Robotics: Advanced Concepts & Analysis

Module 5 - Velocity and Static Analysis of Manipulators

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NPTEL, 2010

OUTLINE

INTRODUCTION
REVIEW

- Position kinematics $\rightarrow$ position & orientation of links, workspace, mobility etc.
- Change of position and orientation with respect to time $\rightarrow$ velocity kinematics
- Linear velocity as derivative of position vector.
- Angular velocity in terms of derivative of rotation matrix.
- Topics in velocity kinematics include
  - Linear and angular velocities of links
  - Manipulator Jacobian(s)
  - Singularities in velocity domain
- Static equilibrium
  - Relation between external forces & moments and joint torques & forces.
  - Singularities in force domain

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OUTLINE

Linear Velocity of Rigid Body

- The linear velocity of O_i with respect to $\{0\}$ is defined as

$$ {}^0\mathbf{V}_{O_i} \triangleq \frac{d}{dt}\, {}^0\mathbf{O}_i(t) = \lim_{\Delta t \to 0} \frac{{}^0\mathbf{O}_i(t+\Delta t) - {}^0\mathbf{O}_i(t)}{\Delta t} \quad (1) $$

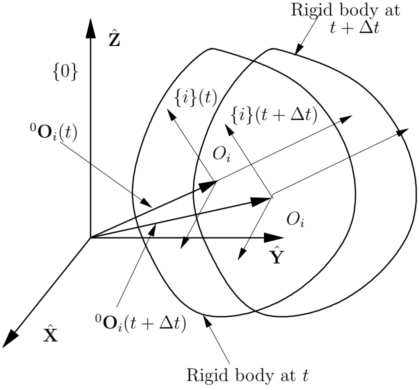

Figure 1: Linear velocity of a rigid body

- '0' denote the coordinate system $\{0\}$ where the limit is taken.
- The linear velocity vector can be *described* in $\{j\}$ as

$$ {}^j\left({}^0\mathbf{V}_{O_i}\right) = {}^j_0[R]\,{}^0\mathbf{V}_{O_i} \quad (2) $$

- Two different coordinate system involved: *where differentiation done*, and *where described*!

- Angular velocity *cannot* be obtained as a time derivative of 3 quantities representing orientation.
- Angular velocity can be derived from time derivative of rotation matrix.
  - Recall

$$ {}^0_i[R] \, {}^0_i[R]^T = [U], \qquad [U] \text{ is a } 3 \times 3 \text{ identity matrix} $$

  - Differentiate with respect to time t

$$ {}^0_i[\dot{R}] \, {}^0_i[R]^T + {}^0_i[R] \, {}^0_i[\dot{R}]^T = [0] $$

where *derivative of a matrix implies derivative of all components of the matrix*.

  - Above equation can be written as

$$ {}^0_i[\dot{R}] \, {}^0_i[R]^T + ({}^0_i[\dot{R}] \, {}^0_i[R]^T)^T = [0] $$

- Define a 3×3 skew symmetric matrix

$$ {}^0_i[\Omega]_R \triangleq {}^0_i[\dot{R}] \, {}^0_i[R]^T $$

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- *Skew-symmetric* matrix in detail

$$ {}^0_i[\Omega]_R = \begin{pmatrix} 0 & -\omega_z^s & \omega_y^s \\ \omega_z^s & 0 & -\omega_x^s \\ -\omega_y^s & \omega_x^s & 0 \end{pmatrix} \tag{3} $$

- The product of ${}^O_i[\Omega]_R$ and a vector $(p_x, p_y, p_z)^T \in \Re^3$ is a cross-product

$$ {}^0_i[\Omega]_R (p_x, p_y, p_z)^T = \begin{pmatrix} \omega_y^s p_z & - & \omega_z^s p_y \\ \omega_z^s p_x & - & \omega_x^s p_z \\ \omega_x^s p_y & - & \omega_y^s p_x \end{pmatrix} = {}^0\omega_i{}^s \times {}^0\mathbf{p} \tag{4} $$

- ${}^0_i[\Omega]_R$ called angular velocity matrix
- ${}^0\omega_i{}^s$ called angular velocity vector of $\{i\}$ with respect to $\{0\}$.
- In contrast to linear velocity, angular velocity vector is not a straightforward differentiation!

- *Skew-symmetric* matrix in detail

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ANGULAR VELOCITY OF RIGID BODY – SKEW SYMMETRIC MATRIX

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- Angular velocity in terms of Z-Y-Z Euler angles.
- Recall for α, β and γ as the Z-Y-Z Euler angles

$$ {}^A_B[R] = \begin{pmatrix} C_\alpha C_\beta C_\gamma - S_\alpha S_\gamma & -C_\alpha C_\beta S_\gamma - S_\alpha C_\gamma & C_\alpha S_\beta \\ S_\alpha C_\beta C_\gamma + C_\alpha S_\gamma & -S_\alpha C_\beta S_\gamma + C_\alpha C_\gamma & S_\alpha S_\beta \\ -S_\beta C_\gamma & S_\beta S_\gamma & C_\beta \end{pmatrix} \quad (5) $$

- Obtain ${}^A_B[\dot{R}] \, {}^A_B[R]^T$
- The X, Y and Z components of the angular velocity vector

$$ \begin{aligned} {\omega_x}^s &= \dot{\gamma} \cos\alpha \sin\beta - \dot{\beta} \sin\alpha \\ {\omega_y}^s &= \dot{\gamma} \sin\alpha \sin\beta + \dot{\beta} \cos\alpha \\ {\omega_z}^s &= \dot{\gamma} \cos\beta + \dot{\alpha} \end{aligned} \quad (6) $$

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- ${}^0_i[\Omega]_R$ called *right-invariant* – derived from *right multiplication* ${}^0_i[R]\ {}^0_i[R]^T = [U]$.
- ${}^0\omega_i{}^s$ called the *space-fixed* angular velocity – superscript s.
- ${}^0_i[R]^T\ {}^0_i[R] = [U] \rightarrow$ another skew-symmetric matrix

$$ {}^0_i[\Omega]_L \triangleq {}^0_i[R]^T\ {}^0_i[\dot{R}] = \begin{pmatrix} 0 & -\omega_z^b & \omega_y^b \\ \omega_z^b & 0 & -\omega_x^b \\ -\omega_y^b & \omega_x^b & 0 \end{pmatrix} \quad (7) $$

- Define an angular velocity vector ${}^0\omega_i{}^b$ from the three components $(\omega_x^b, \omega_y^b, \omega_z^b)$.

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- Define an angular velocity vector ${}^0\omega_i{}^b$ from the three components $(\omega_x^b, \omega_y^b, \omega_z^b)$.

- For the Z-Y-Z rotation the three components are

$$\begin{aligned}
{\omega_x}^b &= -\dot{\alpha}\cos\gamma\sin\beta + \dot{\beta}\sin\gamma \\
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\end{aligned} \tag{8}$$

- ${}^0_i[\Omega]_L$ called *left-invariant* angular velocity matrix.
- ${}^0{\omega_i}^b$ called *body-fixed* angular velocity vector of $\{i\}$ with respect to $\{0\}$ – superscript b.
- The two skew-symmetric matrices are related like two tensors

$${}^0_i[\Omega]_R = {}^0_i[R]\ {}^0_i[\Omega]_L\ {}^0_i[R]^T \tag{9}$$

- The two angular velocities are related as

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Angular Velocity of Rigid Body – left invariant

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ANGULAR VELOCITY OF RIGID BODY – LEFT INVARIANT

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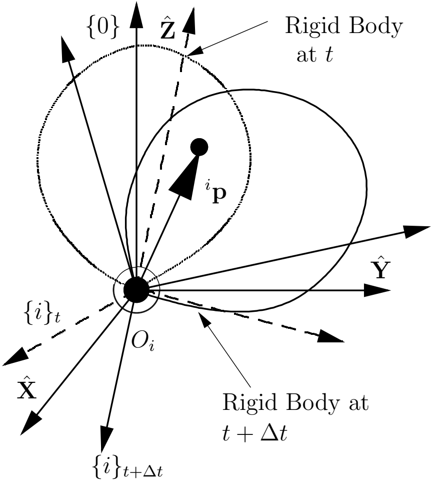

Figure 2: Angular velocity of a rigid body

- More on two forms of angular velocity matrix and vectors.
- Pure rotation – $^0\mathbf{O}_i(t)$ and $^0\mathbf{O}_i(t+\Delta t)$ are coincident and only the elements of the rotation matrix $^i_0[R]$ change with time.
- Point P located by $^i\mathbf{p}$, and *fixed* in $\{i\}$

- Location of P in $\{0\}$

$$^0\mathbf{p} = {}^0_i[R]\,^i\mathbf{p}$$

and since P is fixed in $\{i\}$

$$^0\dot{\mathbf{p}} \stackrel{\Delta}{=} {}^0\mathbf{V}_p = {}^0_i[\dot{R}]\ ^i\mathbf{p}$$

and since ${}^0_i[R]^{-1} = {}^0_i[R]^T$,

$$\begin{aligned} {}^0\mathbf{V}_p &= {}^0_i[\dot{R}]\ {}^0_i[R]^T\ {}^0\mathbf{p} \\ &= {}^0_i[\Omega]_R\ {}^0\mathbf{p} \\ &= {}^0\omega_i{}^s \times {}^0\mathbf{p} \end{aligned} \qquad (11)$$

- The coordinate system $\{i\}$ does not appear *except* in denoting that rigid body $\{i\}$ is being considered.
- Space-fixed angular velocity vector is said to be *independent* of the choice of the body coordinate system.

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- Space-fixed angular velocity vector is said to be *independent* of the choice of the body coordinate system.

- Using relation between ${}^0_i[\Omega]_R$ and ${}^0_i[\Omega]_L$

$$\begin{aligned} {}^0\mathbf{V}_p &= {}^0_i[R]\ {}^0_i[\Omega]_L\ {}^0_i[R]^T\ {}^0\mathbf{p} \\ &= {}^0_i[R]{}^0_i[\Omega]_L\ {}^i\mathbf{p} \end{aligned}$$

and get

$${}^0_i[R]^{-1}\ {}^0\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p}$$

which yields

$${}^i\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p} = {}^0\omega_i{}^b \times {}^i\mathbf{p} \qquad (12)$$

- Again *except* for denoting the reference coordinate system, the coordinate system $\{0\}$ does not appear!
- Body-fixed angular velocity vector is said to be *independent* of the choice of the fixed coordinate system.
- Unless explicitly stated, *space-fixed* angular velocity vector derived from ${}^0_i[\dot{R}]\ {}^0_i[R]^T$ is **normally** used.

Linear and Angular Velocity of Rigid Body

Angular Velocity of Rigid Body (Contd.)

- Using relation between ${}^0_i[\Omega]_R$ and ${}^0_i[\Omega]_L$

$$\begin{aligned} {}^0\mathbf{V}_p &= {}^0_i[R]\ {}^0_i[\Omega]_L\ {}^0_i[R]^T\ {}^0\mathbf{p} \\ &= {}^0_i[R]{}^0_i[\Omega]_L\ {}^i\mathbf{p} \end{aligned}$$

and get

$${}^0_i[R]^{-1}\ {}^0\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p}$$

which yields

$${}^i\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p} = {}^0\omega_i{}^b \times {}^i\mathbf{p} \tag{12}$$

- Again *except* for denoting the reference coordinate system, the coordinate system $\{0\}$ does not appear!
- Body-fixed angular velocity vector is said to be *independent* of the choice of the fixed coordinate system.
- Unless explicitly stated, *space-fixed* angular velocity vector derived from ${}^0_i[\dot{R}]\ {}^0_i[R]^T$ is **normally** used.

- Using relation between ${}^0_i[\Omega]_R$ and ${}^0_i[\Omega]_L$

$$\begin{aligned} {}^0\mathbf{V}_p &= {}^0_i[R]\ {}^0_i[\Omega]_L\ {}^0_i[R]^T\ {}^0\mathbf{p} \\ &= {}^0_i[R]{}^0_i[\Omega]_L\ {}^i\mathbf{p} \end{aligned}$$

and get

$${}^0_i[R]^{-1}\ {}^0\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p}$$

which yields

$${}^i\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p} = {}^0\omega_i{}^b \times {}^i\mathbf{p} \tag{12}$$

- Again *except* for denoting the reference coordinate system, the coordinate system $\{0\}$ does not appear!
- Body-fixed angular velocity vector is said to be *independent* of the choice of the fixed coordinate system.
- Unless explicitly stated, *space-fixed* angular velocity vector derived from ${}^0_i[\dot{R}]\ {}^0_i[R]^T$ is **normally** used.

- Using relation between ${}^0_i[\Omega]_R$ and ${}^0_i[\Omega]_L$

$$
\begin{aligned}
{}^0\mathbf{V}_p &= {}^0_i[R]\ {}^0_i[\Omega]_L\ {}^0_i[R]^T\ {}^0\mathbf{p} \\
&= {}^0_i[R]{}^0_i[\Omega]_L\ {}^i\mathbf{p}
\end{aligned}
$$

and get

$$
{}^0_i[R]^{-1}\ {}^0\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p}
$$

which yields

$$
{}^i\mathbf{V}_p = {}^0_i[\Omega]_L\ {}^i\mathbf{p} = {}^0\omega_i{}^b \times {}^i\mathbf{p} \tag{12}
$$

- Again *except* for denoting the reference coordinate system, the coordinate system $\{0\}$ does not appear!
- Body-fixed angular velocity vector is said to be *independent* of the choice of the fixed coordinate system.
- Unless explicitly stated, *space-fixed* angular velocity vector derived from ${}^0_i[\dot{R}]\ {}^0_i[R]^T$ is **normally** used.

LINEAR AND ANGULAR VELOCITY OF LINKS

ANGULAR VELOCITY IN SERIAL MANIPULATOR – ROTARY (R) JOINT

- For two links connected by a rotary (R) joint (see Module 2, Lecture 2)

$$ {}^{0}_{i}[R] = {}^{0}_{i-1}[R] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)] $$

- The time derivative operation

$$ {}^{0}_{i}[\dot{R}] \; {}^{0}_{i}[R]^T = \frac{d}{dt}({}^{0}_{i-1}[R] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]) \;\; ({}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]^T \; {}^{0}_{i-1}[R]^T) $$

- Rewrite above equation as

$$ {}^{0}_{i}[\Omega]_R = {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i-1}[R] \; ({}^{i-1}_{i}[\dot{R}(\hat{\mathbf{k}}, \theta_i)] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]^T) \; {}^{0}_{i-1}[R]^T $$

- To simplify, use the result

$$ {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)] = e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)} $$

${}^{i-1}_{i}[\mathscr{K}]$ is the skew-symmetric form of the rotation axis vector $\hat{\mathbf{k}}$ and θ_i is the rotation at the rotary joint (see Module 2, Lecture 2).

