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ANALYTICAL DETERMINATION OF PRINCIPAL TWISTS AND SINGULAR DIRECTIONS IN ROBOT MANIPULATORS

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ABSTRACT

The identification of principal twists of the end-effector of a manipulator undergoing multi-degree-of-freedom motion is considered to be one of the central problems in kinematics. In this paper, we use dual velocity vectors to parameterize $se(3)$, the space of twists, and define an inner product of two dual velocities as a dual number analog of a *Riemannian metric* on $SE(3)$. We show that the principal twists can be obtained from the solution of an eigenvalue problem associated with this dual metric. It is shown that the computation of principal twists for any degree-of-freedom of rigid-body motion, requires the solution of at most a cubic dual characteristic equation. Furthermore, the special nature of the coefficients yields simple analytical expressions for the roots of the dual cubic, and this is turn leads to compact analytical expressions for the principle twists. We also show that the method of computation allows us to separately identify the rotational and translational degrees-of-freedom lost or gained at singular configurations. The theory is applicable to serial, parallel, and hybrid manipulators, and is illustrated by obtaining the principal twists and singular directions for a three-degree-of-freedom parallel, and a hybrid six-degree-of-freedom manipulator.

Introduction

It is well known that rigid-body displacements in $\Re^3$ form a 6-dimensional smooth manifold, which is also a Lie group called the *Special Euclidean Group* (denoted by $SE(3)$). The tangent space to $SE(3)$ at its identity element forms the associated Lie algebra (denoted by $se(3)$), which contains the linear and angular velocities of the rigid body [1]. The analysis of $SE(3)$ and $se(3)$ is essential for kinematic analysis and synthesis of serial and parallel manipulators and closed-loop mechanisms, analysis of singularities and algorithmic motion planning. A central problem of kinematics of manipulators is the determination of its principal screws at a given configuration, which characterizes the motion of an end-effector instantaneously. Ball [2] and Hunt [3] have used geometric arguments to identify the principal screws of rigid-body motion. More recently, Fang and Huang [4] have presented an analytical approach for 3-degree-of-freedom motions based on theory of degenerate conic sections. In this paper, we make use of the group structure of $SE(3)$, and the dual-vector representation of its Lie algebra elements. We reduce the problem of identification of principal twists to the solution of an eigen problem of a dual matrix arising out of the dual inner-product of the input screws. Dual vectors and matrices capture the semi-direct product structure of $SE(3) (= SO(3) \otimes \Re^3)$ naturally, hence our results have the advantage of being very compact as well as comprehensive. Following [5], we parameterize $SE(3)$ by dual orthogonal matrices, and arrive at the corresponding parameterization of $se(3)$, the space of twists. Using the isomorphism of twists and dual vectors, we define an inner product on $se(3)$ as a *dual number* arising out of the scalar product of dual vectors. Using analogous arguments presented in [6], the investigation of the extremal values of the allowable twists under the unit speed constraint leads to the eigen problem of a symmetric dual matrix of inner products. We then present the solution of the dual

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eigen problem, which is new to our knowledge. The significant advantages of the approach are:

1. The approach is *exact*, i.e., all the results are obtained in closed, analytical form. This is possible since the dual characteristic polynomial is at most a cubic *irrespective the degree-of-freedom of rigid-body motion considered* (see Appendix A) which has exact solutions. It may be noted that our method is consistent for degree-of-freedom greater than 3, and we do not use the concept of reciprocal screw systems.
2. Our treatment of rigid-body motion leads to the identification of the rotational and translational degrees-of-freedom separately, which we term as *degree-of-freedom-partitioning*. This, we believe, is a unique feature of our approach, and has tremendous potential for applications in the analysis and design of manipulators.
3. The formulation handles singularities in a natural way, and analytical identification of the lost or gained twists at a singularity is possible.
4. The computation involved is of purely algebraic nature, and hence fast and automated computer implementation is possible for *serial, parallel or hybrid* manipulator architectures. We perform all our symbolic manipulations using the package *Mathematica* [7].

