Dynamically isotropic Gough-Stewart platform design using a pair of triangles

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Abstract. An ideal dynamically isotropic Gough Stewart platform (GSP) has its first six modes of vibration as same, and this enables one to use effectively designed identical dampers to attenuate vibration from a source to a sensitive payload. A modified GSP (MGSP) capable of being dynamically isotropic is considered in this work. For a dynamically isotropic MGSP, the use of a force transformation matrix leads to a set of coupled transcendental equations in terms of design variables. It was observed that all the design variables for an MGSP were related by pairs of triangles. This work develops a general analytical closed-form solution for a dynamically isotropic MGSP using geometrical relations between these triangle pairs. The presented closed-form algebraic solutions can be used directly and supersede the need for any other complex algorithms. Additionally, the design variable in their explicit form offers straightforward solutions, flexible design and can ensure mechanical feasibility. The designs obtained in this work were validated by numerical simulations results done in ANSYS.

Keywords: Gough-Stewart platform · Dynamic isotropy · Natural frequency · Modified Gough-Stewart platform

1 Introduction

A Gough Stewart Platform (GSP) based isolator is proposed in the literature for micro-vibration control in spacecraft [1, 2]. A dynamically isotropic GSP is of particular interest to this application as it has equal first six natural frequencies [2, 3, 4]. It is easy to tune a damper for passive vibration control, given all resonance peaks lying close to each other (ideally the same). This is also an effective vibration isolation condition as the region of isolation associated with any degree of freedom (DOF) is not affected by resonance peaks of the cross DOFs in the amplitude versus frequency curve [2]. Moreover, the use of a dynamically isotropic configuration converts a multi-input-multi-output (MIMO) system into several single-input-single-output (SISO) systems, and we can make use of a decoupled control strategy simplifying active vibration control [5]. In a non-isotropic design, the coupling among the six DOFs complicates the controller design leading to a reduction in control accuracy [5]. Apart from the

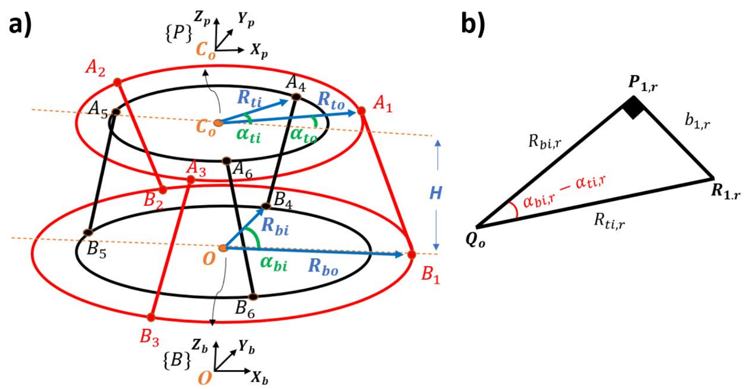

Fig. 1. a) Modified GSP **b)** Right angle triangle case in previous work [2].

vibration isolation standpoint, a dynamically isotropic configuration is singularity free. Dynamic isotropy also implies the maximization of the lowest natural frequency [3], which is a favorable criterion for stability.

A modified GSP/ two-radii GSP (MGSP) was studied for dynamic isotropy [2, 4, 5, 6, 7] as a recourse to a traditional 6-6 GSP, which cannot be dynamically isotropic [3, 4, 7]. In such MGSPs, the anchor points are described on two radii instead of one radius in a traditional 6-6 GSP (refer to Fig. 1(a)), and this can be shown to achieve dynamic isotropy [2, 4, 5, 6, 7]. Afzali et al. [6] and others [4, 5, 7] presented different approaches to developing an analytical solution for an MGSP. In all these cases, arriving at a design is challenging as the presented solutions are in implicit form or coupled. Moreover, such approaches lacked addressing design flexibility and feasibility to satisfy space constraints or the intersection of legs in the 3D space.