- For two links connected by a rotary (R) joint (see Module 2, Lecture 2)

$$ {}^{0}_{i}[R] = {}^{0}_{i-1}[R] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)] $$

- The time derivative operation

$$ {}^{0}_{i}[\dot{R}] \; {}^{0}_{i}[R]^T = \frac{d}{dt}({}^{0}_{i-1}[R] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]) \;\; ({}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]^T \; {}^{0}_{i-1}[R]^T) $$

- Rewrite above equation as

$$ {}^{0}_{i}[\Omega]_R = {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i-1}[R] \; ({}^{i-1}_{i}[\dot{R}(\hat{\mathbf{k}}, \theta_i)] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]^T) \; {}^{0}_{i-1}[R]^T $$

- To simplify, use the result

$$ {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)] = e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)} $$

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- For two links connected by a rotary (R) joint (see Module 2, Lecture 2)

$$ {}^{0}_{i}[R] = {}^{0}_{i-1}[R] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)] $$

- The time derivative operation

$$ {}^{0}_{i}[\dot{R}] \; {}^{0}_{i}[R]^T = \frac{d}{dt}({}^{0}_{i-1}[R] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]) \;\; ({}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]^T \; {}^{0}_{i-1}[R]^T) $$

- Rewrite above equation as

$$ {}^{0}_{i}[\Omega]_R = {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i-1}[R] \; ({}^{i-1}_{i}[\dot{R}(\hat{\mathbf{k}}, \theta_i)] \; {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)]^T) \; {}^{0}_{i-1}[R]^T $$

- To simplify, use the result

$$ {}^{i-1}_{i}[R(\hat{\mathbf{k}}, \theta_i)] = e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)} $$

${}^{i-1}_{i}[\mathscr{K}]$ is the skew-symmetric form of the rotation axis vector $\hat{\mathbf{k}}$ and θ_i is the rotation at the rotary joint (see Module 2, Lecture 2).

- For two links connected by a rotary (R) joint (see Module 2, Lecture 2)

$$ {}_{i}^{0}[R] = {}_{i-1}^{0}[R] \; {}_{i}^{i-1}[R(\hat{\mathbf{k}}, \theta_i)] $$

- The time derivative operation

$$ {}_{i}^{0}[\dot{R}] \; {}_{i}^{0}[R]^T = \frac{d}{dt}({}_{i-1}^{0}[R] \; {}_{i}^{i-1}[R(\hat{\mathbf{k}}, \theta_i)]) \;\; ({}_{i}^{i-1}[R(\hat{\mathbf{k}}, \theta_i)]^T \; {}_{i-1}^{0}[R]^T) $$

- Rewrite above equation as

$$ {}_{i}^{0}[\Omega]_R = {}_{i-1}^{0}[\Omega]_R + {}_{i-1}^{0}[R] \; ({}_{i}^{i-1}[\dot{R}(\hat{\mathbf{k}}, \theta_i)] \; {}_{i}^{i-1}[R(\hat{\mathbf{k}}, \theta_i)]^T) \; {}_{i-1}^{0}[R]^T $$

- To simplify, use the result

$$ {}_{i}^{i-1}[R(\hat{\mathbf{k}}, \theta_i)] = e^{({}_{i}^{i-1}[\mathscr{K}]\theta_i)} $$

${}_{i}^{i-1}[\mathscr{K}]$ is the skew-symmetric form of the rotation axis vector $\hat{\mathbf{k}}$ and θ_i is the rotation at the rotary joint (see Module 2, Lecture 2).

Linear and Angular Velocity of Links

Angular Velocity Propagation in Serial Manipulators – R joint

- $\hat{\mathbf{k}}$ is *fixed* in $\{i-1\}$ and $\{i\}$ $\rightarrow \frac{d}{dt} e^{({}_i^{i-1}[\mathscr{K}]\theta_i)} = {}_i^{i-1}[\mathscr{K}]\dot{\theta}_i \; e^{({}_i^{i-1}[\mathscr{K}]\theta_i)}$
- From above and properties of a rotation matrix,

$$\begin{aligned} {}_i^0[\Omega]_R &= {}_{i-1}^{0}[\Omega]_R + {}_{i-1}^{0}[R] \; {}_i^{i-1}[\mathscr{K}] \; {}_{i-1}^{0}[R]^T \; \dot{\theta}_i \\ &= {}_{i-1}^{0}[\Omega]_R + {}_i^0[\mathscr{K}]\dot{\theta}_i \end{aligned}$$

and in terms of the *space-fixed* angular velocity ${}^0\omega_{(\cdot)}$

$$ {}^0\omega_i = {}^0\omega_{i-1} + {}^0\hat{\mathbf{k}}_i \dot{\theta}_i $$

- Serial manipulators $\rightarrow$ R joint axis is chosen along the $Z-$ axis.
- Pre-multiply both sides by ${}_0^i[R]$ and simplify to get

$$ {}^i\omega_i = {}_{i-1}^{i}[R]^{i-1}\omega_{i-1} + \dot{\theta}_i (0 \; 0 \; 1)^T \tag{13} $$

${}^i\omega_i$ denotes ${}_0^i[R]^0\omega_i$ – ${}^i\omega_i$ **not** necessarily **0**.

- Angular velocity propagation in serial manipulators – links connected by R joints

Linear and Angular Velocity of Links

Angular Velocity Propagation in Serial Manipulators – R joint

- $\hat{\mathbf{k}}$ is *fixed* in $\{i-1\}$ and $\{i\} \rightarrow \frac{d}{dt} e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)} = {}^{i-1}_{i}[\mathscr{K}]\dot{\theta}_i \; e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)}$
- From above and properties of a rotation matrix,

$$\begin{aligned} {}^{0}_{i}[\Omega]_R &= {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i-1}[R] \; {}^{i-1}_{i}[\mathscr{K}] \; {}^{0}_{i-1}[R]^T \; \dot{\theta}_i \\ &= {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i}[\mathscr{K}]\dot{\theta}_i \end{aligned}$$

and in terms of the *space-fixed* angular velocity ${}^{0}\omega_{(\cdot)}$

$$ {}^{0}\omega_i = {}^{0}\omega_{i-1} + {}^{0}\hat{\mathbf{k}}_i\dot{\theta}_i $$

- Serial manipulators $\rightarrow$ R joint axis is chosen along the $Z-$ axis.
- Pre-multiply both sides by ${}^{i}_{0}[R]$ and simplify to get

$$ {}^{i}\omega_i = {}^{i}_{i-1}[R]^{i-1}\omega_{i-1} + \dot{\theta}_i(0\ 0\ 1)^T \qquad (13)$$

${}^{i}\omega_i$ denotes ${}^{i}_{0}[R]^{0}\omega_i$ – ${}^{i}\omega_i$ **not** necessarily **0**.

- Angular velocity propagation in serial manipulators – links connected by R joints

- $\hat{\mathbf{k}}$ is *fixed* in $\{i-1\}$ and $\{i\}$ $\rightarrow$ $\frac{d}{dt} e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)} = {}^{i-1}_{i}[\mathscr{K}]\dot{\theta}_i \; e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)}$
- From above and properties of a rotation matrix,

$$
\begin{aligned}
{}^{0}_{i}[\Omega]_R &= {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i-1}[R] \; {}^{i-1}_{i}[\mathscr{K}] \; {}^{0}_{i-1}[R]^T \; \dot{\theta}_i \\
&= {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i}[\mathscr{K}]\dot{\theta}_i
\end{aligned}
$$

  and in terms of the *space-fixed* angular velocity ${}^{0}\boldsymbol{\omega}_{(\cdot)}$

$$
{}^{0}\boldsymbol{\omega}_i = {}^{0}\boldsymbol{\omega}_{i-1} + {}^{0}\hat{\mathbf{k}}_i\dot{\theta}_i
$$

- Serial manipulators $\rightarrow$ R joint axis is chosen along the $Z-$ axis.
- Pre-multiply both sides by ${}^{i}_{0}[R]$ and simplify to get

$$
{}^{i}\boldsymbol{\omega}_i = {}^{i}_{i-1}[R]^{i-1}\boldsymbol{\omega}_{i-1} + \dot{\theta}_i (0 \; 0 \; 1)^T \tag{13}
$$

  ${}^{i}\boldsymbol{\omega}_i$ denotes ${}^{i}_{0}[R]^{0}\boldsymbol{\omega}_i$ – ${}^{i}\boldsymbol{\omega}_i$ **not** necessarily $\mathbf{0}$.

- Angular velocity propagation in serial manipulators – links connected by R joints

Linear and Angular Velocity of Links

Angular Velocity Propagation in Serial Manipulators – R joint

- $\hat{\mathbf{k}}$ is *fixed* in $\{i-1\}$ and $\{i\} \rightarrow \frac{d}{dt} e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)} = {}^{i-1}_{i}[\mathscr{K}]\dot{\theta}_i \; e^{({}^{i-1}_{i}[\mathscr{K}]\theta_i)}$
- From above and properties of a rotation matrix,

$$\begin{aligned} {}^{0}_{i}[\Omega]_R &= {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i-1}[R] \; {}^{i-1}_{i}[\mathscr{K}] \; {}^{0}_{i-1}[R]^T \; \dot{\theta}_i \\ &= {}^{0}_{i-1}[\Omega]_R + {}^{0}_{i}[\mathscr{K}]\dot{\theta}_i \end{aligned}$$

and in terms of the *space-fixed* angular velocity ${}^{0}\boldsymbol{\omega}_{(\cdot)}$

$${}^{0}\boldsymbol{\omega}_i = {}^{0}\boldsymbol{\omega}_{i-1} + {}^{0}\hat{\mathbf{k}}_i\dot{\theta}_i$$

- Serial manipulators $\rightarrow$ R joint axis is chosen along the $Z-$ axis.
- Pre-multiply both sides by ${}^{i}_{0}[R]$ and simplify to get

$${}^{i}\boldsymbol{\omega}_i = {}^{i}_{i-1}[R]^{i-1}\boldsymbol{\omega}_{i-1} + \dot{\theta}_i(0\ 0\ 1)^T \qquad (13)$$

${}^{i}\boldsymbol{\omega}_i$ denotes ${}^{i}_{0}[R]^{0}\boldsymbol{\omega}_i$ – ${}^{i}\boldsymbol{\omega}_i$ **not** necessarily **0**.

- Angular velocity propagation in serial manipulators – links connected by R joints

Linear and Angular Velocity of Links

Linear Velocity Propagation in Serial Manipulator – R joint

- For two consecutive links in a serial manipulator (see Module 2, Lecture 2)

$$ {}^0\mathbf{O}_i = {}^0\mathbf{O}_{i-1} + {}^0_{i-1}[R]\,{}^{i-1}\mathbf{O}_i $$

- Taking derivatives on both sides

$$ {}^0\mathbf{V}_{O_i} = {}^0\mathbf{V}_{O_{i-1}} + {}^0\omega_{i-1} \times {}^0_{i-1}[R]\,{}^{i-1}\mathbf{O}_i $$

- Simplify and rewrite above as

$$ {}^i\mathbf{V}_i = {}^i_{i-1}[R]({}^{i-1}\mathbf{V}_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}\mathbf{O}_i) \tag{14} $$

  Note: ${}^i\mathbf{V}_i$ and ${}^{i-1}\mathbf{V}_{i-1}$ denote ${}^i_0[R]{}^0\mathbf{V}_i$ and ${}^{i-1}_0[R]{}^0\mathbf{V}_{i-1}$, respectively. They are **not** necessarily **0**!

- Linear velocity vector propagation in links of a serial manipulator – Rotary joint.

- For two consecutive links in a serial manipulator (see Module 2, Lecture 2)

$$^0\mathbf{O}_i = ^0\mathbf{O}_{i-1} + ^0_{i-1}[R]^{i-1}\mathbf{O}_i$$

- Taking derivatives on both sides

$$^0\mathbf{V}_{O_i} = ^0\mathbf{V}_{O_{i-1}} + ^0\boldsymbol{\omega}_{i-1} \times ^0_{i-1}[R]^{i-1}\mathbf{O}_i$$

- Simplify and rewrite above as

$$^i\mathbf{V}_i = ^i_{i-1}[R](^{i-1}\mathbf{V}_{i-1} + ^{i-1}\boldsymbol{\omega}_{i-1} \times ^{i-1}\mathbf{O}_i) \tag{14}$$

Note: $^i\mathbf{V}_i$ and $^{i-1}\mathbf{V}_{i-1}$ denote $^i_0[R]^0\mathbf{V}_i$ and $^{i-1}_0[R]^0\mathbf{V}_{i-1}$, respectively. They are **not** necessarily **0**!

- Linear velocity vector propagation in links of a serial manipulator – Rotary joint.

- For two consecutive links in a serial manipulator (see Module 2, Lecture 2)

$$^0\mathbf{O}_i = ^0\mathbf{O}_{i-1} + ^0_{i-1}[R]^{i-1}\mathbf{O}_i$$

- Taking derivatives on both sides

$$^0\mathbf{V}_{O_i} = ^0\mathbf{V}_{O_{i-1}} + ^0\boldsymbol{\omega}_{i-1} \times ^0_{i-1}[R]^{i-1}\mathbf{O}_i$$

- Simplify and rewrite above as

$$^i\mathbf{V}_i = ^i_{i-1}[R](^{i-1}\mathbf{V}_{i-1} + ^{i-1}\boldsymbol{\omega}_{i-1} \times ^{i-1}\mathbf{O}_i) \tag{14}$$

Note: $^i\mathbf{V}_i$ and $^{i-1}\mathbf{V}_{i-1}$ denote $^i_0[R]^0\mathbf{V}_i$ and $^{i-1}_0[R]^0\mathbf{V}_{i-1}$, respectively. They are **not** necessarily **0**!

- Linear velocity vector propagation in links of a serial manipulator – Rotary joint.

- For two consecutive links in a serial manipulator (see Module 2, Lecture 2)

$$^0\mathbf{O}_i = ^0\mathbf{O}_{i-1} + ^{0}_{i-1}[R]^{i-1}\mathbf{O}_i$$

- Taking derivatives on both sides

$$^0\mathbf{V}_{O_i} = ^0\mathbf{V}_{O_{i-1}} + ^0\boldsymbol{\omega}_{i-1} \times ^{0}_{i-1}[R]^{i-1}\mathbf{O}_i$$

- Simplify and rewrite above as

$$^i\mathbf{V}_i = ^{i}_{i-1}[R](^{i-1}\mathbf{V}_{i-1} + ^{i-1}\boldsymbol{\omega}_{i-1} \times ^{i-1}\mathbf{O}_i) \quad (14)$$

Note: $^i\mathbf{V}_i$ and $^{i-1}\mathbf{V}_{i-1}$ denote $^{i}_{0}[R]^0\mathbf{V}_i$ and $^{i-1}_{0}[R]^0\mathbf{V}_{i-1}$, respectively. They are **not** necessarily **0**!