The paper is organized as follows: In section 1, we briefly present the mathematical preliminaries related to the dual representation of lines, screws and twists, and introduce the notion of dual metric and its properties. In section 2, we present the analytical expressions for the principal twists and the principal singular directions. In section 3, we illustrate our method with the help of parallel and hybrid manipulators. In the appendices, we present details of the solution of the dual eigen value problem and also show that our approach yields the classical equations of the cylindroid and the pitch hyperboloid in the cases of two- and three-degree-of-freedom motions respectively.

Mathematical Formulation

A dual number, $\hat{a}$, has the form $a+\varepsilon a_0$, where $a,\ a_0 \in \Re$ and ε stands for the *dual unit*, with the properties $\varepsilon \neq 0,\ \varepsilon^2 = 0$. The properties of dual numbers are detailed in [8]. We note here only that the dual numbers over the real field form a *ring* (denoted by Δ), and dual n-vectors form a *free module* over this ring [9] , which is denoted by $\mathcal{D}^n$. We can define an inner product on $\mathcal{D}^3$, i.e., the space of 3-dimensional dual vectors, as follows:

$$\begin{aligned}\langle \hat{x},\hat{y}\rangle &= x\cdot y+\varepsilon(x\cdot y_0+y\cdot x_0) \\ &= -\frac{1}{4}\langle \hat{x},\hat{y}\rangle_{Killing}+\varepsilon\,\langle \hat{x},\hat{y}\rangle_{Klein}\end{aligned} \tag{1}$$

where '$\cdot$' denotes the usual inner product in the Euclidean space, $\hat{x} = x+\varepsilon x_0\ \in \mathcal{D}^3$ and $\langle\cdot,\cdot\rangle_{Killing}$ and $\langle\cdot,\cdot\rangle_{Klein}$ are the *Killing* and *Klein forms* on $SE(3)$ respectively [1]. Both these forms are known to posses frame-invariance, and hence the dual inner product is frame-invariant. The inner product is positive semi-definite, as the Killing form is negative semi-definite. Using the inner-product, we can define the norm $||\hat{x}||$ of $\hat{x}$ as $\langle \hat{x},\hat{x}\rangle^{1/2}$ when $x \neq 0$. Then we obtain $||\hat{x}|| = ||x||+\varepsilon\frac{x\cdot x_0}{||x||},\ \ ||x|| \neq 0$ A dual vector $\hat{x}$ with norm $1+\varepsilon 0$ is called a *dual unit vector* and $\hat{x}$ is a dual unit vector *iff* following relations $||x|| = 1, x.x_0 = 0$ hold.

Lines and Screws as Dual Vectors

A line in $\Re^3$ can be described in terms of a dual vector as $\hat{\mathcal{L}} = Q+\varepsilon Q_0$, where $(Q;Q_0)$ is the Plücker vector associated with the line[1]. There are four independent parameters in Q and Q_0, since $||Q|| = 1$ and $Q\cdot Q_0 = 0$ and thus there is a one-to-one correspondence between lines in $\Re^3$ and dual unit vectors. The location of a line in space is uniquely determined by the the foot of the perpendicular from the origin and is obtained as $r_0 = Q\times Q_0$. The inner product of two lines follows from the properties of dual vectors, and is given by

$$\begin{aligned}\langle \hat{\mathcal{L}}_1,\hat{\mathcal{L}}_2\rangle &= Q_1\cdot Q_2+\varepsilon(Q_1\cdot Q_{02}+Q_2\cdot Q_{01}) \\ &= \cos\phi-\varepsilon d\sin\phi = \cos\hat{\phi}\end{aligned} \tag{2}$$

where ϕ and d are the angle and the shortest distance between the two lines respectively, and $\hat{\phi} = \phi+\varepsilon d$ denotes the dual angle between the lines.

A *screw* has five independent parameters and may be identified with a line (which is called the *axis* of the screw) and a pitch. The screw can be described by a dual vector $\hat{\mathcal{S}} = S+\varepsilon S_0$, where $S = Q$ and $S_0 = Q_0+hQ$. The pitch of the screw, h, is given by

$$h = \frac{S\cdot S_0}{S\cdot S}, \quad ||S|| \neq 0 \tag{3}$$

If the magnitude of the real part of $\hat{\mathcal{S}}$ is 0, and that of the dual part is non-zero, then the pitch is infinite, signifying a pure translation.