In our previous work, we presented closed-form solutions in their explicit form by adopting a partial geometry-based approach [2]. A set of solutions was obtained by using geometrical relation in a right-angled triangle, resulting in valuable observations for a dynamically isotropic MGSP. This paper is based on the development of a general analytical closed-form solution using a purely geometric approach. This work concludes that the 3-D dynamically isotropic MGSP design problem can be simplified into sets of triangles in 2-D space related by certain geometrical relationships. All the design parameters of an MGSP can be deduced from these triangles, and this leads to an algebraic solution that can be used directly, superseding the need for any other complex algorithms to solve the set of transcendental dynamic equations for this problem.

2 Formulation

A modified two-radii Gough-Stewart Platform (MGSP) consist of a top mobile platform, a fixed base, and six prismatic actuated legs in between them as shown in Fig. 1(a). The anchor points are on two radii with variables R_{bi} and R_{bo} denoting the inner and outer radius of the base platform while R_{ti} and R_{to} representing the inner and outer radii of the mobile platform, respectively. There are two sets of three identical legs having rotational symmetry of 120°. The symbol H denotes the height between the two platforms. The coordinates of a point on the base frame $\{B\}$ are represented by $\{x_b, y_b, z_b\}$ and on the moving frame $\{P\}$ are represented by $\{x_p, y_p, z_p\}$. The vector $\boldsymbol{OB_1}$ (magnitude equal to R_{bo}) is chosen along $\boldsymbol{X_b}$. The variable α_{bi}, α_{to}, α_{ti} denote angles made by vectors $\boldsymbol{OB_4}$, $\boldsymbol{C_oA_1}$, $\boldsymbol{C_oA_4}$ with $\boldsymbol{X_b}$, respectively. Each leg is connected to the mobile platform through a spherical joint and to the base platform through a spherical or universal joint.

We made use of the force transformation matrix ($[\mathbf{B}]$) for an MGSP [1, 2]. With all legs assumed to have an axial stiffness of k in their joint space, the stiffness matrix $[\mathbf{K}_T]$ in the task space [1, 2] is given by

$$[\mathbf{K}_T] = k[\mathbf{B}][\mathbf{B}]^T \tag{1}$$

The force transformation matrix ($[\mathbf{B}]$) for MGSP is given by

$$[\mathbf{B}]_{6\times 6} = \begin{bmatrix} \boldsymbol{s_1} & \cdots & \boldsymbol{s_6} \\ ({}^B[\mathbf{R}]_P({}^P\boldsymbol{p_1})) \times \boldsymbol{s_1} & \ldots & ({}^B[\mathbf{R}]_P({}^P\boldsymbol{p_6})) \times \boldsymbol{s_6} \end{bmatrix} \tag{2}$$

where $\boldsymbol{s_j} = \dfrac{{}^B\boldsymbol{t} + {}^B[\mathbf{R}]_P\, {}^P\boldsymbol{p_j} - {}^B\boldsymbol{b_j}}{l_j}$, $j = 1, \ldots, 6$. The vector ${}^B\boldsymbol{t}$ is directed from centre of the base platform to the centre of the mobile platform, $\boldsymbol{S}_j$ $(= l_j\boldsymbol{s}_j)$ is a vector along the respective leg of an MGSP with length l_j while ${}^P\boldsymbol{p_j}$ and ${}^B\boldsymbol{b_j}$ are the coordinates of an anchor point on the mobile and base platform expressed in their respective frames. All the formulations are considered at the neutral position of the platform with ${}^B[\mathbf{R}]_P = [\mathbf{I}]$ and ${}^B\boldsymbol{t} = [\ 0\ 0\ H\ ]^T$. This is a fair assumption for precise control applications like micro-vibration isolation that require small motion. The mass/inertia of the legs can be neglected owing to their small values. With proper choice of the coordinate system, the mass matrix can be written as $[\mathbf{M}] = \mathrm{diag}[\ m_p,\ m_p,\ m_p,\ I_{xx},\ I_{yy},\ I_{zz}]$, where m_p denote the payloads' mass (including mobile platform) and I_{xx}, I_{yy}, and I_{zz} denotes its principal moment of inertia along each direction with respect to its centre of mass (COM). For a neutral configuration, $[\mathbf{B}]$ is a constant matrix and will have an infinite number of possible configurations [7] as attachment points on the mobile platform are not uniquely determined. Using Eqs. (1) and (2), we can write the natural frequency matrix in task space [2, 4, 7] as:

$$[\mathbf{G}] = [\mathbf{M}]^{-1}[\mathbf{K_T}] = [\mathbf{M}]^{-1}k[\mathbf{B}][\mathbf{B}]^T = \begin{bmatrix} [\mathbf{P}] & [\mathbf{T}] \\ [\mathbf{T}]^T & [\mathbf{U}] \end{bmatrix} \tag{3}$$

with $[\mathbf{P}] = \mathrm{diag}(\,\lambda_1,\, \lambda_2,\, \lambda_3\,)$, $[\mathbf{U}] = \mathrm{diag}(\,\lambda_4,\, \lambda_5,\, \lambda_6\,)$ and $[\mathbf{T}] = \begin{bmatrix} \mu_{11} & -\mu_{12} & 0 \\ \mu_{12} & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix}$

The expressions for λ_i and μ_{ij} are given in appendix. The leg length ratio, a, relates the lengths of two sets of legs by

$$l_2 = al_1 \tag{4}$$

where $l_1 = |\boldsymbol{S_1}| = |\boldsymbol{S_2}| = |\boldsymbol{S_3}|$ and $l_2 = |\boldsymbol{S_4}| = |\boldsymbol{S_5}| = |\boldsymbol{S_6}|$ (see appendix for details).

For the dynamic isotropy at the neutral position, all six eigenvalues of the natural frequency matrix in Eq. (3) must be equal. To obtain the design parameters, we have to solve a set of coupled transcendental equations generated from conditions of dynamic isotropy given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \omega^2 \text{ and } \mu_{11} = \mu_{12} = \mu_{33} = 0 \tag{5}$$

where, ω is the natural frequency of the MGSP and λ_1 to λ_6 are the eigenvalues of the matrix $[\mathbf{G}]$.

3 Design of dynamically isotropic MGSP

The unknowns (design parameters) for an MGSP are R_{bo}, R_{to}, R_{bi}, R_{ti}, H, a, α_{to}, and $(\alpha_{bi} - \alpha_{ti})$. Simplifications of dynamic isotropy conditions in Eq. (5) give rise to several useful geometrical observations, which can be seen in Table 1 and Fig. 2(a). If any set of geometries satisfies conditions in Table 1, the design parameters obtained will represent a dynamically isotropic configuration. It is to be noted that $K(= I_{xx}/I_{zz})$ and $Q(= I_{xx}/m_p)$ are payload properties that are known to us. The necessary condition of $\lambda_4 = \lambda_5$ is practically valid for symmetrical payloads ($I_{xx} = I_{yy}$). The challenge to obtain the sides for any general triangles in Fig. 2(a) is overcome by obtaining sides of the corresponding right angle triangle $\Delta Q_o P_{1,r} R_1$ in Fig. 2(b) (denoted by subscript 'r' for right angle case). We make use of analytical results for a right-angled triangle (see Fig. 1(b)) obtained in our previous work [2]. The relation in Fig. 1(b) was used to initiate the solution, and all other variables were obtained analytically (using Eq. (5)) [2] as:

$$\begin{gathered} R_{ti,r} = \sqrt{QC_2}, R_{to,r} = \frac{\sqrt{QC_1}}{a}, H_r = \sqrt{\frac{Q(KC_1C_2 - a^2)}{KC_1^2}}, \\ \theta = \arctan\left(\frac{a}{\sqrt{KC_1C_2 - a^2}}\right) \text{ and } (\alpha_{bi,r} - \alpha_{ti,r}) = 90^\circ - \theta \end{gathered} \tag{6}$$

where the leg length ratio a can be taken as input (free variable) with K and Q as known.