- Linear velocity vector propagation in links of a serial manipulator – Rotary joint.

Linear and Angular Velocity of Links

Velocity Propagation – Prismatic Joints

- Two links connected by a prismatic (P) joint (see Module 2, Lecture 2)
- Prismatic joint *allows* relative translation between $\{1-i\}$ and $\{i\}$ $\rightarrow$ angular velocity is same
- Relative translation is along $Z-$ axis $\rightarrow$ $\dot{d}_i(0\ 0\ 1)^T$
- Velocity propagation for P joint

$$\begin{aligned}&\text{Angular velocity}\\&{}^{i}\omega_i = {}^{i}_{i-1}[R]^{i-1}\omega_{i-1} \qquad (15)\\&\text{Linear velocity}\\&{}^{i}\mathbf{V}_i = {}^{i}_{i-1}[R]({}^{i-1}\mathbf{V}_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}\mathbf{O}_i) + \dot{d}_i(0\ 0\ 1)^T \qquad (16)\end{aligned}$$

where ${}^{i}_{i-1}[R]^{i-1}\omega_i \triangleq {}^{i}\omega_i$ and ${}^{i}_{i-1}[R]^{i-1}\mathbf{V}_i \triangleq {}^{i}\mathbf{V}_i$.

Velocity Propagation – Prismatic Joints

- Two links connected by a prismatic (P) joint (see Module 2, Lecture 2)
- Prismatic joint *allows* relative translation between $\{1-i\}$ and $\{i\}$ $\rightarrow$ angular velocity is same
- Relative translation is along $Z-$ axis $\rightarrow$ $\dot{d}_i(0\ 0\ 1)^T$
- Velocity propagation for P joint

Angular velocity

$$ {}^{i}\omega_i = {}^{i}_{i-1}[R]^{i-1}\omega_{i-1} \tag{15} $$

Linear velocity

$$ {}^{i}\mathbf{V}_i = {}^{i}_{i-1}[R]({}^{i-1}\mathbf{V}_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}\mathbf{O}_i) + \dot{d}_i(0\ 0\ 1)^T \tag{16} $$

where ${}^{i}_{i-1}[R]^{i-1}\omega_i \triangleq {}^{i}\omega_i$ and ${}^{i}_{i-1}[R]^{i-1}\mathbf{V}_i \triangleq {}^{i}\mathbf{V}_i$.

- Two links connected by a prismatic (P) joint (see Module 2, Lecture 2)
- Prismatic joint *allows* relative translation between $\{1-i\}$ and $\{i\}$ $\rightarrow$ angular velocity is same
- Relative translation is along $Z-$ axis $\rightarrow$ $\dot{d}_i(0\ 0\ 1)^T$
- Velocity propagation for P joint

Angular velocity
$$^{i}\omega_i = {}^{i}_{i-1}[R]^{i-1}\omega_{i-1} \tag{15}$$
Linear velocity
$$^{i}\mathbf{V}_i = {}^{i}_{i-1}[R]({}^{i-1}\mathbf{V}_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}\mathbf{O}_i) + \dot{d}_i(0\ 0\ 1)^T \tag{16}$$

where ${}^{i}_{i-1}[R]^{i-1}\omega_i \triangleq {}^{i}\omega_i$ and ${}^{i}_{i-1}[R]^{i-1}\mathbf{V}_i \triangleq {}^{i}\mathbf{V}_i$.

Velocity Propagation – Prismatic Joints

- Two links connected by a prismatic (P) joint (see Module 2, Lecture 2)
- Prismatic joint *allows* relative translation between $\{1 - i\}$ and $\{i\} \rightarrow$ angular velocity is same
- Relative translation is along $Z-$ axis $\rightarrow \dot{d}_i (0\ 0\ 1)^T$
- Velocity propagation for P joint

$$\text{Angular velocity}$$

$$^{i}\omega_i = {}^{i}_{i-1}[R]^{i-1}\omega_{i-1} \tag{15}$$

$$\text{Linear velocity}$$

$$^{i}\mathbf{V}_i = {}^{i}_{i-1}[R]({}^{i-1}\mathbf{V}_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}\mathbf{O}_i) + \dot{d}_i(0\ 0\ 1)^T \tag{16}$$

where ${}^{i}_{i-1}[R]^{i-1}\omega_i \triangleq {}^{i}\omega_i$ and ${}^{i}_{i-1}[R]^{i-1}\mathbf{V}_i \triangleq {}^{i}\mathbf{V}_i$.

Linear and Angular Velocity of Links

Velocity Propagation – Planar 3R manipulator

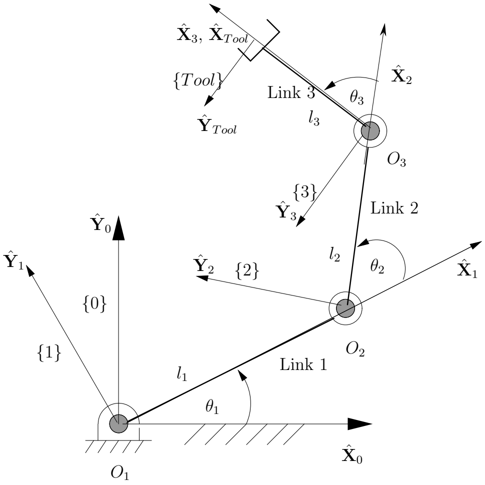

Figure 3: The planar 3R manipulator – revisited

- All joint axis are parallel and coming out of page.
- {0} is fixed →

$$
\begin{aligned}
{}^0\omega_0 &= \mathbf{0} \\
{}^0\mathbf{V}_0 &= \mathbf{0}
\end{aligned}
$$

- Links connected by rotary (R) joint → Equations (13) and (14) give velocities of all links.

Linear and Angular Velocity of Links

Velocity Propagation – Planar 3R manipulator (Contd.)

- For **i=1**

$$
\begin{aligned}
{}^1\omega_1 &= (0\ 0\ \dot{\theta}_1)^T \\
{}^1\mathbf{V}_1 &= \mathbf{0}
\end{aligned}
$$

- For **i=2**

$$
\begin{aligned}
{}^2\omega_2 &= (0\ 0\ \dot{\theta}_1 + \dot{\theta}_2)^T \\
{}^2\mathbf{V}_2 &= \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{pmatrix}
\end{aligned}
$$

- For **i=3**

$$
\begin{aligned}
{}^3\omega_3 &= (0\ 0\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^T \\
{}^3\mathbf{V}_3 &= \begin{pmatrix} (l_1 s_{23} + l_2 s_3)\dot{\theta}_1 + l_2 s_3 \dot{\theta}_2 \\ (l_1 c_{23} + l_2 c_3)\dot{\theta}_1 + l_2 c_3 \dot{\theta}_2 \\ 0 \end{pmatrix}
\end{aligned}
$$

- For **i=1**

$$
\begin{aligned}
{}^1\omega_1 &= (0 \; 0 \; \dot{\theta}_1)^T \\
{}^1\mathbf{V}_1 &= \mathbf{0}
\end{aligned}
$$

- For **i=2**

$$
\begin{aligned}
{}^2\omega_2 &= (0 \; 0 \; \dot{\theta}_1 + \dot{\theta}_2)^T \\
{}^2\mathbf{V}_2 &= \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{pmatrix}
\end{aligned}
$$

- For **i=3**

$$
\begin{aligned}
{}^3\omega_3 &= (0 \; 0 \; \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^T \\
{}^3\mathbf{V}_3 &= \begin{pmatrix} (l_1 s_{23} + l_2 s_3)\dot{\theta}_1 + l_2 s_3 \dot{\theta}_2 \\ (l_1 c_{23} + l_2 c_3)\dot{\theta}_1 + l_2 c_3 \dot{\theta}_2 \\ 0 \end{pmatrix}
\end{aligned}
$$

Linear and Angular Velocity of Links

Velocity Propagation – Planar 3R manipulator (Contd.)

- For **i=1**

$$
\begin{aligned}
{}^1\omega_1 &= (0\ 0\ \dot{\theta}_1)^T \\
{}^1\mathbf{V}_1 &= \mathbf{0}
\end{aligned}
$$

- For **i=2**

$$
\begin{aligned}
{}^2\omega_2 &= (0\ 0\ \dot{\theta}_1 + \dot{\theta}_2)^T \\
{}^2\mathbf{V}_2 &= \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{pmatrix}
\end{aligned}
$$

- For **i=3**

$$
\begin{aligned}
{}^3\omega_3 &= (0\ 0\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^T \\
{}^3\mathbf{V}_3 &= \begin{pmatrix} (l_1 s_{23} + l_2 s_3)\dot{\theta}_1 + l_2 s_3 \dot{\theta}_2 \\ (l_1 c_{23} + l_2 c_3)\dot{\theta}_1 + l_2 c_3 \dot{\theta}_2 \\ 0 \end{pmatrix}
\end{aligned}
$$

- For $\mathbf{i} = \mathbf{Tool}$

$$
\begin{aligned}
{}^{Tool}\omega_{Tool} &= (0\ 0\ \dot\theta_1 + \dot\theta_2 + \dot\theta_3)^T \\
{}^{Tool}\mathbf{V}_{Tool} &= \begin{pmatrix} (l_1 s_{23} + l_2 s_3)\dot\theta_1 + l_2 s_3 \dot\theta_2 \\ (l_1 c_{23} + l_2 c_3 + l_3)\dot\theta_1 + (l_2 c_3 + l_3)\dot\theta_2 + l_3\dot\theta_3 \\ 0 \end{pmatrix}
\end{aligned}
$$

- Linear and angular velocity in $\{0\}$

$$ {}^0\omega_{Tool} = (0\ 0\ \dot\theta_1 + \dot\theta_2 + \dot\theta_3)^T \tag{17} $$

and

$$ {}^0\mathbf{V}_{Tool} = \begin{pmatrix} -l_1 s_1 \dot\theta_1 - l_2 s_{12}(\dot\theta_1 + \dot\theta_2) - l_3 s_{123}(\dot\theta_1 + \dot\theta_2 + \dot\theta_3) \\ l_1 c_1 \dot\theta_1 + l_2 c_{12}(\dot\theta_1 + \dot\theta_2) + l_3 c_{123}(\dot\theta_1 + \dot\theta_2 + \dot\theta_3) \\ 0 \end{pmatrix} \tag{18} $$

LINEAR AND ANGULAR VELOCITY OF LINKS

VELOCITY PROPAGATION – PLANAR 3R MANIPULATOR (CONTD.)

- For **i = Tool**

$$
\begin{aligned}
{}^{Tool}\omega_{Tool} &= (0 \;\; 0 \;\; \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^T \\
{}^{Tool}\mathbf{V}_{Tool} &= \begin{pmatrix} (l_1 s_{23} + l_2 s_3)\dot{\theta}_1 + l_2 s_3 \dot{\theta}_2 \\ (l_1 c_{23} + l_2 c_3 + l_3)\dot{\theta}_1 + (l_2 c_3 + l_3)\dot{\theta}_2 + l_3 \dot{\theta}_3 \\ 0 \end{pmatrix}
\end{aligned}
$$

- Linear and angular velocity in $\{0\}$

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$$

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$$
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$$

OUTLINE

- Linear and angular velocity of $\{Tool\}$ (Equations (17) and (18)) can be written in a compact form as

$$
{}^{0}\mathcal{V}_{Tool} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ -- & -- & -- \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} \tag{19}
$$

- ${}^{0}\mathcal{V}_{Tool}$ is a 6×1 entity – ${}^{0}\mathcal{V}_{Tool} \triangleq \begin{pmatrix} {}^{0}\mathbf{V}_{Tool} \\ -- \\ {}^{0}\omega_{Tool} \end{pmatrix}$

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- ${}^0\mathscr{V}_{Tool}$ *is not* a 6×1 vector[1] – contains linear velocity and the angular velocity which have different units!
- Use '–' or ';' to separate the linear and angular velocities & to remind that ${}^0\mathscr{V}_{Tool}$ or $({}^0\mathbf{V}_{Tool};{}^0\omega_{Tool})^T$ is not a vector.
- Matrix in square brackets, ${}^0_{Tool}[J(\Theta)]$, is called the *Jacobian* matrix for the planar 3R manipulator.
    - ${}^0_{Tool}[J(\Theta)]$ relate the linear and angular velocities of the tool with the joint velocities.
    - Jacobian matrix is for the end-effector or the $\{Tool\}$ – see subscript $Tool$.
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[1] In theoretical kinematics, $({}^0\omega_{Tool};{}^0\mathbf{V}_{Tool})$ is called *twist* (see Additional Material in Module 2).

Serial Manipulator Jacobian Matrix

Introduction

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- ${}^{0}_{Tool}[J(\Theta)]$ is not a *proper* matrix.
  - The first and the last three rows represent linear and angular velocity,
  - Elements of the first three rows have units of length,
  - Elements of last three rows have *no* units.
- Similar to ${}^{0}\mathscr{V}_{Tool}$, top and bottom halves of a Jacobian matrix are separated by '–'.
- Many matrix operations makes no sense – the condition number[2] of this matrix changes with the choice of length units.
- ${}^{0}_{Tool}[J(\Theta)]$ is best thought of as a map ${}^{0}_{Tool}[J(\Theta)] : \dot{\Theta} \rightarrow {}^{0}\mathscr{V}_{Tool}$

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Serial Manipulator Jacobian Matrix

Properties of Jacobian matrix

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Serial Manipulator Jacobian Matrix

Properties of Jacobian matrix (contd.)

- The Jacobian matrix can be derived for any serial manipulator with rotary and prismatic joints.
  - Compute the linear and angular velocities using propagation equations
  - Rearrange in a matrix equation as done for the planar 3R manipulator.
- Jacobian can be defined for any differentiable vector function.
- Direct kinematics equations – differentiable vector function $\mathscr{X} = \Psi(\Theta)$
  - $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$ denotes the n joint variables
  - Position and orientation of end-effector are denoted by $\mathscr{X}$[3].
- $[J(\Theta)]$ is the matrix of first partial derivatives of Ψ with respect to θ_i – i^{th} column of $[J(\Theta)]$ is the partial derivatives of Ψ with respect to θ_i.

$$[J(\Theta)] = \left[\frac{\partial \Psi}{\partial \theta_1} \; \frac{\partial \Psi}{\partial \theta_2} \; \ldots \; \frac{\partial \Psi}{\partial \theta_n} \right]$$

[3] For example, $\mathscr{X}$ denotes the three Cartesian position variables $(x,\ y,\ z)$ and the three Euler angles $(\alpha,\ \beta, \gamma)$.