The inner product of two screws is computed as

$$\begin{aligned}\langle \hat{\mathcal{S}}_1,\hat{\mathcal{S}}_2\rangle &= S_1\cdot S_2+\varepsilon(S_1\cdot S_{02}+S_2\cdot S_{01}) \\ &= \cos\phi+\varepsilon((h_1+h_2)\cos\phi-d\sin\phi)\end{aligned} \tag{4}$$

where h_1 and h_2 are the pitches associated with the two screws, d and ϕ have the meanings as explained above.

[1] Q denotes direction of the line, and $Q_0 = r\times Q$ is the moment of the line with r as position vector of an arbitrary point on the line from an origin.

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Rigid-body Motion and Twists

We parameterize $SE(3)$, the space of rigid-body displacements, in terms of dual orthogonal matrices of the form $\hat{A} = R + \varepsilon DR$, where $R \in SO(3)$ gives the orientation of the moving frame attached to the rigid-body with respect to some fixed reference frame, and D is the 3×3 skew-symmetric matrix associated with $d \in \Re^3$, the displacement of the origin of the moving frame with respect to the fixed frame [5]. For n-degree-of-freedom motions of the rigid-body, we can associate n independent real *motion parameters*, $\theta_i, i = 1, \ldots n$ via a smooth map, $\psi : \Re^n \to SE(3)$ such that $\psi(\theta) = \hat{A} \in SE(3)$. The motion parameters, θ, may be assumed to be functions of time t alone, and thus the vector function $\theta(t)$ describes the motion in $\Re^n$. As $\theta(t)$ evolves smoothly, it traces a *curve* $c(t) = \psi(\theta(t))$ on the manifold $SE(3)$, to each point of which we can associate a tangent space, which contains the velocity $\dot{c}(t)$ of the curve. The tangent vector $\dot{c}(t)$ may be obtained from the push-forward map $\psi_* : \Re^n \to T_{\hat{A}}SE(3)$ such that $\psi_*(\dot{\theta}) = \dot{\hat{A}}(\theta(t)) = \dot{R} + \varepsilon(\dot{D}R + D\dot{R}) \in T_{\hat{A}}SE(3)$. We can translate this tangent vector to the tangent-space at the *identity* element of $SE(3)$ by left or right translations by $\hat{A}^{-1} (= \hat{A}^T)$ to obtain the Lie algebra associated with the group, where the multiplication is given by the Lie bracket, denoted by $[\cdot,\cdot]$. In kinematics literature, $se(3)$ is well known as the algebra of *twists* [1]. Depending upon the translation used to take them to the identity, we can get a *left-invariant* twist or a right-invariant twist. In this paper, we use the right-invariant twists[2], whose explicit form is $\hat{\Omega} = \dot{\hat{A}}\hat{A}^T = \Omega + \varepsilon([D,\Omega] + \dot{D})$ where $\Omega = \dot{R}R^T \in so(3)$ denotes the right-invariant angular velocity of the rigid-body. Using the isomorphism of the algebras $(so(3), [\cdot,\cdot])$ and $(\Re^3, \times)$, we express the twist in terms of a dual vector:

$$\hat{\mathcal{V}} = \omega + \varepsilon(\dot{d} + d \times \omega) \tag{5}$$

where $\omega, \dot{d}$ and $d \times \omega$ are the counterparts of $\Omega, \dot{D}$, and $[D,\Omega]$ respectively in $\Re^3$. The quantity $\hat{\mathcal{V}}$ is also known as a *motor*, and may be thought of as a screw together with a magnitude [10]. In terms of line coordinates, $\hat{\mathcal{V}} = ||\omega||(Q + \varepsilon(Q_0 + hQ))$, where $||\omega||$, the magnitude of the angular velocity vector also denotes the magnitude of the twist.