A general analytical solution in an explicit form is tedious due to coupling among variables. However, the study of the specific cases for $\Delta Q_o P_{1,r} R_1$ (Right-angle case) led to several useful observations and geometrical interpretations. The geometries, i.e., $\Delta Q_o P_{1,r} R_1$ and $\Delta Q_o P_{2,r} R_2$ shown in Fig. 2(b) can be constructed from the known results in Eq. (6) (for other design variables in $\Delta Q_o P_{2,r} R_2$ for right angle triangle case, substitute $f = 1$ in Table 2). Now, these geometries are perturbed to generate the new geometries, i.e., $\Delta Q_o P_1 R_1$ and $\Delta Q_o P_2 R_2$ as shown in Fig. 2(b). The perturbations done along line $\overline{P_{1,r} R_1}$ will keep the leg length ratio a constant. This means such perturbations along line $\overline{P_{1,r} R_1}$ to generate $\overline{P_1 R_1}$(hence $\Delta Q_o P_1 R_1$) also keeps few design variables such as $R_{ti}(= R_{ti,r})$, $R_{to}(= R_{to,r})$ and $\theta_1 \backslash \theta_2 = \theta$ the same (see Eq. (6)). Different a values generate new sets of triangles corresponding to their respective right-angled triangle (see Fig. 3). The new height H for the general case will now be given as $H = f H_r$ (see Eq. (6)) because $\overline{P_1 R_1} = f\ \overline{P_{1,r} R_1}$ (their lengths

Table 1. Geometrical interpretations from dynamic isotropy conditions

Case	Condition used	Simplified Equation	Geometrical Interpretation in Fig. 2(a)
1	$\lambda_1 = \lambda_2$ and Eq. (4)	${b_1}^2 = R_{ti}^2 + R_{bi}^2 - 2R_{ti}R_{bi}\ \cos(\alpha_{bi} - \alpha_{ti})$	$\Delta Q_o P_1 R_1$:triangle with sides R_{bi}, R_{ti} and b_1 (cosine rule)
2	$\lambda_1 = \lambda_2$ and Eq. (4)	${b_2}^2 = R_{to}^2 + R_{bo}^2 - 2R_{to}R_{bo}\ \cos(\alpha_{to})$	$\Delta Q_o P_2 R_2$:triangle with sides R_{bo}, R_{to} and b_2
3	$\mu_{11} = 0$	$R_{ti}R_{bi}\sin(\alpha_{bi} - \alpha_{ti}) = a^2\ R_{to}R_{bo}\sin(\alpha_{to})$	Area of $\Delta Q_o P_1 R_1 = a^2 \times$ (Area of $\Delta Q_o P_2 R_2$)
4	$\mu_{12} = 0$	$R_{ti}(R_{ti} - R_{bi}\cos(\alpha_{bi} - \alpha_{ti})) = a^2\ R_{to}(R_{bo}\cos(\alpha_{to}) - R_{to})$	$(\overline{Q_o R_1})P_{g1} = a^2\ (\overline{Q_o R_2})P_{g2}$
5	$\lambda_4 = \lambda_3$ and $\lambda_4 = \lambda_6$	$\dfrac{a^2 R_{to}^2}{a^2+1} + \dfrac{R_{ti}^2}{a^2+1} = 2Q$	$\overline{P_3 R_3}$ is a constant magnitude line treating R_{to} and R_{ti} as perpendicular basis.
6	$\mu_{12} = 0$ and $\mu_{11} = 0$	$\tan\theta_1 = \tan\theta_2$	$\angle Q_o R_1 P_1 = 180^\circ - \angle Q_o R_2 P_2 \implies \theta_1 = \theta_2 = \theta$
where $b_1 = H\sqrt{C_1}$, $b_2 = \dfrac{H\sqrt{C_2}}{a}$, $C_1 = \left(\dfrac{3a^2+1}{2}\right)$, $C_2 = \left(\dfrac{a^2+3}{2}\right)$			