PROPERTIES OF JACOBIAN MATRIX (CONTD.)

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Serial Manipulator Jacobian Matrix

Properties of Jacobian matrix (contd.)

- ${}^{0}_{Tool}[J(\Theta)]$ very important in velocity kinematics of serial manipulators.
- The elements of the Jacobian matrix are non-linear functions of the joint variables Θ.
  - Manipulator in motion $\rightarrow {}^{0}_{Tool}[J(\Theta)]$ is time varying.
  - At instant with Θ known, ${}^{0}_{Tool}[J(\Theta)]$ relates linear and angular velocities to joint rates.
  - The relationship is linear!
- The Jacobian matrix can be obtained for *any* link – most often for end-effector.
- The Jacobian matrix is always with respect to a coordinate system – where the linear and angular velocities are obtained.
- Most often Jacobian matrix is with respect to fixed $\{0\}$.
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SERIAL MANIPULATOR JACOBIAN MATRIX

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$$\dot{\Theta} = {}^0_{Tool}[J(\Theta)]^{-1}\ {}^0\mathcal{V}_{Tool} \tag{20}$$

- Above relationship gives joint velocities required for a desired linear and angular velocities of $\{Tool\}$.
- Direct velocity kinematics – ${}^0\mathcal{V}_{Tool} = {}^0_{Tool}[J(\Theta)]\dot{\Theta}$
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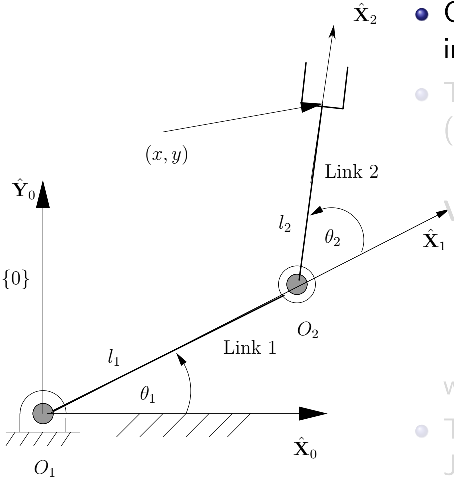


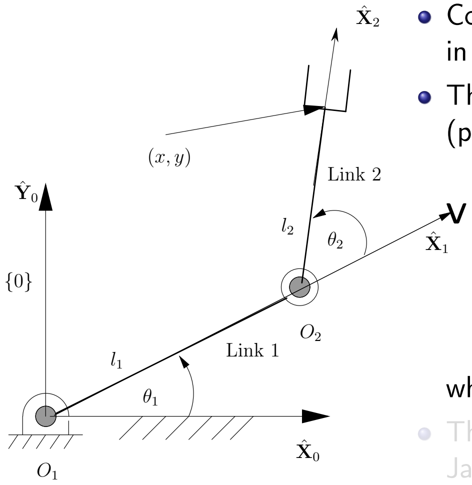
Figure 4: A planar 2R manipulator

- Consider a planar 2R manipulator shown in in figure 4.
- The linear velocity **V** of the end-effector (point (x, y)) is

$$\mathbf{V} \triangleq \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

where $\dot{\theta}_1$, $\dot{\theta}_2$ are the two joint rates.

- The matrix inside square brackets is the Jacobian matrix in $\{0\}$.



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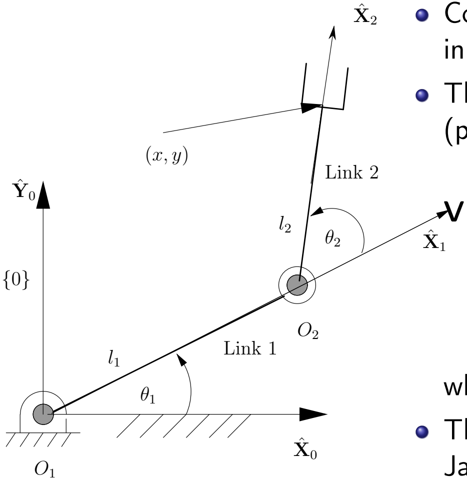

Figure 4: A planar 2R manipulator

- Consider a planar 2R manipulator shown in in figure 4.
- The linear velocity $\mathbf{V}$ of the end-effector (point (x, y)) is

$$\triangleq \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

where $\dot{\theta}_1$, $\dot{\theta}_2$ are the two joint rates.

- The matrix inside square brackets is the Jacobian matrix in $\{0\}$.

- Magnitude of linear velocity vector

$$\mathbf{V}^2 \triangleq \mathbf{V}\cdot\mathbf{V} = g_{11}\dot{\theta}_1^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2^2 \quad (21)$$

- g_{ij}, $i,j = 1,2$, are the elements of a matrix $[\ g\ ] = [J(\Theta)]^T[J(\Theta)]$.
- For the planar 2R manipulator the g_{ij}'s are

$$\begin{aligned} g_{11} &= l_1^2 + l_2^2 + 2l_1 l_2 c_2 \\ g_{12} = g_{21} &= l_2^2 + l_1 l_2 c_2 \\ g_{22} &= l_2^2 \end{aligned} \quad (22)$$

- The elements g_{ij}'s are functions of θ_2 alone and g_{22} is a constant.
- g_{ij}'s could in general be function of all joint variables.

- Magnitude of linear velocity vector

$$\mathbf{V}^2 \stackrel{\Delta}{=} \mathbf{V}\cdot\mathbf{V} = g_{11}\dot{\theta}_1^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2^2 \tag{21}$$

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- Maximum and minimum $\mathbf{V}^2$ subject to constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$[5]
- Solve $\partial {\mathbf{V}^*}^2 / \partial \dot{\theta}_i = 0$, $i = 1, 2$, where

$$ {\mathbf{V}^*}^2 = g_{11}\dot{\theta}_1^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2^2 - \lambda(\dot{\theta}_1^1 + \dot{\theta}_2^2 - 1) $$

- Partial differentiation reduces to an eigenvalue problem

$$ [\ g\ ]\dot{\Theta} - \lambda\dot{\Theta} = 0 \qquad (23) $$

- The eigenvalues are

$$ \lambda_{1,2} = (1/2)\{(g_{11} + g_{22}) \pm [(g_{11} + g_{22})^2 - 4(g_{11}g_{22} - g_{12}^2)]^{1/2}\} $$

[5] Without any constraint $\mathbf{V} \in \Re^2$ and fills up $\Re^2$. The constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ is similar to the *unit speed* constraint in differential geometry of space curves.

Serial Manipulator Jacobian Matrix

Geometric Interpretation of Jacobian matrix

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- [g] real, symmetric and positive definite → eigenvalues are always real and positive.
- For $\lambda_1 > \lambda_2$,

$$|\mathbf{V}|_{\max} = \sqrt{\lambda_1}, \quad |\mathbf{V}|_{\min} = \sqrt{\lambda_2}$$

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- For a planar 2R manipulator, $([J][\ g\ ]^{-1})([J][\ g\ ]^{-1})^T$ is symmetric and of rank 2.
- Hence for $\dot{\Theta}^T \dot{\Theta} = 1$, $(\dot{x},\dot{y})^T([J][\ g\ ]^{-1})([J][\ g\ ]^{-1})^T(\dot{x},\dot{y}) = 1$.
- $\mathbf{x}^T[A]\mathbf{x} = 1$, with $[A]$ symmetric and non-singular, describes an ellipse.
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Serial Manipulator Jacobian Matrix

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- Eigenvalues of [g] are only functions of θ_2 $\rightarrow$ shape and size of ellipse will change with θ_2.
- Can plot ellipses at all points in the workspace
- Recall: workspace of a planar 2R is the area between two circles of radii $l_1 + l_2$ and $l_1 - l_2$.
- Ellipse independent of θ_1 $\rightarrow$ All ellipses at a chosen radius (in the annular region) are same!

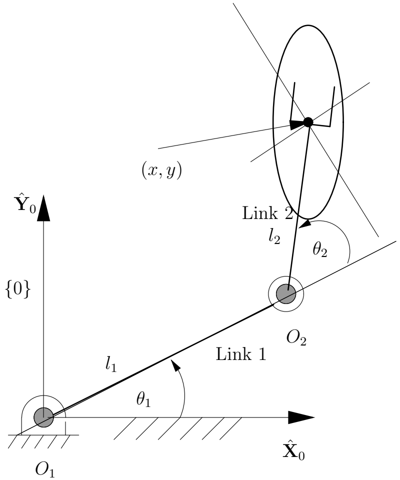

Figure 5: Velocity ellipse for a planar 2R manipulator

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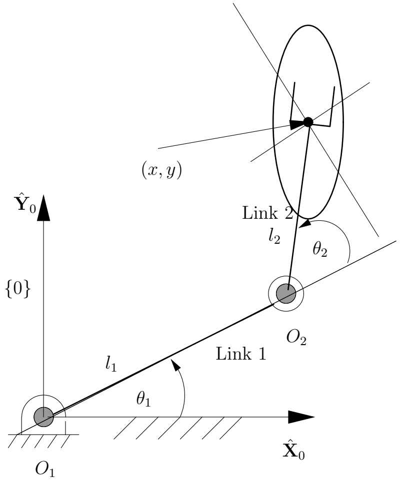

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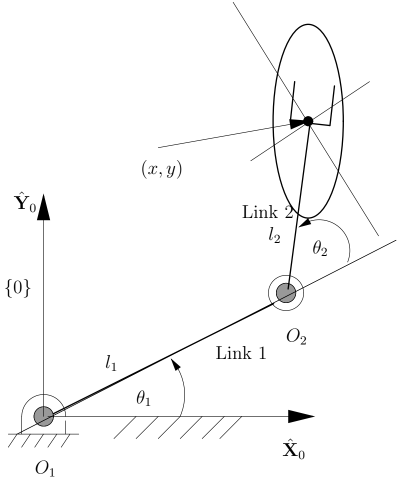

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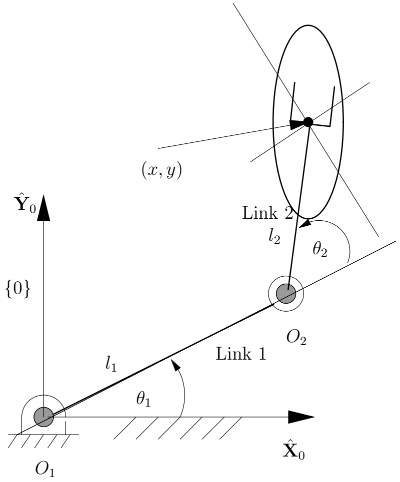

Figure 5: Velocity ellipse for a planar 2R manipulator

- The shape of the velocity ellipse indicates which directions are 'easier' to move for given joint rates
- $|\mathbf{V}|$ is larger along major axis $\rightarrow$ Easier to move along major axis.
- Less easier to move along the minor axis.
- Ellipse reduces to a circle $\rightarrow$ equally easy to move in all directions.
- All points in the workspace, where the ellipse is a circle, are called *isotropic* (see Salisbury 1982)

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- Isotropic configuration – eigenvalues of $[J(\Theta)]$ (or $[\ g\ ]$) are equal.
- For planar 2R, eigenvalues equal only if

$$g_{11} = g_{22} \quad \text{and} \quad g_{12} = 0$$

- From the expressions of g_{ij}'s above conditions imply that

$$l_1^2 + 2l_1 l_2 c_2 = 0 \quad \text{and} \quad l_2^2 + l_1 l_2 c_2 = 0$$

and this is only possible if

$$l_1 = \sqrt{2} l_2 \quad \text{and} \quad c_2 = -\frac{1}{\sqrt{2}}$$

- A planar 2R manipulator can posses isotropic configurations *only* if the link lengths have a ratio of $\sqrt{2}$, and $\theta_2 = 135°$.
- Since $\theta_1 \in [0, 2\pi]$, all the isotropic configurations lie on a circle.
- Degenerate form of velocity ellipse $\rightarrow$ singular configuration (see Lecture 4).

SERIAL MANIPULATOR JACOBIAN MATRIX

GEOMETRIC INTERPRETATION OF JACOBIAN MATRIX (CONTD.)

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Serial Manipulator Jacobian Matrix

Geometric Interpretation of Jacobian matrix (contd.)

- Isotropic configuration – eigenvalues of $[J(\Theta)]$ (or $[\ g\ ]$) are equal.
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- Redundant systems (see Module 3, Lecture 3) Jacobian matrix is *not square* $\rightarrow$ number of joint variables more than 6 (for $\Re^3$) or more than 3 (for $\Re^2$).
- Jacobian matrix cannot be inverted to obtain joint rates given linear and angular velocity of end-effector.
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Serial Manipulator Jacobian Matrix

Resolution of redundancy at velocity level

- Some properties of pseudo-inverse
  - Dimension of $[J(\Theta)]^{\#}$ is $n \times m$ – not square!
  - Left inverse $\rightarrow$ $[J(\Theta)][J(\Theta)]^{\#} = [U]$ – $[U]$ is an identity matrix.
  - Not a right inverse $[J(\Theta)]^{\#}[J(\Theta)] \neq [U]$
  - General solution to ${}^0\mathscr{V}_{Tool} = {}^0_{Tool}\, [J(\Theta)]\dot{\Theta}$ is

$$\dot{\Theta} = [J(\Theta)]^{\#\,0}\mathscr{V}_{Tool} + ([U] - [J(\Theta)]^{\#}[J(\Theta)])\dot{\mathscr{W}}$$

  - $([U] - [J(\Theta)]^{\#}[J(\Theta)])\dot{\mathscr{W}}$ lies in the null-space of $[J(\Theta)]$.
- The pseudo-inverse solution, without the null-space term, *minimises* $\dot{\Theta}^T\dot{\Theta}$.
- The null-space term used to avoid obstacles, joint limits and to maximise manipulability index $\det([[J(\Theta)][J(\Theta)]^T]^{1/2})$(see Nakamura 1991).
- Disadvantages
  - Local numerical scheme – cannot obtain global or analytical results.
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OUTLINE

- Parallel manipulator – actuated *and* passive joints – $\mathbf{q} = (\theta, \phi)^T$
- Loop-closure constraint equations do not contain all joint variables.
- No natural choice of end-effector $\{Tool\}$ $\rightarrow$ no velocity propagation
- Platform type parallel manipulator – position of centroid & orientation of platform $\{Tool\}$ is of interest.
- Linear and angular velocity of centroid and $\{Tool\}$

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- For m passive variables, m constraint equations $\eta_i(q_1, ..., q_{n+m}) = 0, i = 1, ..., m$ or in a vector form

$$\eta(\mathbf{q}) = \eta(\theta, \phi) = \mathbf{0} \tag{25}$$

- Differentiate equation (25) with respect to t, and rearrange

$$[K(\mathbf{q})]\dot{\theta} + [K^*(\mathbf{q})]\dot{\phi} = \mathbf{0} \tag{26}$$

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PARALLEL MANIPULATOR JACOBIAN MATRIX

ELIMINATION OF PASSIVE JOINT RATES (CONTD.)