The *Dual Metric*

The resultant twist in (5) may be re-written in terms of the dual Jacobian $\hat{J}$ as

$$\hat{\mathcal{V}} = \hat{J}\dot{\theta} = (J_\omega + \varepsilon J_v)\dot{\theta} = \sum_{i=1}^{n} \hat{\$}_i \dot{\theta}_i \tag{6}$$

[2] Analogous results can be obtained for left-invariant twists.

where $\hat{\$}_i$, the ith column of $\hat{J}$ is the ith input screw which can be computed as the vector form of the dual skew-symmetric matrix $\frac{\partial \hat{A}}{\partial \theta_i}\hat{A}^T \dot{\theta}_i$, and J_ω, J_v represent the Jacobians corresponding to the angular and linear velocities respectively. At a *non-singular* configuration, the columns of $\hat{J}$ are linearly independent, and as such they form a *basis set* for for a subspace of $se(3)$ spanned by the permissible twists of the manipulator end-effector. The principal basis-set, consisting of the principal twists describe completely the first-order instantaneous kinematics of rigid-body motion.

Following the results for point trajectories [6], we seek extremal magnitudes of the resultant twist, $||\hat{\mathcal{V}}||$, subject to a *unit speed* constraint, $||\dot{\theta}|| = 1$. Form equation (6), $||\hat{\mathcal{V}}|| = \dot{\theta}^T \hat{g} \dot{\theta}$, where $\hat{g} = \hat{J}^T \hat{J} = J_\omega^T J_\omega + \varepsilon(J_\omega^T J_v + J_v^T J_\omega)$. Under the unit speed constraint, the vecotr objective function becomes

$$f(\dot{\theta}) = \hat{g}_{ij}\dot{\theta}_i\dot{\theta}_j - \hat{\lambda}_i(\dot{\theta}_i^2 - 1) \quad i,j = 1,\ldots,n$$

where $\hat{g}_{ij} = \left\langle \hat{\$}_i, \hat{\$}_j \right\rangle$ is the element (i,j) of the matrix $\hat{g}$, and $\hat{\lambda}_i \in \Delta$ are the unknown *Lagrange multipliers*. As shown in [6], the solution of this n-dimensional extremization problem in n-dimensions reduces to the following eigen problem:

$$\hat{g}\dot{\theta} = \hat{\lambda}\dot{\theta} \tag{7}$$

which can be solved as described in the Appendix A. Here, we list some of the properties of the matrix $\hat{g}$ and its eigen system.

Symmetry: The matrix $\hat{g}$ is *symmetric* and this follows from the definition of $\hat{g}$.

Bi-invariance: Since the dual inner-product is frame-invariant (the Klein and Killing forms are both frame invariant), $\hat{g}$ is independent of the reference frame, and so are the principal twists.

Positive Definiteness over Δ:

From the properties of dual eigenvalues (see Appendix A), we see that the real part of an eigenvalue of $\hat{g}$ is eigenvalue of the real part of $\hat{g}$, i.e., $J_\omega^T J_\omega$. The matrix $J_\omega^T J_\omega$ can admit only non-negative real eigenvalues, hence the dual eigenvalues are also non-negative [8]. We also note that the $rank_{\Re}(J_\omega)$, hence that of $J_\omega^T J_\omega$ can be at most 3, hence at most 3 of the dual eigenvalues of $\hat{g}$ are non-zero. The rank of a dual matrix over $\Delta^{3\times 3}$ can at most be 3 [11], hence if $rank_\Delta(\hat{g}) = n, n \leq 3$ or $rank_\Delta(\hat{g}) = 3, n > 3$, $\hat{g}$ is nondegenerate over Δ.

These properties allow us to consider $\hat{g}$ as a dual analog of a *Riemannian metric* on $SE(3)$. We also observe that the characteristic polynomial of $\hat{g}$ is at most a cubic due to the last property, which renders the eigen problem solvable analytically. In summary, for degree-of-freedom(n) = $1,2,3$, at a non-singular configuration, n eigenvalues of $\hat{g}$ are positive, and for $n > 3$, at non-singular configurations, 3 of them are non-zero, and $n-3$ of them are zeros.