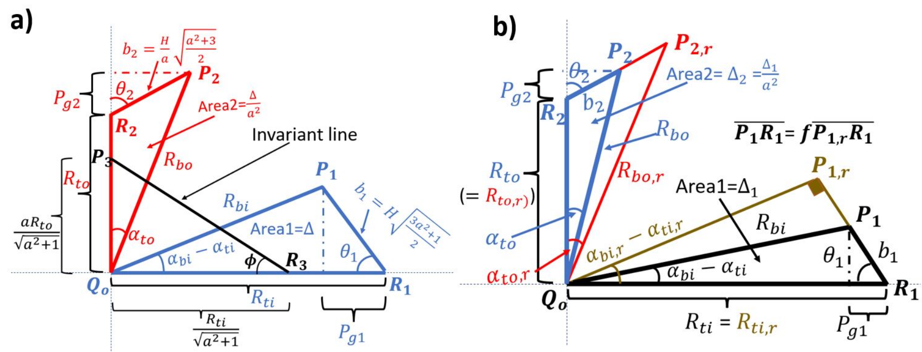

Fig. 2. **a)** Geometrical interpretation of design variables **b)** General triangle from a right-angled triangle.

are directly proportional to H, see Fig. 2(b)), where f is a ratio (free variable) whose value could be less than 1, equal to 1 (right-angle case) or greater than 1 (extrapolation of line) depending on the design values we want for an MGSP.

Interestingly, the new triangles due to scaling ($\Delta Q_o P_1 R_1$ and $\Delta Q_o P_2 R_2$) satisfy all the isotropic/geometric conditions in Table 1 and represents a dynamically isotropic design. The design variables for the general case ($R_{bi},\ \alpha_{bi} - \alpha_{ti}, R_{bo},\ \alpha_{to}$) which are different from the right-angled triangle case can be computed analytically in terms of ratios f, a and payload properties (K, Q). This can be done by making use of geometrical relations in Fig. 2(b) using known variables ($R_{ti}, R_{to}, \theta_1 \backslash \theta_2$) and the new value of H. For, eg., R_{bi} can be obtained as $\sqrt{(R_{ti} - b_1\cos\theta_1)^2 + (b_1\sin\theta_1)^2}$, which gives :

$$R_{bi} = \sqrt{\frac{QKC_1C_2 + fQ(KC_1C_2 - a^2)(f-2)}{KC_1}} \tag{7}$$

Table 2. General solution for design variables in their explicit form

Variable	$R_{ti}(=\overline{Q_oR_1})$	$R_{to}(=\overline{Q_oR_2})$	$H\ (\propto \overline{P_1R_1}$ or $\propto \overline{P_2R_2})$
Solution	Same as Eq. (6)	Same as Eq. (6)	$=fH_r$ (see *Eq.* (6))
Variable	$R_{bi}(=\overline{Q_oP_1})$	$R_{bo}(=\overline{Q_oP_2})$	
Solution	See Eq. (7)	$\sqrt{\dfrac{QC_1}{a^2}+\dfrac{fQ(KC_1C_2-a^2)(2C_1+fC_2)}{KC_1^2a^2}}$	
Variable	$\alpha_{bi}-\alpha_{ti}\ (=\angle R_1Q_oP_1)$		$\alpha_{to}\ (=\angle R_2Q_oP_2)$
Solution	$\arctan\left(\dfrac{fa\sqrt{(KC_1C_2\ -\ a^2)}}{KC_1C_2\ -\ f(KC_1C_2-a^2)}\right)$		$\arctan\left(\dfrac{b_2\ sin(\theta_2)}{R_{to}+\ b_2\ cos(\theta_2)}\right)$