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$$\begin{aligned} ^0\omega_{Tool} &= ([J_\omega] - [J^*_\omega][K^*]^{-1}[K])\dot{\theta} \\ ^0\mathbf{V}_{Tool} &= ([J_V] - [J^*_V][K^*]^{-1}[K])\dot{\theta} \end{aligned}$$

- Define equivalent $[J_\omega]_{eq}$ and $[J_\mathbf{V}]_{eq}$

$$[J_\mathbf{V}]_{eq} \stackrel{\Delta}{=} [J_\mathbf{V}] - [J^*_\mathbf{V}][K^*]^{-1}[K] \tag{28}$$

$$[J_\omega]_{eq} \stackrel{\Delta}{=} [J_\omega] - [J^*_\omega][K^*]^{-1}[K] \tag{29}$$

- If $\det([K^*]) \neq 0$,

$$\dot{\phi} = -[K^*]^{-1}[K]\dot{\theta} \tag{27}$$

- The angular and linear velocity can partitioned as

$$^0\omega_{Tool} = [J_\omega]\dot{\theta} + [J^*_\omega]\dot{\phi}, \quad {}^0\mathbf{V}_{Tool} = [J_\mathbf{V}]\dot{\theta} + [J^*_\mathbf{V}]\dot{\phi}$$

- Substitute $\dot{\phi}$ to get

$$\begin{aligned} {}^0\omega_{Tool} &= ([J_\omega] - [J^*_\omega][K^*]^{-1}[K])\dot{\theta} \\ {}^0\mathbf{V}_{Tool} &= ([J_V] - [J^*_V][K^*]^{-1}[K])\dot{\theta} \end{aligned}$$

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- Using $[J_{\mathbf{V}}]_{eq}$ and $[J_{\omega}]_{eq}$

$$^0\mathscr{V}_{Tool} \triangleq \begin{pmatrix} ^0\mathbf{V}_{Tool} \\ -- \\ ^0\omega_{Tool} \end{pmatrix} = {}^0_{Tool}[J_{eq}]\dot{\theta} \qquad (30)$$

- The $6 \times n$ matrix, ${}^0_{Tool}[J_{eq}]$, consists of $3 \times n$ rows from $[J_{\mathbf{V}}]_{eq}$ and $3 \times n$ rows from $[J_{\omega}]_{eq}$.
- The matrix ${}^0_{Tool}[J_{eq}]$ is the Jacobian matrix for parallel manipulators.
- At a known $\mathbf{q}$, equation (30) relate actuated joint rates $\dot{\theta}$ to the linear and angular velocity of chosen end-effector $\{Tool\}$.

- Using $[J_{\mathbf{v}}]_{eq}$ and $[J_{\omega}]_{eq}$

$$ {}^0\mathcal{V}_{Tool} \triangleq \begin{pmatrix} {}^0\mathbf{V}_{Tool} \\ -- \\ {}^0\omega_{Tool} \end{pmatrix} = {}^0_{Tool}[J_{eq}]\dot{\theta} \tag{30} $$

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$$[\ g_{\mathbf{V}}\ ]_{eq} = ([J_{\mathbf{V}}] - [J_{\mathbf{V}}{}^{*}][K^{*}]^{-1}[K])^{T}([J_{\mathbf{V}}] - [J_{\mathbf{V}}{}^{*}][K^{*}]^{-1}[K]) \qquad (31)$$

- $[\ g_{\mathbf{V}}\ ]_{eq}$ is symmetric and positive definite.
- Similar to a serial manipulator, the tip of the linear velocity vector lies on an ellipse or an ellipsoid.
- Much more complicated than in serial manipulators!
- $[\ g_{\omega}\ ]_{eq}$ defined using $[J_{\omega}]_{eq} \rightarrow$ angular velocity ellipse or ellipsoid.
- The above geometrical description is valid if $\det[K^{*}] \neq 0$.

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- Constraint equation of a four-bar (see Module 4, Lecture 1)

$$\eta_1(\mathbf{q}) \triangleq l_1 \cos\theta_1 + l_2\cos(\theta_1+\phi_2) - l_0 - l_3\cos\phi_1 = 0$$
$$\eta_2(\mathbf{q}) \triangleq l_1 \sin\theta_1 + l_2\sin(\theta_1+\phi_2) - l_3\sin\phi_1 = 0$$

- θ_1 is the actuated joint variable and (ϕ_1, ϕ_2) are the passive joint variables.
- Derivative of constraint equations with respect to time t gives

$$\begin{pmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\phi_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\phi_2) \end{pmatrix}\dot{\theta}_1 + \begin{pmatrix} l_3\sin\phi_1 & -l_2\sin(\theta_1+\phi_2) \\ -l_3\cos\phi_1 & l_2\cos(\theta_1+\phi_2) \end{pmatrix}\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = 0$$

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- $[K]$ and $[K^*]$ matrices for the planar 4-bar are

$$[K] = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \phi_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \phi_2) \end{pmatrix}$$

$$[K^*] = \begin{bmatrix} l_3 \sin\phi_1 & -l_2 \sin(\theta_1 + \phi_2) \\ -l_3 \cos\phi_1 & l_2 \cos(\theta_1 + \phi_2) \end{bmatrix}$$

- The matrix $[K^*]$ is a square 2×2 matrix.
- $[K]$ and $[K^*]$ matrices are functions of the actuated and passive variables.
- Fairly simple to calculate for planar 4-bar.
- Multi- degree-of-freedom spatial mechanisms $\rightarrow$ use *symbolic algebra* software such as MAPLE.

PARALLEL MANIPULATOR JACOBIAN MATRIX

EXAMPLE – PLANAR 4-BAR MECHANISM

- $[K]$ and $[K^*]$ matrices for the planar 4-bar are

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Parallel Manipulator Jacobian Matrix

Example – planar 4-bar mechanism

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- For the 3-RPS manipulator, loop-closure equations are

$$
\begin{aligned}
\eta_1(\mathbf{q}) = 3 - 3a^2 + l_1^2 + l_2^2 + l_1 l_2 c_1 c_2 - 2 l_1 l_2 s_1 s_2 - 3 l_1 c_1 - 3 l_2 c_2 &= 0 \\
\eta_2(\mathbf{q}) = 3 - 3a^2 + l_2^2 + l_3^2 + l_2 l_3 c_2 c_3 - 2 l_2 l_3 s_2 s_3 - 3 l_2 c_2 - 3 l_3 c_3 &= 0 \\
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\end{aligned}
$$

- Perform the derivative of $\eta_i(\mathbf{q})$, $i = 1, 2, 3$, with respect to time and rearrange to obtain $[K]$ and $[K^*]$
- $[K]$ involves derivative with respect to the actuated variables $\mathbf{l}$

$$
\begin{bmatrix}
2l_1 - 3c_1 + l_2 c_1 c_2 - 2 l_2 s_1 s_2 & 2l_2 - 3c_2 + l_1 c_1 c_2 - 2 l_1 s_1 s_2 & 0 \\
0 & 2l_2 - 3c_2 + l_3 c_2 c_3 - 2 l_3 s_2 s_3 & 2l_3 - 3c_3 + l_2 c_2 c_3 - 2 l_2 s_2 s_3 \\
2l_1 - 3c_1 + l_3 c_1 c_3 - 2 l_3 s_1 s_3 & 0 & 2l_3 - 3c_3 + l_1 c_1 c_3 - 2 l_1 s_1 s_3
\end{bmatrix}
$$

Parallel Manipulator Jacobian Matrix

Example – 3-RPS parallel manipulator

- For the 3-RPS manipulator, loop-closure equations are

$$
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0 & 2l_2 - 3c_2 + l_3 c_2 c_3 - 2 l_3 s_2 s_3 & 2l_3 - 3c_3 + l_2 c_2 c_3 - 2 l_2 s_2 s_3 \\
2l_1 - 3c_1 + l_3 c_1 c_3 - 2 l_3 s_1 s_3 & 0 & 2l_3 - 3c_3 + l_1 c_1 c_3 - 2 l_1 s_1 s_3
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- Perform the derivative of $\eta_i(\mathbf{q}),\ i = 1,2,3$, with respect to time and rearrange to obtain $[K]$ and $[K^*]$
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\begin{bmatrix}
2l_1 - 3c_1 + l_2 c_1 c_2 - 2 l_2 s_1 s_2 & 2l_2 - 3c_2 + l_1 c_1 c_2 - 2 l_1 s_1 s_2 & 0\\
0 & 2l_2 - 3c_2 + l_3 c_2 c_3 - 2 l_3 s_2 s_3 & 2l_3 - 3c_3 + l_2 c_2 c_3 - 2 l_2 s_2 s_3\\
2l_1 - 3c_1 + l_3 c_1 c_3 - 2 l_3 s_1 s_3 & 0 & 2l_3 - 3c_3 + l_1 c_1 c_3 - 2 l_1 s_1 s_3
\end{bmatrix}
$$

Parallel Manipulator Jacobian Matrix

Example – 3-RPS parallel manipulator (contd.)

- $[K^*]$ involves derivative with respect to passive joint variables, θ

$$\begin{bmatrix} 3l_1s_1 - l_1l_2s_1c_2 - 2l_1l_2c_1s_2 & 3l_2s_2 - l_1l_2c_1s_2 - 2l_1l_2s_1c_2 & 0 \\ 0 & 3l_2s_2 - l_2l_3s_2c_3 - 2l_2l_3c_2s_3 & 3l_3s_3 - l_2l_3c_2s_3 - 2l_2l_3s_2c_3 \\ 3l_1s_1 - l_1l_3s_1c_3 - 2l_1l_3c_1s_3 & 0 & 3l_3s_3 - l_1l_3c_1s_3 - 2l_1l_3s_1c_3 \end{bmatrix}$$

- For the centroid, $[J_V]$ and $[J_V^*]$, are

$$[J_V] = (1/3)\begin{bmatrix} -c_1 & (1/2)c_2 & (1/2)c_3 \\ 0 & (-\sqrt{3}/2)c_2 & (\sqrt{3}/2)c_3 \\ s_1 & s_2 & s_3 \end{bmatrix}$$

and

$$[J_V^*] = (1/3)\begin{bmatrix} l_1s_1 & -(1/2)l_2s_2 & (-1/2)l_3s_3 \\ 0 & (\sqrt{3}/2)l_2s_2 & (-\sqrt{3}/2)l_3s_3 \\ l_1c_1 & l_2c_2 & l_3c_3 \end{bmatrix}$$

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$$\begin{bmatrix} 3l_1s_1 - l_1l_2s_1c_2 - 2l_1l_2c_1s_2 & 3l_2s_2 - l_1l_2c_1s_2 - 2l_1l_2s_1c_2 & 0 \\ 0 & 3l_2s_2 - l_2l_3s_2c_3 - 2l_2l_3c_2s_3 & 3l_3s_3 - l_2l_3c_2s_3 - 2l_2l_3s_2c_3 \\ 3l_1s_1 - l_1l_3s_1c_3 - 2l_1l_3c_1s_3 & 0 & 3l_3s_3 - l_1l_3c_1s_3 - 2l_1l_3s_1c_3 \end{bmatrix}$$

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and

$$[J_{\mathbf{V}}{}^*] = (1/3)\begin{bmatrix} l_1s_1 & -(1/2)l_2s_2 & (-1/2)l_3s_3 \\ 0 & (\sqrt{3}/2)l_2s_2 & (-\sqrt{3}/2)l_3s_3 \\ l_1c_1 & l_2c_2 & l_3c_3 \end{bmatrix}$$

- To obtain $[J\omega]$ and $[J\omega^*]$, compute $\frac{d}{dt}({}^{Base}_{Top}[R])\ {}^{Base}_{Top}[R]^T$ and then rearrange.
- Expressions are too large – for $l_1 = 2/3$, $l_2 = 3/5$, $l_3 = 3/4$ and corresponding $\theta_1 = 0.7593$, $\theta_2 = 0.2851$, $\theta_3 = 0.8028$ radians,

$$[J\omega] = \begin{pmatrix} -1.4147 & 1.3103 & 1.6929 \\ 0.2092 & 0.7130 & 0.8537 \\ -1.3628 & -0.1286 & 0.6743 \end{pmatrix}, \quad [J\omega^*] = \begin{pmatrix} 0.6682 & 0.8774 & -0.3316 \\ -1.6372 & 0.4175 & 0.1672 \\ 1.4206 & 0.0226 & 0.1321 \end{pmatrix}$$

- Expressions for $[Jv]_{eq}$ and $[J\omega]_{eq}$ are more harder to obtain – $[K^*]^{-1}$ is needed. For above numerical values

$$[Jv]_{eq} = \begin{pmatrix} -0.2313 & 0.5372 & 0.0114 \\ 0.0722 & -0.6758 & 0.1951 \\ 1.1765 & -1.6830 & 0.9223 \end{pmatrix}, \quad [J\omega]_{eq} = \begin{pmatrix} 2.1409 & -6.4331 & 0.4665 \\ 0.0072 & -4.1216 & 1.6048 \\ 0.1565 & 0.4570 & -0.3285 \end{pmatrix}$$

Example – 3-RPS parallel manipulator (Contd.)

- To obtain $[J\omega]$ and $[J\omega^*]$, compute $\frac{d}{dt}(^{Base}_{Top}[R])\ ^{Base}_{Top}[R]^T$ and then rearrange.
- Expressions are too large – for $l_1 = 2/3,\ l_2 = 3/5,\ l_3 = 3/4$ and corresponding $\theta_1 = 0.7593,\ \theta_2 = 0.2851,\ \theta_3 = 0.8028$ radians,

$$[J\omega] = \begin{pmatrix} -1.4147 & 1.3103 & 1.6929 \\ 0.2092 & 0.7130 & 0.8537 \\ -1.3628 & -0.1286 & 0.6743 \end{pmatrix}, \quad [J\omega^*] = \begin{pmatrix} 0.6682 & 0.8774 & -0.3316 \\ -1.6372 & 0.4175 & 0.1672 \\ 1.4206 & 0.0226 & 0.1321 \end{pmatrix}$$

- Expressions for $[J_V]_{eq}$ and $[J\omega]_{eq}$ are more harder to obtain – $[K^*]^{-1}$ is needed. For above numerical values

$$[J_V]_{eq} = \begin{pmatrix} -0.2313 & 0.5372 & 0.0114 \\ 0.0722 & -0.6758 & 0.1951 \\ 1.1765 & -1.6830 & 0.9223 \end{pmatrix}, \quad [J\omega]_{eq} = \begin{pmatrix} 2.1409 & -6.4331 & 0.4665 \\ 0.0072 & -4.1216 & 1.6048 \\ 0.1565 & 0.4570 & -0.3285 \end{pmatrix}$$

Parallel Manipulator Jacobian Matrix

Example – 3-RPS parallel manipulator (contd.)