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The eigenvectors, $\dot{\theta}_i$, obtained from the solution of equation(7) form the basis of the row space (for $\hat{\lambda} \neq 0$) and nullspace (for $\hat{\lambda} = 0$) of $\hat{J}$. From equation (6), it is clear that the principal twists lie in the column space (for $\hat{\lambda} \neq 0$) and the left nullspace of $\hat{J}$ (for $\hat{\lambda} = 0$), which may be obtained by constructing the set of vectors $\{\hat{\mathcal{V}}_i\} = \hat{J}\dot{\theta}_i,\ i = 1, \ldots, n$. We look at the two spaces separately, and propose the concept of *degree-of-freedom partitioning*.

Column-space of $\hat{J}$

The non-zero eigen values of $\hat{J}^T\hat{J}$ (or $\hat{g}$) are equal to those of $\hat{J}\hat{J}^T$, and square of the singular values of $\hat{J}$. This allows us to write the principal twists corresponding to the column-space of $\hat{J}$ as

$$\hat{\mathcal{V}}_i^* = \sqrt{\hat{\lambda}_i}\hat{Q}_i^* \tag{8}$$

where $\hat{Q}_i^* = Q_i^* + \varepsilon Q_{0i}^*$ is a dual eigenvector of $\hat{J}\hat{J}^T$. Expanding into real and dual parts, we get

$$\hat{\mathcal{V}}_i^* = \sqrt{\lambda_i}\left(Q_i^* + \varepsilon\left(\frac{\lambda_{0i}}{2\lambda_i}Q_i^* + Q_{0i}^*\right)\right) \quad \lambda_i \neq 0 \tag{9}$$

Comparing with the expression for dual velocity, $\hat{\mathcal{V}}_i^* = \omega_i^* + \varepsilon v_i^*$ we get

$$\begin{aligned} \omega_i^* &= \sqrt{\lambda_i}Q_i^* \\ v_i^* &= \sqrt{\lambda_i}\left(\frac{\lambda_{0i}}{2\lambda_i}Q_i^* + Q_{0i}^*\right) \end{aligned} \tag{10}$$

We deduce two important results from the last equation. Firstly, the principal pitch is given by

$$h_i^* = \frac{\lambda_{0i}}{2\lambda_i} \tag{11}$$

Secondly, the magnitudes of the principal twists, given by $||\omega_i|| = \sqrt{\lambda_i}$ is related to the principal pitch by the relation

$$\sqrt{\hat{\lambda}_i} = ||\omega_i||(1 + \varepsilon h_i^*) \tag{12}$$

Left-nullspace of $\hat{J}$

If $n > 3$, or $rank_{\Re}(J_\omega) < 3$, one or more of the principal twists will lie in the left nullspace of $\hat{J}$. These twists may be computed from equation (6), where $\dot{\theta}_i$ are the eigenvectors corresponding to the vanishing eigenvalues of $\hat{g}$. Expressed as dual vectors, these twists are of the form $0 + \varepsilon v_i^*, \quad (i = 1, \ldots, n-3)$ for n-degree-of-freedom motion ($n > 3$). These twists have infinite pitches, and they signify *pure translational motion* of the rigid-body. To motivate this point, we show below the dual-vector form of the principal twists for the general case of $n = 6$, where the principal twists are the columns of the following matrix:

$$\hat{\mathcal{B}} = \begin{pmatrix} \omega_{1x} + \varepsilon v_{1x} & \omega_{2x} + \varepsilon v_{2x} & \omega_{3x} + \varepsilon v_{3x} & \varepsilon v_{1x}^* & \varepsilon v_{2x}^* & \varepsilon v_{3x}^* \\ \omega_{1y} + \varepsilon v_{1y} & \omega_{2y} + \varepsilon v_{2y} & \omega_{3y} + \varepsilon v_{3y} & \varepsilon v_{1y}^* & \varepsilon v_{2y}^* & \varepsilon v_{3y}^* \\ \omega_{1z} + \varepsilon v_{1z} & \omega_{2z} + \varepsilon v_{2z} & \omega_{3z} + \varepsilon v_{3z} & \varepsilon v_{1z}^* & \varepsilon v_{2z}^* & \varepsilon v_{3z}^* \end{pmatrix} \tag{13}$$

