2 = 0 \\
\eta_9(\mathbf{q}) &= |{}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_5|^2 - d_{15}^2 = 0 \\
\eta_{10}(\mathbf{q}) &= ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_3) \times ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4) \cdot ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_2) = 0 \\
\eta_{11}(\mathbf{q}) &= ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4) \times ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_5) \cdot ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_3) = 0 \\
\eta_{12}(\mathbf{q}) &= ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_5) \times ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_6) \cdot ({}^{B_0}\mathbf{p}_1 - {}^{B_0}\mathbf{p}_4) = 0
\end{aligned}
\tag{33}
$$

- d_{ij} is the known distance between the spherical joints S_i and S_j on the top platform.

- 12 non-linear equations in twelve passive variables $\phi_i, \psi_i,\ i = 1,...,6$, and six actuated joint variables $l_i,\ i = 1,...,6$.
- All equations do not contain *all* passive variables $\rightarrow$ first equation in (32) is a function of only ϕ_1, ψ_1, l_1, ϕ_2, ψ_2, and l_2.
- 12 equations are not unique and one can have other combinations.
- For direct kinematics, eliminate 11 passive variables from these 12 equations.
- Very hard and not yet done!!
- Direct kinematics of Gough-Stewart platform easier with *task* space variables.

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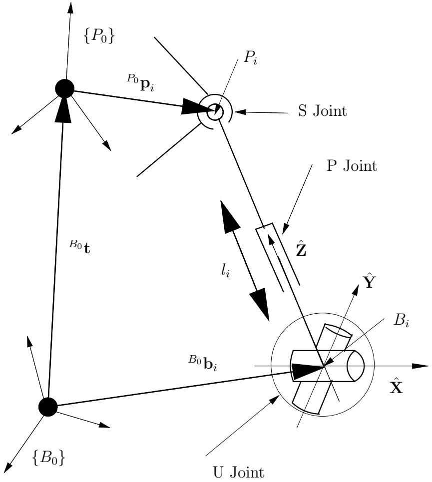

Figure 15: A leg of a Stewart platform -revisited

- The point P_i in $\{B_0\}$

$$ {}^{B_0}\mathbf{p}_i = {}^{B_0}_{P_0}[R]\, {}^{P_0}\mathbf{p}_i + {}^{B_0}\mathbf{t} \qquad (34) $$

where ${}^{P_0}\mathbf{p}_i = (p_{i_x}, p_{i_y}, 0)^T$.

- Denoting point B_i by ${}^{B_0}\mathbf{B}_i$, the leg vector ${}^{B_0}\mathbf{S}_i$ is

$$ {}^{B_0}\mathbf{S}_i = {}^{B_0}_{P_0}[R]\, {}^{P_0}\mathbf{p}_i + {}^{B_0}\mathbf{t} - {}^{B_0}\mathbf{b}_i \qquad (35) $$

where ${}^{B_0}\mathbf{b}_i = (b_{i_x}, b_{i_y}, 0)^T$.

- The magnitude of the leg vector is

$$
\begin{aligned}
l_i^2 &= (r_{11}p_{i_x} + r_{12}p_{i_y} + t_x - b_{i_x})^2 + (r_{21}p_{i_x} + r_{22}p_{i_y} + t_y - b_{i_y})^2 \\
&\quad + (r_{31}p_{i_x} + r_{32}p_{i_y} + t_z - b_{i_z})^2 \qquad (36)
\end{aligned}
$$

- Using properties of the elements r_{ij}, get

$$
\begin{aligned}
&(t_x^2 + t_y^2 + t_z^2) + 2p_{i_x}(r_{11}t_x + r_{21}t_y + r_{31}t_z) + 2p_{i_y}(r_{12}t_x + r_{22}t_y + r_{32}t_z) \\
&\quad -2b_{i_x}(t_x + p_{i_x}r_{11} + p_{i_y}r_{12}) - 2b_{i_y}(t_y + p_{i_x}r_{21} + p_{i_y}r_{22}) \\
&\quad + b_{i_x}^2 + b_{i_y}^2 + p_{i_x}^2 + p_{i_y}^2 - l_i^2 = 0 \qquad (37)
\end{aligned}
$$

- For the six legs, $i = 1, \ldots, 6$, six equations of the type shown above.
- Additional 3 constraints

$$
\begin{aligned}
r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\
r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \qquad (38) \\
r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0
\end{aligned}
$$

- The magnitude of the leg vector is

$$
\begin{aligned}
l_i^2 \quad = \quad & (r_{11} p_{i_x} + r_{12} p_{i_y} + t_x - b_{i_x})^2 + (r_{21} p_{i_x} + r_{22} p_{i_y} + t_y - b_{i_y})^2 \\
& + (r_{31} p_{i_x} + r_{32} p_{i_y} + t_z - b_{i_z})^2 \qquad (36)
\end{aligned}
$$