- To obtain $[J\omega]$ and $[J\omega^*]$, compute $\frac{d}{dt}({}^{Base}_{Top}[R])\ {}^{Base}_{Top}[R]^T$ and then rearrange.
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- Expressions for $[J\mathbf{v}]_{eq}$ and $[J\omega]_{eq}$ are more harder to obtain – $[K^*]^{-1}$ is needed. For above numerical values

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- For $a = 1/2$, and $(l_1, l_2, l_3) = (0.5, 1.0, 2.0)$ meters $\rightarrow$ $(\theta_1, \theta_2, \theta_3) = (0.4, 0.7535, 0.2402)$ radians by direct kinematics (see Module 4, Lecture 2).
- Tip of linear velocity vector of centroid lies on an ellipsoid – shown in figure 6 as three sectional views and a 3D plot.
- Maximum, intermediate, and minimum velocities along the principal axes of the ellipsoid are $0.3724, 0.3162, 0.2031$ $\mathrm{m/sec}$, respectively.
- The directions of principal axes are $(0.9921, -0.0394, 0.1187)^T$, $(0.1166, 0.6338, -0.7646)^T$ and $(-0.0452, 0.7724, 0.6335)^T$, respectively.

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PARALLEL MANIPULATOR JACOBIAN MATRIX

EXAMPLE – 3-RPS PARALLEL MANIPULATOR (CONTD.)

- For $a = 1/2$, and $(l_1, l_2, l_3) = (0.5, 1.0, 2.0)$ meters $\rightarrow$ $(\theta_1, \theta_2, \theta_3) = (0.4, 0.7535, 0.2402)$ radians by direct kinematics (see Module 4, Lecture 2).
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Parallel Manipulator Jacobian Matrix

Example – 3-RPS parallel manipulator (contd.)

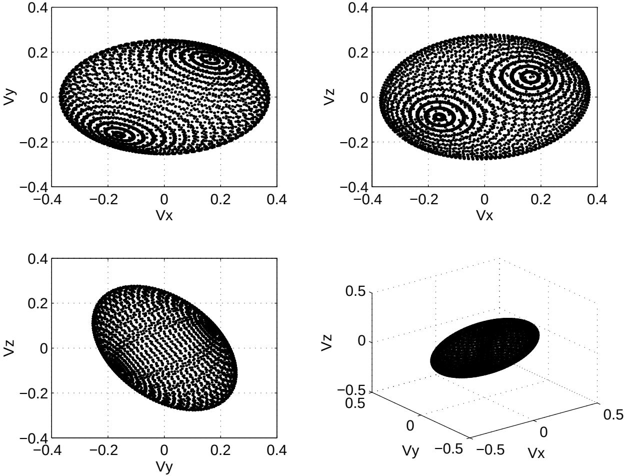

Figure 6: Velocity ellipsoid at a non-singular point

OUTLINE

Singularities in Serial and Parallel Manipulators

Serial manipulators – review

- Direct velocity kinematics

$$^0\mathscr{V}_{Tool} = {}^0_{Tool}[J(\Theta)]\dot{\Theta}$$

  - For known Θ and $\dot{\Theta}$, linear and angular velocity of end-effector obtained from above equation.
  - $^0\mathscr{V}_{Tool}$ always exists
- Inverse velocity kinematics

$$\dot{\Theta} = {}^0_{Tool}[J(\Theta)]^{-1}\ {}^0\mathscr{V}_{Tool}$$

  - Joint rates can be obtained when Jacobian matrix is square, **and**
  - $\det({}^0_{Tool}[J(\Theta)]) = 0$
- $\det({}^0_{Tool}[J(\Theta)]) = 0 \rightarrow$ Loss of rank of ${}^0_{Tool}[J(\Theta)] \rightarrow$ *Singular configuration.*
- At singular configuration, $\dot{\Theta}$ *cannot* be obtained for given linear and angular velocity.

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Singularities in Serial and Parallel Manipulators

Serial manipulators – singularity in planar 2R manipulator

- For a planar 2R manipulator

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

- The Jacobian matrix is

$$^{0}_{Tool}[J(\Theta)] = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

- $\det(^{0}_{Tool}[J(\Theta)]) = 0 \rightarrow \sin\theta_2 = 0.$
- This implies $\theta_2 = 0, \pi \rightarrow$ the second link is stretched completely or folded on top of first link.

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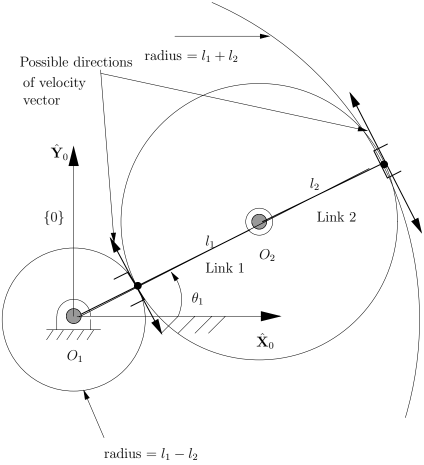

Figure 7: Singular configurations for a planar 2R manipulator

- Planar 2R manipulator for $\theta_2 = 0, \pi$.
- End-effector can *only* move perpendicular to the line $O_1 - O_2$ connecting the two rotary joints.
- The end-effector *cannot* have a velocity component *along* the second link.
- *Instantaneous loss* (only at this configuration) of one degree of freedom.

Singularities in Serial and Parallel Manipulators

Serial manipulators – singularity in planar 2R manipulator

- At $\theta_2 = 0, \pi$, the velocity ellipse degenerates to a line – along the possible direction of motion as shown in figure 7.
- For the 2R planar example, the Jacobian matrix can be inverted easily, and

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \frac{1}{l_1 l_2 s_2} \begin{pmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

- $\sin\theta_2$ is in the denominator and as $\theta_2 \rightarrow 0$ or π, $(\dot{\theta}_1, \dot{\theta}_2)^T \rightarrow \infty$.
- Knowledge of singularity is important – when $\det({}^0_{Tool}[J(\Theta)])$ is *close to zero*, joint velocities tend to become large and cause problems for the servo controller of the robot.
- Singularities occurs in all serial manipulator and *not only in planar 2R*
- Planar 2R singularity *only* at workspace boundaries – in other manipulators can happen elsewhere also!

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Singularities in Serial and Parallel Manipulators

Parallel manipulators – review

- In parallel manipulators, *equivalent Jacobian* need to be used.
- For parallel manipulators, the linear and angular velocity Jacobians are

$$[J_{\mathbf{v}}]_{eq} \stackrel{\Delta}{=} [J_{\mathbf{v}}] - [J^*_{\mathbf{v}}][K^*]^{-1}[K]$$

$$[J_{\omega}]_{eq} \stackrel{\Delta}{=} [J_{\omega}] - [J^*_{\omega}][K^*]^{-1}[K]$$

- Equivalent Jacobian

$${}^{0}_{Tool}[J]_{eq} = \begin{pmatrix} [J_{\mathbf{v}}]_{eq} \\ -- \\ [J_{\omega}]_{eq} \end{pmatrix}$$

- $\det({}^{0}_{Tool}[J_{eq}]) = 0 \rightarrow$ End-effector *looses* one or more degrees of freedom $\rightarrow$ actuated joint rates $\dot{\Theta} \rightarrow \infty$.
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- In addition to *loss* singularity, second kind of singularity in parallel manipulators.
- $\dot{\theta} = \mathbf{0} \rightarrow$ Actuated joints locked $\rightarrow$ mechanism *becomes a structure*.
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$$[K^*(\mathbf{q})]\dot{\phi} = \mathbf{0}$$

- From linear algebra, above equation can have a *non-zero* solution $\dot{\phi}^*$ when $\det([K^*]) = 0$.
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Singularities in Serial and Parallel Manipulators

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PARALLEL MANIPULATORS – GAIN SINGULARITY (CONTD.)

- Geometric picture: Non-singular configuration.
  - At non-singular configurations, $\dot{\theta} = \mathbf{0} \rightarrow \dot{\phi} = \mathbf{0} \rightarrow {}^0\mathbf{V}_{Tool} = \mathbf{0}$.
  - At a *non-singular position* velocity ellipsoid is of *zero* size.
- Geometric picture: Gain singularity configuration.
  - Loss of rank of $[K^*]$.
  - If rank is $(m-1) \rightarrow$ there exists non zero eigenvector $\dot{\phi}_1$ for the zero eigenvalue of $[K^*]$
  - $C_1\dot{\phi}$ also an eigenvector with C_1 a scaling constant.
  - For $\dot{\theta} = 0$

$${}^0\mathbf{V}_{Tool} = C_1[J_{\mathbf{V}}{}^*]\dot{\phi}_1$$

- There can be motion along $[J_{\mathbf{V}}{}^*]\dot{\phi}_1$!
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Singularities in Serial and Parallel Manipulators

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Singularities in Serial and Parallel Manipulators

Parallel manipulators – gain singularity

- Gain singularity occurs in all parallel and hybrid manipulators.
- In fully-parallel six- degree-of-freedom manipulator (end-effector directly connected to base by *one* actuated joint – Stewart-Gough platform) only *gain* singularity possible (Hunt, 1991).
- In six- degree-of-freedom hybrid parallel manipulator (example six- degree-of-freedom three-fingered hand, see Module 2, Lecture 3 and Module 4, Lecture 2) both loss and gain singularity possible.
- Gain singularity is related to capability of resisting external force or moments (see Lecture 5).
- Large amount of literature on singularity analysis of parallel manipulators (see, for example, Hunt 1986, Litvin et. al 1990, Merlet 1991, Gosselin and Angeles 1990, Zlatanov 1995, Park and Kim 1999).
- Singularity analysis & uses of singularity analysis is an active topic of research (see Bandyopadhyay and Ghosal 2009, Ranganath et al. 2004)

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Singularities in Serial and Parallel Manipulators

- Special link lengths and geometry $\rightarrow$ *gain* over *finite* range of motion.
- Passive link can show instantaneous and finite *dwell*.

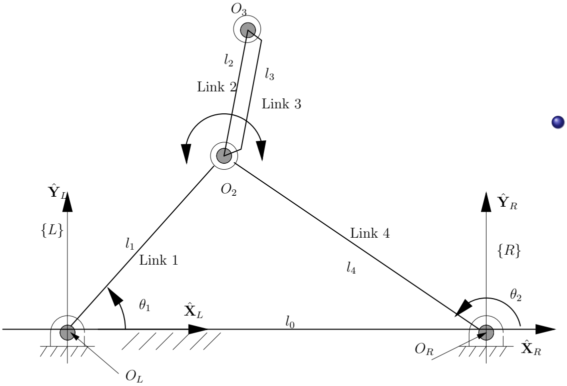

Figure 8: Finite motion at gain singularity

- Link 2 and Link 3 can rotate from 0 to 2π with θ_1 and θ_2 locked (see Bandyopadhyay and Ghosal (2004a) for details).

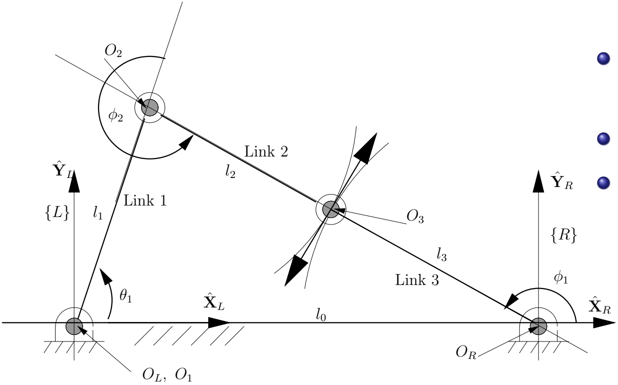

Figure 9: Singular configuration for a planar four-bar mechanism

- $\det([K^*]) = 0$ gives $l_2 l_3 \sin(\theta_1 + \phi_2 - \phi_1) = 0$
- $\Rightarrow \theta_1 + \phi_2 - \phi_1 = n\pi \rightarrow \phi_3 = 2\pi$
- Link 2 and 3 are parallel.
- Instantaneous gain: θ_1 locked, point O_2 is fixed $\rightarrow$ Link 2 and Link 3 along a straight line $\rightarrow$ O_3 can have instantaneous velocity along the common tangent.

Example – The 3-RPS parallel manipulator (see Basu and Ghosal, 1997

- For the 3-RPS parallel manipulator,
  - $\det([J_{\mathbf{V}}]_{eq}) = 0 \rightarrow$ linear velocity ellipsoid described by the centroid of the top platform degenerates to an ellipse[7]
  - For $(l_1, l_2, l_3) = (0.5, 1.0, 1.9710)$ meters and $(\theta_1, \theta_2, \theta_3) = (1.1691, 0.4781, 0.2355)$ radians $\rightarrow \det([J_{\mathbf{V}}]_{eq}) = 0$.
- The linear velocity ellipse at this configuration is shown in sectional and a 3D view in figure 10.
- Not a contradiction to result by Hunt (1991)
  - The 3-RPS parallel manipulator is not six- degree-of-freedom manipulator,
  - Only the linear velocity vector of the centroid is considered.

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Singularities in Serial and Parallel Manipulators

Singularities in parallel manipulators – Examples

Example – The 3-RPS parallel manipulator

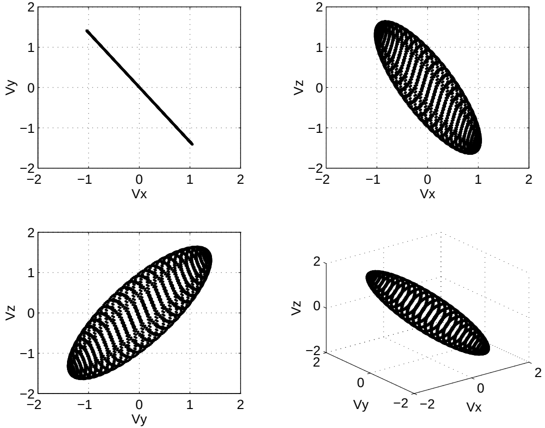

Figure 10: Linear velocity ellipse at a loss singular point

Example – 3-RPS parallel manipulator

- *Gain* one or more degrees-of-freedom when $\det([K^*]) = 0$ i.e.,

$$\det([K^*]) =$$
$$(3l_1s_1 - l_1l_2s_1c_2 - 2l_1l_2c_1s_2) \times (3l_2s_2 - l_2l_3s_2c_3 - 2l_2l_3c_2s_3) \times$$
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$$(3l_3s_3 - l_2l_3c_2s_3 - 2l_2l_3s_2c_3) = 0$$

- $\det([K^*]) = 0$ is a function of all (θ, ϕ)
- $\det([K^*]) = 0$ and three loop-closure equations $\rightarrow$ four equations in six variables $\rightarrow$ a 2D surface.
- Difficult to eliminate (see Module 3, Lecture 4) and get analytical expression.