The first three columns have the non-zero principal twists forming the column space of $\hat{J}$, and the pure dual twists in the last three columns correspond to the left-nullspace of $\hat{J}$. These two sets divide the rigid-body motion into two parts, namely, one consisting of both translation and rotation, but *independent* of the pure translational modes of motion, and another consisting of purely translational motion, and independent of the rotational motion of the rigid-body. This decoupling occurs due to the fact that the dual inner product of two pure translational twists is zero, and hence the pure translations lie in the left-nullspace of $\hat{J}$, and it is well known in linear algebra that the null space is the orthogonal complement of the column space. The three pure translations account for the three degrees-of-freedom of the rigid-body as the three non-zero twists imply only three rotational degrees-of-freedom. The translation associated with these rotational degrees-of-freedom due to the non-zero pitches are not *independent* and do *not* add to the degree-of-freedom of the rigid-body since the equation (12) states that they are related to the rotational motion. We call this decoupling *degree-of-freedom partitioning*, which allows us to study the rotational and translational modes of rigid-body motion independent of each other, and has great potential for application in design of robots where the objective can be split into these two modes explicitly. We also note here that if $n > 6$, then the number of principal twists in the left null space of $\hat{J}$ will be more than 3, and their dual parts will be linearly dependent. However, we can always construct an orthogonal basis set for subspace of $\Re^3$ spanned by the dual parts, which will give us the distribution of pure translational motions of the rigid body.

Analytical Expressions of Principal Twists for Multi-degree-of-freedom Rigid-body Motion

We now present the most important results of the paper, namely, the analytical expressions for the principal twists of multi-degree-of-freedom rigid-body motion.

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One-degree-of-freedom Rigid Body Motion

The simplest case of rigid-body motion is that of one-degree-of-freedom motion, and the distribution of allowable twists is of the form $\hat{\mathcal{V}} = \hat{\$}_1\dot{\theta}_1$. The single input screw $\hat{\$}_1$ itself may be identified with the principal screw of the system, and transforming to a frame where the X axis is along the screw axis, and the origin is some chosen point on the axis, the principal twist can be written as

$$\hat{\mathcal{V}}_i^* = k(1+\varepsilon h^*)(1,0,0)^T \tag{14}$$

where h^* is the pitch of $\hat{\$}$ and $k \in \Re$.

Two-degrees-of-freedom Rigid Body Motion

For two-degree-of-freedom motion of a rigid-body, let $\theta(t) = (\theta_1(t), \theta_2(t))^T$ represent the two independent motion parameters. Let $\hat{\$}_i = S_i + \varepsilon(h_i S_i + S_{oi})$ represent the ith input screw. The resultant twist can be written as

$$\hat{\mathcal{V}} = \hat{\$}_1\dot{\theta}_1 + \hat{\$}_2\dot{\theta}_2 = \hat{J}\dot{\theta} \tag{15}$$

Following the development in the last section, we obtain the matrix $\hat{g}$, whose elements can also be written as $\hat{g}_{ij} = \left\langle \hat{\$}_i, \hat{\$}_j \right\rangle = \cos\phi_{ij} + \varepsilon((h_i+h_j)\cos\phi_{ij} - d_{ij}\sin\phi_{ij}),\ i,j = 1,2$. In particular, $\hat{g}_{ii} = 1+\varepsilon(2h_i),\ \ i = 1,2$. The dual characteristic equation may be written in its real and dual components as[3]

$$\begin{aligned} \lambda^2 - 2\lambda + \sin^2\phi_{12} &= 0 \\ 2(\lambda-1)\lambda_0 + (h_1+h_2)(d\sin 2\phi_{12} + 2sin^2\phi_{12} - 2\lambda) &= 0 \end{aligned} \tag{16}$$

Solving for λ, λ_0, we finally obtain the two eigenvalues as

$$\begin{aligned} \hat{\lambda}_1 &= 2\cos^2\phi_{12}/2(1+\varepsilon(h_1+h_2-d_{12}\tan(\phi_{12}/2)) \\ \hat{\lambda}_2 &= 2\sin^2\phi_{12}/2(1+\varepsilon(h_1+h_2+d_{12}\cot(\phi_{12}/2)) \end{aligned} \tag{17}$$