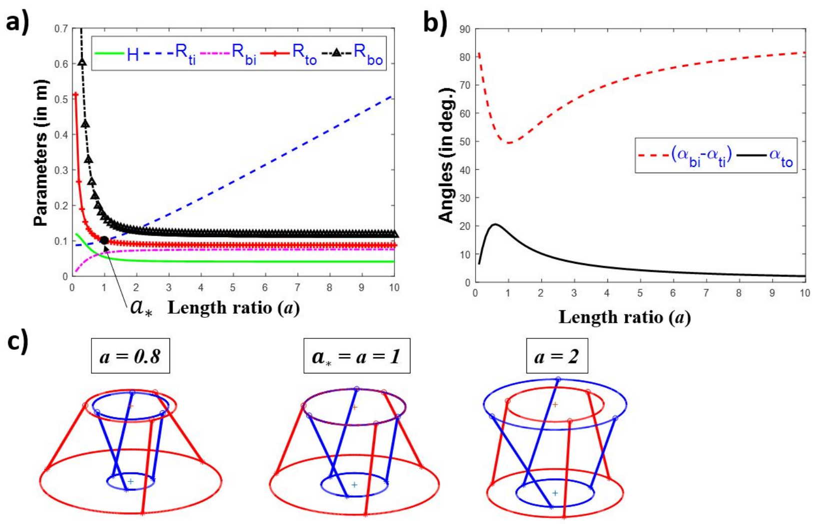

Fig. 3. a) & **b)** Variation of parameters for $f=1$, **c)** Configuration transition.

Similar to R_{bi}, general analytical solutions to all the variables can be obtained using geometry and are listed in Table 2. The result obtained in Table 2 can be directly used to design a dynamically isotropic MGSP.

4 Observation

1) Parameter variation: The variation of design parameters for right-angled triangle case ($f=1$) with respect to leg length ratio a can be seen in Fig. 3(a) and (b) for a typical payload of $K=0.590887$, $Q=5.089\times10^{-3}m^2$, and $m_p=5$ Kg.. Each a corresponds to a different right-angled triangle case, which served as the base/initial formulation in Fig. 2(b). An interesting observation from the expressions in Table 2 and Fig. 3(a) can be seen at $a=1$. At this point, $R_{to}=R_{ti}$ or the two radii on the top platform become equal (irrespective of f). This is also a point of configuration transition (a_*) where there is a transition from outer-outer (inner-inner) type leg connections to outer-inner (cross leg) type legs, as shown in Fig. 3(c). Choosing a design with $a<1$ avoids interference of legs in the 3D space, and in this case, the legs will never intersect, ensuring feasibility. For $a\ >1$, such interference needs to be investigated. The parameter variation

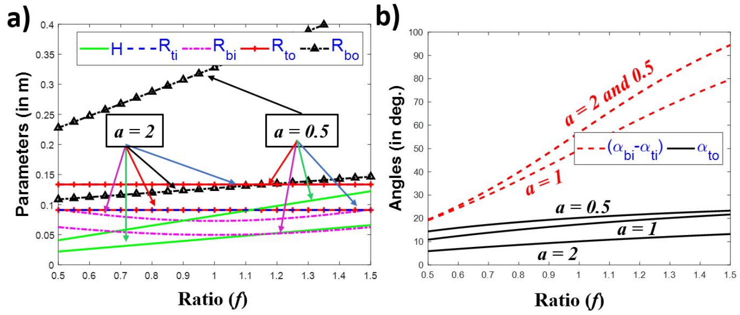

Fig. 4. **a)** Variation of geometrical parameters for $a = 0.5, 2$ **b)** Angle variation.

at $a = 2,\ 0.5$ with different scaling values f is shown in Fig. 4. The value of R_{ti} and R_{to} can be seen interchanged for $a = 2$ and $a = 0.5$ due to the reciprocal property between a and its inverse. These graphs can help us to design MGSP within our design constraints using free variables a and f.

2) Dynamically isotropic natural frequency: On substituting closed form expression for all design variables in Eq. (3), we obtain an equal value of all six natural frequencies (ω) of the platform as

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \omega^2 \text{ and } \omega = \sqrt{2k/m_p} \tag{8}$$

3) Invariant line: It can be observed from Case 5 (Table 1) and Fig. 2(a) that there is an invariant line $\overline{P_3R_3}$ whose length remains constant ($= \sqrt{2Q}$) for any value of a (any triangle). However angle ϕ ($= \arctan(\sqrt{C_1/C_2})$) varies with a.