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SINGULARITIES IN PARALLEL MANIPULATORS – EXAMPLES

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Singularities in parallel manipulators – Examples

Example – 3-RPS parallel manipulator

- For $(l_1, l_2, l_3) = (0.575, 0.483, 0.544)$, and $(\theta_1, \theta_2, \theta_3) = (-0.3441, -0.0138, 0.2320)$ radians, $\det[K^*] \approx 0$.
- The eigenvalues of $[K^*]$ are approximately -0.5565, 0 and 0.4509.
- The three corresponding eigenvectors are $(-0.8098, 0.3571, -0.4656)^T$, $(-0.3109, -0.8743, -0.3727)^T$ and $(-0.0877, -0.4781, -0.8739)^T$.
- Gained velocity of centroid is

$$
{}^0\mathbf{V}_{Tool} = \begin{pmatrix} -0.0647 \\ 0 \\ 0.1804 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} 0.0011 \\ -0.0019 \\ 0.1610 \end{pmatrix} \dot{\theta}_2 + \begin{pmatrix} -0.0208 \\ -0.0361 \\ 0.1763 \end{pmatrix} \dot{\theta}_3
$$

where $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T = \alpha \times (-0.3109, -0.8743, -0.3727)^T$ with α arbitrary.

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{}^0\mathbf{V}_{Tool} = \begin{pmatrix} -0.0647 \\ 0 \\ 0.1804 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} 0.0011 \\ -0.0019 \\ 0.1610 \end{pmatrix} \dot{\theta}_2 + \begin{pmatrix} -0.0208 \\ -0.0361 \\ 0.1763 \end{pmatrix} \dot{\theta}_3
$$

where $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T = \alpha \times (-0.3109, -0.8743, -0.3727)^T$ with α arbitrary.

Singularities in parallel manipulators – Examples

Example – 3-RPS parallel manipulator

- For $(l_1, l_2, l_3) = (0.575, 0.483, 0.544)$, and $(\theta_1, \theta_2, \theta_3) = (-0.3441, -0.0138, 0.2320)$ radians, $\det[K^*] \approx 0$.
- The eigenvalues of $[K^*]$ are approximately -0.5565, 0 and 0.4509.
- The three corresponding eigenvectors are $(-0.8098, 0.3571, -0.4656)^T$, $(-0.3109, -0.8743, -0.3727)^T$ and $(-0.0877, -0.4781, -0.8739)^T$.
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Singularities in Serial and Parallel Manipulators

Singularities in parallel manipulators – Examples

Example – 3-RPS parallel manipulator

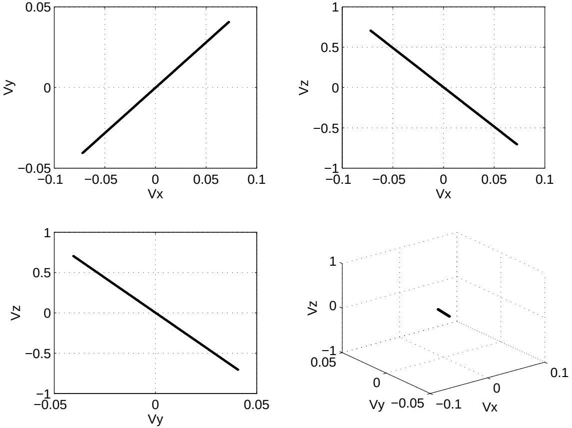

Figure 11: Velocity at a gain singular point

Example – 3-RPS parallel manipulator

- At $(l_1, l_2, l_3) = (1.9363, 2.9998, 1.9363)$ meters
- Corresponding $(\theta_1, \theta_2, \theta_3) = (1.3096, 0.9817, 1.3096)$ radians,
- $\det[K^*] \approx 0 \rightarrow$ eigenvalues are approximately 0, 0, 3.9680.
- At this configuration, gain of *two* degrees of freedom.
- The singularities corresponding to gain of two degrees of freedom lie on a curve in $\Re^3 \rightarrow$ difficult to get analytical expression.

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Singularities in parallel manipulators – Examples

Example – 3-RPS parallel manipulator

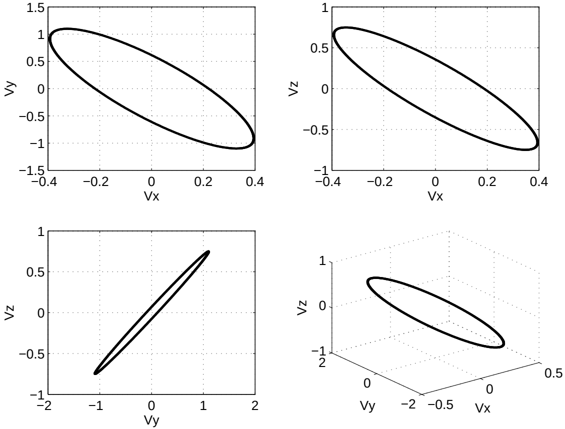

Figure 12: Velocity ellipse at a gain singular point

Outline

- Joints of a serial manipulator are locked $\rightarrow$ the manipulator becomes a structure.
- Forces and moments acting at joints when manipulator structure is subjected to external forces and moments.
- External forces and moments on end-effector if pushing some object or carrying a payload.
- Useful to know joint forces or torques which can maintain the static equilibrium.
- Use free-body diagram.

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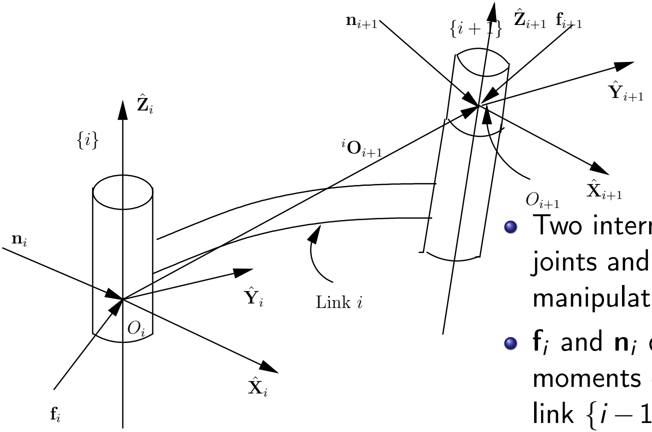

Figure 13: Free-body diagram of a link

- Two intermediate rotary (R) joints and a link of a manipulator.
- $\mathbf{f}_i$ and $\mathbf{n}_i$ denote the forces and moments exerted on link $\{i\}$ by link $\{i-1\}$.

- For static equilibrium of $\{i\}$, $\Sigma \mathbf{F} = \mathbf{0}$

$$ {}^{i}\mathbf{f}_i - {}^{i}\mathbf{f}_{i+1} = 0 $$

  - $\mathbf{f}_{i+1}$ is the *force on link* $\{i+1\}$ *exerted by link* $\{i\}$ $\rightarrow$ Force on link $\{i\}$ exerted by link $\{i+1\}$ will be equal and of opposite sign.
  - The leading superscript i signifies that the vectors are described in $\{i\}$.

- For static equilibrium of $\{i\}$, $\Sigma \mathbf{M} = \mathbf{0}$

$$ {}^{i}\mathbf{n}_i - {}^{i}\mathbf{n}_{i+1} - {}^{i}\mathbf{O}_{i+1} \times {}^{i}\mathbf{f}_{i+1} = 0 $$

  - ${}^{i}\mathbf{O}_{i+1}$ is the vector from O_i to O_{i+1}.
  - Negative sign due to same reason as for forces.

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  - $^{i}\mathbf{O}_{i+1}$ is the vector from O_i to O_{i+1}.
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- Equilibrium equations can be written as

$$
\begin{aligned}
{}^{i}\mathbf{f}_i &= {}^{i}_{i+1}[R]\ {}^{i+1}\mathbf{f}_{i+1} \\
{}^{i}\mathbf{n}_i &= {}^{i}_{i+1}[R]\ {}^{i+1}\mathbf{n}_{i+1} + {}^{i}\mathbf{O}_{i+1} \times {}^{i}\mathbf{f}_i
\end{aligned}
\qquad (32)
$$

- *Inward Recursion* for forces and moments on each link.
  - Forces and moments at the end-effector: ${}^{n+1}\mathbf{f}_{n+1} = {}^{n+1}\mathbf{n}_{n+1} = 0$ if not in contact with environment.
  - ${}^{n+1}\mathbf{f}_{n+1}$, ${}^{n+1}\mathbf{n}_{n+1}$ known otherwise.
  - Recursively compute ${}^{i}\mathbf{f}_i$, ${}^{i}\mathbf{n}_i$ for $i : n \rightarrow 1$ using equation (32).
- Joint can only apply force or moment along $\hat{Z}-$ axis – all other components resisted by structure/bearings.
- Torque required at joint i to maintain equilibrium

$$
\begin{aligned}
\tau_i &= {}^{i}\mathbf{n}_i \cdot {}^{i}\hat{\mathbf{Z}}_i \quad \text{(joint } i \text{ is rotary (R))} \\
\tau_i &= {}^{i}\mathbf{f}_i \cdot {}^{i}\hat{\mathbf{Z}}_i \quad \text{(joint } i \text{ is prismatic (P))}
\end{aligned}
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\end{aligned}
\tag{33}
$$

- 3R planar manipulator applying force

$$
\begin{aligned}
{}^0\mathbf{f}_{Tool} &= (f_x,\ f_y,\ 0)^T \\
{}^0\mathbf{n}_{Tool} &= (0,\ 0,\ n_z)^T
\end{aligned}
$$

- In $\{Tool\}$ coordinate system

$$
\begin{pmatrix} f'_x \\ f'_y \\ 0 \end{pmatrix} = \begin{bmatrix} c_{123} & s_{123} & 0 \\ -s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} f_x \\ f_y \\ 0 \end{pmatrix}
$$

and $(0,\ 0,\ n'_z)^T = (0,\ 0,\ n_z)^T$

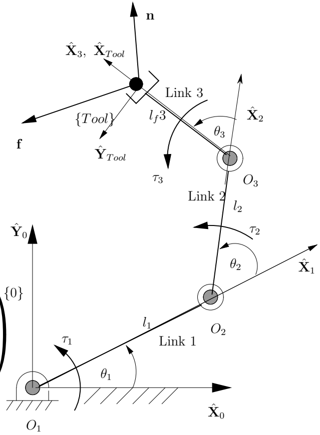

Figure 14: A 3R manipulator applying force and moment

i=3

$$
\begin{aligned}
{}^{3}\mathbf{f}_3 &= (f'_x,\ f'_y,\ 0)^T \\
{}^{3}\mathbf{n}_3 &= (0,\ 0,\ n'_z + l_3 f'_y)^T
\end{aligned}
$$

i=2

$$
\begin{aligned}
{}^{2}\mathbf{f}_2 &= (c_3 f'_x - s_3 f'_y,\ s_3 f'_x + c_3 f'_y,\ 0)^T \\
{}^{2}\mathbf{n}_2 &= (0,\ 0,\ n'_z + l_2(s_3 f'_x + c_3 f'_y) + l_3 f'_y)^T
\end{aligned}
$$

i=1

$$
\begin{aligned}
{}^{1}\mathbf{f}_1 &= (c_{23} f'_x - s_{23} f'_y,\ s_{23} f'_x + c_{23} f'_y,\ 0)^T \\
{}^{1}\mathbf{n}_1 &= (0,\ 0,\ n'_z + l_1(s_{23} f'_x + c_{23} f'_y) + l_2(s_3 f'_x + c_3 f'_y) + l_3 f'_y)^T
\end{aligned}
$$

- Finally, the joint torques required to maintain equilibrium

$$
\begin{aligned}
\tau_1 = {}^1\mathbf{n}_1 \cdot {}^1\hat{\mathbf{Z}}_1 &= n'_z + f'_x(l_1 s_{23} + l_2 s_3) + f'_y(l_1 c_{23} + l_2 c_3 + l_3) \\
\tau_2 = {}^2\mathbf{n}_2 \cdot {}^2\hat{\mathbf{Z}}_2 &= n'_z + f'_x l_2 s_3 + f'_y(l_2 c_3 + l_3) \\
\tau_3 = {}^3\mathbf{n}_3 \cdot {}^3\hat{\mathbf{Z}}_3 &= n'_z + f'_y l_3
\end{aligned}
$$

- Above equations can be re-arranged as

$$
\tau = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & l_1 c_1 + l_2 c_{12} + l_3 c_{123} & 0 & 0 & 0 & 1 \\ -l_2 s_{12} - l_3 s_{123} & l_2 c_{12} + l_3 c_{123} & 0 & 0 & 0 & 1 \\ -l_2 s_{123} & l_3 c_{123} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} f_x \\ f_y \\ 0 \\ 0 \\ 0 \\ n_z \end{pmatrix} \tag{34}
$$

- Term in the square bracket is the transpose of the Jacobian matrix (see equation (19)).
- As in velocities, denote forces and moments acting on the end-effector by

$$
{}^0\mathscr{F}_{Tool} \triangleq \begin{pmatrix} {}^0\mathbf{f}_{Tool} \\ -- \\ {}^0\mathbf{n}_{Tool} \end{pmatrix} = (f_x \; f_y \; f_z ; \; n_x \; n_y \; n_z)^T \qquad (35)
$$

- Note: ${}^0\mathscr{F}_{Tool}$ is not a 6×1 vector since forces and moments have different units.
- ${}^0\mathscr{F}_{Tool}$ is called a *wrench* in theoretical kinematics, and a wrench can be thought of as *screw* with a magnitude which has units of force.