The principal magnitude and pitches are obtained from the last equation as

$$\begin{aligned} ||\omega_1^*|| &= \sqrt{\lambda_1} = \sqrt{2}\cos\phi_{12}/2 \\ ||\omega_2^*|| &= \sqrt{\lambda_2} = \sqrt{2}\sin\phi_{12}/2 \\ h_1^* &= \frac{\lambda_{01}}{2\lambda_1} = 1/2((h_1+h_2) - d_{12}\tan(\phi_{12}/2)) \\ h_2^* &= \frac{\lambda_{02}}{2\lambda_2} = 1/2((h_1+h_2) + d_{12}\cot(\phi_{12}/2)) \end{aligned} \tag{18}$$

The real eigenvectors of $\hat{g}$ are given by $1/\sqrt{2}(1\pm 1)^T$, and they map to the principal twists as

$$\hat{\mathcal{V}}_{1,2}^* = \frac{1}{\sqrt{2}}(\hat{\$}_1 \pm \hat{\$}_2) \tag{19}$$

It may be verified by direct computation that the inner products of the screws, as well as their axes are zero. This implies that the principal twists intersect each other orthogonally. If we transform to a new coordinate system, in which the X- and Y-axes are along the axes of $\hat{\mathcal{V}}_1^*$ and $\hat{\mathcal{V}}_2^*$ respectively, and the origin is the point of intersection of the two, then we can write the principal twists as

$$\begin{aligned} \hat{\mathcal{V}}_1^* &= ||\omega_1^*||(1+\varepsilon h_1^*)(1,0,0)^T \\ \hat{\mathcal{V}}_2^* &= ||\omega_2^*||(1+\varepsilon h_2^*)(0,1,0)^T \end{aligned} \tag{20}$$

Using this form of the principal twists, we can easily derive the equation of the cylindroid, which describes the distribution of the principle screws, and also compute the the distribution of the magnitude of twists as an ellipse (see Appendix B for the derivation).

Three-degrees-of-freedom Rigid Body Motion

The dual velocity vector for a rigid body undergoing three-degree-of-freedom motion, with motion parameters $\theta(t) = (\theta_1(t), \theta_2(t), \theta_3(t))^T$, may be represented by

$$\hat{\mathcal{V}} = \hat{\$}_1\dot{\theta}_1 + \hat{\$}_2\dot{\theta}_2 + \hat{\$}_3\dot{\theta}_3 \tag{21}$$

where $\hat{\$}_i = Q_i + \varepsilon(h_i Q_i + Q_{oi})$ is the screw associated with the motion parameter θ_i. We obtain $\hat{g}$ as above, and the real part of the dual characteristic equation of $\hat{g}$ in this case is obtained as[4]

$$\begin{aligned} &\lambda^3 - 3\lambda^2 + (3 - c_{12}^2 - c_{23}^2 - c_{31}^2)\lambda \\ &+ (c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1) = 0 \end{aligned} \tag{22}$$

This cubic equation can be solved analytically using Cardan's formula and using the fact that the roots are real quantities (being the eigenvalues of a symmetric real matrix g), the analytical expressions for the roots are obtained as

$$\lambda_i = 1 + \frac{2}{\sqrt{3}}\sqrt{c_{12}^2 + c_{23}^2 + c_{31}^2}\cos\left(\frac{\phi + (i-1)2\pi}{3}\right) \tag{23}$$

[3] We use d_{ij} and ϕ_{ij} to denote the distance and angle between the ith and jth screw axes.

[4] We use $c_{(\cdot)}$ and $s_{(\cdot)}$ to denote $\cos(\cdot)$ and $\sin(\cdot)$ respectively throughout the paper.