4) Traditional GSP: The known fact that a traditional GSP ($R_{bi} = R_{bo}$ and $R_{to} = R_{ti}$) fails to give dynamic isotropy can also be verified from this approach. For this, $\Delta Q_oP_1R_1$ and $\Delta Q_oP_2R_2$ must be congruent, implying the angle θ_2 to be obtuse and the condition $\tan\theta_1 = \tan\theta_2$ thus cannot be satisfied.

5 Validation

Simulations were performed in ANSYS for the above-mentioned designs treating mobile and base platforms as rigid bodies (shell 181 element) and legs as ideal springs (link 180 elements). In each of the cases, the analytical results closely match the simulation results. For, e.g., assuming $a = 2$ and $f = 0.75$, $m_p = 5$ Kg, $k = 10^5$ N/m, $K = 0.590887$, and $Q = 5.089 \times 10^{-3}\, m^2$ (approximate values for a hardware under design and fabrication), the first six natural frequencies were obtained using the analytical result in Eq. (8) ($= \sqrt{2k/m_p}$) as 31.83 Hz. The first six natural frequency obtained with simulation are 31.68, 31.70, 31.76, 31.82, 31,82, and 31.85 Hz, which closely matches the analytical results, validating our design.

6 Conclusion

A pair of triangles were used to arrive at a general design of a dynamically isotropic MGSP. Using this geometry-based approach, a closed-form solution in its explicit form was derived, providing feasibility and flexibility. Such designs with the first six natural frequencies equal can ease passive and active micro-vibration isolation.

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Appendix

The value of all $\lambda' s$ and $\mu' s$ in Eqn. (3) are given as:

$$\lambda_1 = \lambda_2 = \frac{3k\ \left(l_1^2\Psi_1 + l_2^2\Psi_2\right)}{\left(2m_p l_1^2 l_2^2\right)},\ \lambda_3 = \frac{3kH^2\left(l_1^2 +\ l_2^2\right)}{\left(m_p l_1^2 l_2^2\right)},\quad \lambda_4 = \frac{3k\Psi_5 H^2}{\left(2I_{xx} l_1^2 l_2^2\right)},$$

$$\lambda_5 = \lambda_4 \frac{(I_{xx})}{(I_{yy})},\ \lambda_6 = \frac{3k\left(\Psi_6^2 l_1^2 +\ \Psi_7^2 l_2^2\right)}{\left(I_{zz} l_1^2 l_2^2\right)},\ \mu_{11} = \frac{-3kH\left(-\Psi_6 l_1^2 +\ \Psi_7 l_2^2\right)}{\left(2l_1^2 l_2^2\right)},$$

$$\mu_{33} = -2\mu_{11},\quad \mu_{12} = \frac{3kH\left(\Psi_3 l_1^2 + \Psi_4 l_2^2\right)}{\left(2l_1^2 l_2^2\right)}$$

where,

$\Psi_1 = R_{ti}^2 +\ R_{bi}^2 - 2R_{ti}R_{bi}\cos(\alpha_{bi} - \alpha_{ti}),\ \Psi_2 = R_{to}^2 +\ R_{bo}^2 - 2R_{to}R_{bo}\cos(\alpha_{to}),$
$\Psi_3 = R_{ti}^2 - R_{ti}R_{bi}\cos(\alpha_{bi} - \alpha_{ti}), \Psi_4 = R_{to}^2 - R_{to}R_{bo}\cos(\alpha_{to}), \Psi_5 = R_{ti}^2 l_1^2 +\ R_{to}^2 l_2^2$
$\Psi_6 = R_{ti}R_{bi}\sin(\alpha_{bi} - \alpha_{ti}),\ \Psi_7 = R_{to}R_{bo}\sin(\alpha_{to})$

$$l_1 = \left(\sqrt{R_{to}^2 + R_{bo}^2 - 2R_{to}R_{bo}\cos(\alpha_{to}) + H^2}\right)$$
$$l_2 = \left(\sqrt{R_{ti}^2 + R_{bi}^2 - 2R_{ti}R_{bi}\cos(\alpha_{bi} - \alpha_{ti}) + H^2}\right)$$