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- Consider an infinitesimal Cartesian displacement of the end effector ${}^0\delta\mathscr{X}_{Tool}$[8] and the *virtual work* done by ${}^0\mathscr{F}_{Tool}$
- Equating virtual work done by external force/moment and at joints

$$ {}^0\mathscr{F}_{Tool} \cdot {}^0\delta\mathscr{X}_{Tool} \stackrel{\triangle}{=} {}^0\mathbf{f}_{Tool} \cdot \delta\mathbf{x} + {}^0\mathbf{n}_{Tool} \cdot \delta\theta = \tau \cdot \delta\Theta $$

- Using the definition of Jacobian (${}^0\delta\mathscr{X}_{Tool} = {}^0_{Tool}[J(\Theta)]\delta\Theta$),

$$ {}^0\mathscr{F}_{Tool} \cdot {}^0_{Tool}[J(\Theta)]\delta\Theta = \tau \cdot \delta\Theta $$

- The above equations hold true for all $\delta\Theta$, hence

$$ \tau = {}^0_{Tool}[J(\Theta)]^T \, {}^0\mathscr{F}_{Tool} \qquad (36) $$

- Not surprising transpose of Jacobian appears in statics!!

[8] The quantity ${}^0\delta\mathscr{X}_{Tool}$ is *not* a 6×1 vector and it is like a *twist*. The infinitesimal change in position and orientation could be $(\delta\mathbf{x};\delta\theta)^T$.

- Consider an infinitesimal Cartesian displacement of the end effector ${}^0\delta\mathscr{X}_{Tool}$[8] and the *virtual work* done by ${}^0\mathscr{F}_{Tool}$
- Equating virtual work done by external force/moment and at joints

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- The above equations hold true for all $\delta\Theta$, hence

$$\tau = {}^{0}_{Tool}[J(\Theta)]^{T}\, {}^{0}\mathscr{F}_{Tool} \qquad (36)$$

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- Consider an infinitesimal Cartesian displacement of the end effector ${}^0\delta\mathscr{X}_{Tool}$[8] and the *virtual work* done by ${}^0\mathscr{F}_{Tool}$
- Equating virtual work done by external force/moment and at joints

$$ {}^0\mathscr{F}_{Tool} \cdot {}^0\delta\mathscr{X}_{Tool} \stackrel{\Delta}{=} {}^0\mathbf{f}_{Tool} \cdot \delta\mathbf{x} + {}^0\mathbf{n}_{Tool} \cdot \delta\theta = \tau \cdot \delta\Theta $$

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- Consider an infinitesimal Cartesian displacement of the end effector ${}^0\delta\mathscr{X}_{Tool}$[8] and the *virtual work* done by ${}^0\mathscr{F}_{Tool}$
- Equating virtual work done by external force/moment and at joints

$$ {}^0\mathscr{F}_{Tool} \cdot {}^0\delta\mathscr{X}_{Tool} \stackrel{\Delta}{=} {}^0\mathbf{f}_{Tool} \cdot \delta\mathbf{x} + {}^0\mathbf{n}_{Tool} \cdot \delta\theta = \tau \cdot \delta\Theta $$

- Using the definition of Jacobian (${}^0\delta\mathscr{X}_{Tool} = {}^0_{Tool}[J(\Theta)]\delta\Theta$),

$$ {}^0\mathscr{F}_{Tool} \cdot {}^0_{Tool}[J(\Theta)]\delta\Theta = \tau \cdot \delta\Theta $$

- The above equations hold true for all $\delta\Theta$, hence

$$ \tau = {}^0_{Tool}[J(\Theta)]^T \, {}^0\mathscr{F}_{Tool} \tag{36} $$

- Not surprising transpose of Jacobian appears in statics!!

[8] The quantity ${}^0\delta\mathscr{X}_{Tool}$ is *not* a 6×1 vector and it is like a *twist*. The infinitesimal change in position and orientation could be $(\delta\mathbf{x}; \delta\theta)^T$.

- For serial manipulators $\tau = {}^0_{Tool}[J(\Theta)]^T \, {}^0\mathscr{F}_{Tool}$
- Principle of virtual work equally applicable – for parallel manipulator

$$\tau = {}^0_{Tool}[J_{eq}]^T \, {}^0\mathscr{F}_{Tool}$$

  - $\{Tool\}$ is a *chosen* end-effector.
  - ${}^0_{Tool}[J_{eq}]$ is the equivalent Jacobian – function of $\mathbf{q}$, and
  - τ is the vector of forces or torques applied at the *actuated joints only*.
- Difficult to compute ${}^0_{Tool}[J_{eq}]$ since computation of $[K^*]^{-1}$ (see Lecture 3).
- Inverse problem: Obtaining forces/moments applied by $\{Tool\}$

$${}^0\mathscr{F}_{Tool} = {}^0_{Tool}[J(\mathbf{q})_{eq}]^{-T} \tau$$

- Inverse of Jacobian even more difficult!! Simpler approach for Gough-Stewart platform.

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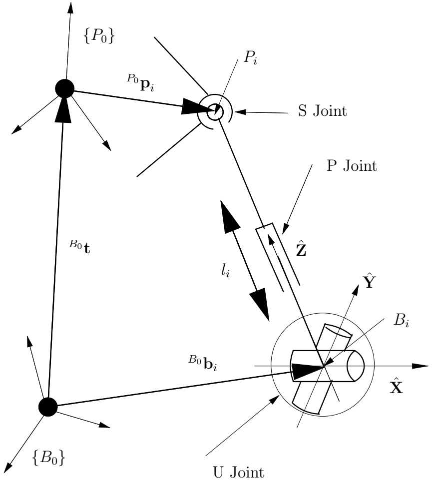

Figure 15: A leg of a Stewart-Gough platform – revisited

- The vector along the leg, ${}^{B_0}\mathbf{S}_i$,

$$ {}^{B_0}\mathbf{S}_i = {}^{B_0}_{P_0}[R]\,{}^{P_0}\mathbf{p}_i + {}^{B_0}\mathbf{t} - {}^{B_0}\mathbf{b}_i $$

- Unit vector along leg ${}^{B_0}\mathbf{s}_i = \frac{{}^{B_0}\mathbf{S}_i}{l_i}$

- The force exerted by actuated prismatic joint is $f_i^{B_0}\mathbf{s}_i$.
- The moment of the force (about the origin B_0) $f_i({}^{B_0}\mathbf{b}_i \times {}^{B_0}\mathbf{s}_i)$.
- Denoting external force and moment by ${}^0\mathscr{F}_{Tool}$,

$$
{}^{B_0}\mathscr{F}_{Tool} \triangleq \begin{pmatrix} {}^{B_0}\mathbf{F}_{Tool} \\ --- \\ {}^{B_0}\mathbf{M}_{Tool} \end{pmatrix} = \begin{bmatrix} \sum_{i=1}^{6} {}^{B_0}\mathbf{s}_i f_i \\ --- \\ \sum_{i=1}^{6} ({}^{B_0}\mathbf{b}_i \times {}^{B_0}\mathbf{s}_i) f_i \end{bmatrix}
$$

  where ${}^{B_0}\mathbf{F}_{Tool}$, ${}^{B_0}\mathbf{M}_{Tool}$ are the 3×1 force and moment vectors acting on $\{Tool\}$.
- In matrix form ${}^{B_0}\mathscr{F}_{Tool} = {}^{B_0}_{Tool}[\ H\ ]\mathbf{f}$

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- The force transformation matrix ${}^{B_0}_{Tool}[\, H \,]$

$$
{}^{B_0}_{Tool}[\, H \,] = \begin{bmatrix} {}^{B_0}\mathbf{s}_1 & {}^{B_0}\mathbf{s}_2 & \dots & {}^{B_0}\mathbf{s}_6 \\ --- & --- & --- & --- \\ ({}^{B_0}\mathbf{b}_1 \times {}^{B_0}\mathbf{s}_1) & ({}^{B_0}\mathbf{b}_2 \times {}^{B_0}\mathbf{s}_2) & \dots & ({}^{B_0}\mathbf{b}_6 \times {}^{B_0}\mathbf{s}_6) \end{bmatrix} \tag{37}
$$

- $\mathbf{f} = (f_1, f_2, ..., f_6)^T$ is the vector of forces applied at the prismatic joints.
- Like the manipulator Jacobian matrix, ${}^{B_0}_{Tool}[\, H \,]$ is not a *proper* matrix in the linear algebra sense.
- ${}^{B_0}_{Tool}[\, H \,] = {}^{0}_{Tool}[J(\mathbf{q})_{eq}]^{-T}$ but ${}^{0}_{Tool}[\, H \,]$ is much simpler to compute!
- Easily extended to any fully parallel manipulator – there are n columns.

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- Direct force analysis: Obtain external force/moment given leg forces

$$^{B_0}\mathscr{F}_{Tool} = {}^{B_0}_{Tool}[\ H\ ]\mathbf{f}$$

- Inverse force analysis: Obtain leg forces given external force/moment

$$\mathbf{f} = {}^{B_0}_{Tool}[\ H\ ]^{-1}\,{}^{B_0}\mathscr{F}_{Tool}$$

- If $\det([\ H\ ]) = 0$, then inverse problem cannot be solved $\rightarrow$ Force singularity.
- As $\det([\ H\ ]) \rightarrow 0$, $\mathbf{f} \rightarrow \infty$ – any external force/moment along certain direction *cannot* be resisted by the parallel manipulator[9].

[9] In velocity singularity, no joint rates can cause motion along certain (singular) directions.

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[9] In velocity singularity, no joint rates can cause motion along certain (singular) directions.

- Force singularity can be visualised using degeneracy of force ellipsoid.
- In a Stewart-Gough platform, external force $\mathbf{F}$ is given by

$$\mathbf{F} = [H_{\mathbf{F}}]\mathbf{f} = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 & \mathbf{s}_4 & \mathbf{s}_5 & \mathbf{s}_6 \end{bmatrix} \mathbf{f}$$

- $|\mathbf{F}|^2$ is $\mathbf{F}^T\mathbf{F} = \mathbf{f}^T[g_{\mathbf{F}}]\mathbf{f}$ where $[g_{\mathbf{F}}] = [H_{\mathbf{F}}]^T[H_{\mathbf{F}}]$.
- The maximum, intermediate and minimum values of $\mathbf{F}^T\mathbf{F}$ subject to a constraint of the form $\mathbf{f}^T\mathbf{f} = 1$ are the eigenvalues of $[g_{\mathbf{F}}]$.
- Since the rank of $[g_{\mathbf{F}}]$ is 3, the tip of the force vector $\mathbf{F}$ lies on an ellipsoid in $\Re^3$.

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- rank($[g_F]$) $= 2 \rightarrow$ force ellipsoid shrinks to an ellipse and the Stewart platform manipulator cannot apply a force normal to the plane of the ellipse.
- rank($[g_F]$) $= 1, 0$, the Stewart platform cannot apply any force in a plane or cannot apply any external force, respectively.
- Example:
  - Stewart-Gough platform with fixed base and moving platform as regular hexagons of same size.
  - Consider the configuration of all legs parallel to the vertical.
  - $[\ H\ ]$ matrix is given by

$$[H] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ b_{1_y} & b_{2_y} & b_{3_y} & b_{4_y} & b_{5_y} & b_{6_y} \\ -b_{1_x} & -b_{2_x} & -b_{3_x} & -b_{4_x} & -b_{5_x} & -b_{6_x} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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- Three rows of ([H]) are zero and the Stewart-Gough platform is in a singular configuration $\rightarrow$ $[g_{\mathbf{F}}]$ has rank 1.
- Tip of the force vector **F** can only lie along a line and only the vertical external force can be resisted.
- The Stewart-Gough platform, in this configuration, has singularity along F_x and F_y.
- Similar analysis can be done for **M** $\rightarrow$ Any M_z cannot be resisted in this configuration.
- Above analysis used by Ranganath et al. 2004 to design sensitive 6 components force-torque sensors (see Module 10, Lecture 2).

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- Above analysis used by Ranganath et al. 2004 to design sensitive 6 components force-torque sensors (see Module 10, Lecture 2).

- Three rows of ([H]) are zero and the Stewart-Gough platform is in a singular configuration $\rightarrow$ $[g_{\mathbf{F}}]$ has rank 1.
- Tip of the force vector **F** can only lie along a line and only the vertical external force can be resisted.
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- If $\det([\ K^*\ ]) = 0 \rightarrow$ parallel manipulator gains one (or more) degree of freedom instantaneously.
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- Relationship between two?
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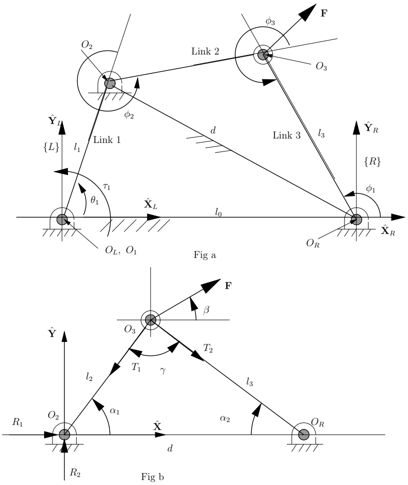

Figure 16: Static force analysis in a four-bar mechanism

- With θ_1 locked $\rightarrow$ point O_2 is fixed.
- For a given θ_1, l_0 and l_1, the length d opposite to θ_1 is known.
- Draw the planar truss structure determined by Link 2, Link 3 and the now fixed side $O_2 - O_R$. Angles α_1 and α_2 can be computed in terms of θ_1, ϕ_1 and ϕ_2.
- Consider a force $\mathbf{F} = F_x, F_y$, acting at an angle β at point O_3.
- Denote axial forces along the links $O_2 - O_3$ and $O_3 - O_R$ of the planar truss by T_1 and T_2, respectively.

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- By using a *free-body* diagram

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{bmatrix} \cos\alpha_1 & -\cos\alpha_2 \\ \sin\alpha_1 & \sin\alpha_2 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

- T_1 and T_2 are

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \frac{1}{\sin(\alpha_1+\alpha_2)} \begin{bmatrix} \sin\alpha_2 & \cos\alpha_2 \\ -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

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- Torque required in joint 1, τ_1, to keep the four-bar mechanism in equilibrium $\tau_1 = R_1 l_1 s_1 - R_2 l_1 c_1$
- $\det([\ H\ ]) = 0$ if $\sin(\alpha_1 + \alpha_2) = 0$ or γ is π radians $\rightarrow$ ϕ_3 2π $\rightarrow$ Link 2 and Link 3 are aligned.
- Gain singularity condition *same* as force singularity condition!
- Gain singularity $\rightarrow$ Instantaneous velocity perpendicular to line along Link 2 and Link 3 (singular direction).
- Force singularity $\rightarrow$ Any force along singular direction gives rise to infinite R_1 and R_2 and infinite τ_1 is required to maintain static equilibrium.

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OUTLINE

Module 5 – Additional Material

- Exercise Problems
- References & Suggested Reading