- Using properties of the elements r_{ij}, get

$$
\begin{aligned}
& (t_x^2 + t_y^2 + t_z^2) + 2p_{i_x}(r_{11} t_x + r_{21} t_y + r_{31} t_z) + 2p_{i_y}(r_{12} t_x + r_{22} t_y + r_{32} t_z) \\
& -2b_{i_x}(t_x + p_{i_x} r_{11} + p_{i_y} r_{12}) - 2b_{i_y}(t_y + p_{i_x} r_{21} + p_{i_y} r_{22}) \\
& + b_{i_x}^2 + b_{i_y}^2 + p_{i_x}^2 + p_{i_y}^2 - l_i^2 = 0 \qquad (37)
\end{aligned}
$$

- For the six legs, $i = 1, \ldots, 6$, six equations of the type shown above.
- Additional 3 constraints

$$
\begin{aligned}
r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\
r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \qquad (38) \\
r_{11} r_{12} + r_{21} r_{22} + r_{31} r_{32} &= 0
\end{aligned}
$$

- The magnitude of the leg vector is

$$l_i^2 = (r_{11}p_{i_x} + r_{12}p_{i_y} + t_x - b_{i_x})^2 + (r_{21}p_{i_x} + r_{22}p_{i_y} + t_y - b_{i_y})^2 + (r_{31}p_{i_x} + r_{32}p_{i_y} + t_z - b_{i_z})^2 \quad (36)$$

- Using properties of the elements r_{ij}, get

$$(t_x^2 + t_y^2 + t_z^2) + 2p_{i_x}(r_{11}t_x + r_{21}t_y + r_{31}t_z) + 2p_{i_y}(r_{12}t_x + r_{22}t_y + r_{32}t_z) - 2b_{i_x}(t_x + p_{i_x}r_{11} + p_{i_y}r_{12}) - 2b_{i_y}(t_y + p_{i_x}r_{21} + p_{i_y}r_{22}) + b_{i_x}^2 + b_{i_y}^2 + p_{i_x}^2 + p_{i_y}^2 - l_i^2 = 0 \quad (37)$$

- For the six legs, $i = 1, ..., 6$, six equations of the type shown above.
- Additional 3 constraints

$$\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\ r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \\ r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0 \end{aligned} \quad (38)$$

- Equations (37) and (38) are nine *quadratic* equations in nine unknowns, t_x, t_y, t_z, r_{11}, r_{12}, r_{21}, r_{22}, r_{31}, and r_{32} (see Dasgupta and Mruthyunjaya, 1994)
- All quadratic terms in equation (37) are square of the magnitude of the translation vector $(t_x^2 + t_y^2 + t_z^2)$, and as X and Y component of the vector ${}^{B_0}\mathbf{t}$, $(r_{11}t_x + r_{21}t_y + r_{31}t_z)$ and $(r_{12}t_x + r_{22}t_y + r_{32}t_z)$, respectively.
- Reduce the set of nine quadratics to *six quadratic* and *three linear* equations in *nine* unknowns $\rightarrow$ Starting point of elimination.
- Very hard to eliminate eight variables from the nine equations to arrive at a univariate polynomial in one unknown.
- Univariate polynomial widely accepted to be of 40th degree (Raghavan 1993).
- Continuing attempts to obtain the simplest explicit expressions for the co-efficients of this 40th-degree polynomial.

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Gough-Stewart Platform Manipulator

Direct kinematics of 6–6 configuration in task space

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- Reduce the set of nine quadratics to *six quadratic* and *three linear* equations in *nine* unknowns $\rightarrow$ Starting point of elimination.
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- All quadratic terms in equation (37) are square of the magnitude of the translation vector $(t_x^2 + t_y^2 + t_z^2)$, and as X and Y component of the vector ${}^{B_0}\mathbf{t}$, $(r_{11}t_x + r_{21}t_y + r_{31}t_z)$ and $(r_{12}t_x + r_{22}t_y + r_{32}t_z)$, respectively.
- Reduce the set of nine quadratics to *six quadratic* and *three linear* equations in *nine* unknowns $\rightarrow$ Starting point of elimination.
- Very hard to eliminate eight variables from the nine equations to arrive at a univariate polynomial in one unknown.
- Univariate polynomial widely accepted to be of 40th degree (Raghavan 1993).
- Continuing attempts to obtain the simplest explicit expressions for the co-efficients of this 40th-degree polynomial.

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