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where $i = 1,2,3$, and $\phi \in [0,2\pi]$ is such that $\sin\phi = (1/27)(c_{12}^2 + c_{23}^2 + c_{31}^2)^3 - c_{12}^2 c_{23}^2 c_{31}^2$ and $\cos\phi = c_{12}c_{23}c_{31}$. Analogous to the two-degree-of-freedom case, the principal magnitudes and pitches associated with these twists are obtained as $||\omega_i^*|| = \sqrt{\lambda_i},\ i = 1,2,3$, and

$$h_i^* = -\frac{a_{20}\lambda_i^2 + a_{10}\lambda_i + a_{00}}{3\lambda_i^3 - 6\lambda_i^2 + (3 - (c_{12}^2 + c_{23}^2 + c_{31}^2))\lambda_i} \tag{24}$$

where using the notation $H = h_1 + h_2 + h_3$,

$$\begin{aligned} a_{20} &= -2H \\ a_{10} &= H(2 - c_{12} - c_{23} - c_{31}) + h_1 c_{23} + h_2 c_{23} + h_3 c_{12} \\ a_{00} &= H(\cos\phi_{12} + \cos\phi_{23} + \cos\phi_{31} - 4c_{12}c_{23}c_{31}) \\ &\quad + 2d_{12}(c_{23}c_{31} - c_{12}) + 2d_{31}(c_{12}c_{23} - c_{31}) \\ &\quad + 2d_{23}(c_{12}c_{31} - c_{23}) \end{aligned}$$

The ith eigenvector of $\hat{g}$, may be obtained as

$$\dot{\theta}_i = (\frac{c_{12}c_{31} + c_{23}(1+\lambda_i)}{(1+\lambda_i)^2 - c_{12}^2}, \frac{c_{12}c_{23} + c_{31}(1+\lambda_i}{(1+\lambda_i)^2 - c_{12}^2}, 1)^T \tag{25}$$

Normalizing these eigenvectors and writing them as $(l_i, m_i, n_i)^T$, the principal twists are obtained from equation (21) as

$$\hat{\mathcal{V}}_i^* = \hat{\$}_1 l_i + \hat{\$}_2 m_i + \hat{\$}_3 n_i \tag{26}$$

It follows from the properties of the dual symmetric matrices that the lines $\hat{\mathcal{L}}_i$ along $\hat{\mathcal{V}}_i$ are mutually orthogonal (for unique eigenvalues), and they meet at a point. Following the treatment in the two-screw case, we transform to the principal basis, in which the axes X, Y and Z are along $\hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2$ and $\hat{\mathcal{L}}_3$, respectively, with the origin at the point of intersection of the three axes. We can write the principal twists in this basis, using the standard basis $\{e_i\}$ of $\Re^3$, as

$$\hat{\mathcal{V}}_i^* = ||\omega_i^*||(1 + \varepsilon h_i^*)e_i,\ i = 1,2,3 \tag{27}$$

Analogous to the case of two-degree-of-freedom rigid-body motion, we derive the distribution of the magnitude of twists as an ellipsoid, and deduce the equation of the pitch hyperboloid associated with a 3-screw system (see Appendix B).

By recovering the classical results of screw theory using our dual metric approach, we have demonstrated the mathematical exactness of our formulation. The key advantage of the dual metric approach is that we are able to derive compact analytical expressions for all the results, and also introduce the concept of distribution of magnitudes of twists.

n-degree-of-freedom Rigid Body Motion, $n > 3$

The general case of n-degree-of-freedom motion can be considered within the same framework as above. As noted earlier, the $rank_{\Re}(J_\omega) \le 3$, and hence $rank_\Delta(\hat{g}) \le 3$, which reduces the characteristic polynomial of $\hat{g}$ to at most a dual cubic. More explicitly, the characteristic equation (43) takes the shape as follows:

$$\hat{\lambda}^{n-3}(\hat{\lambda}^3 + \hat{a}_{n-1}\hat{\lambda}^2 + \hat{a}_{n-2}\hat{\lambda} + \hat{a}_{n-3}) = 0 \tag{28}$$

We conclude from the above that $n-3$ of the eigen values are zeros, and the 3 non-zero ones can be computed from the residual cubic equation, once the coefficients are computed from the dual invariants of $\hat{g}$ (see Appendix A). We also note that $a_{n-1} = -n$, as it is the negative of the trace of g and $\hat{g}_{ii} = 1 + \varepsilon(2h_i)$. Hence the real part of the characteristic equation becomes

$$\lambda^3 - n\lambda^2 + a_{n-2}\lambda + a_{n-3} = 0 \tag{29}$$

This equation requires the computation of only two coefficients, which can be done very efficiently by computing the second and third invariants of g. Thus, by exploiting the algebraic structure of the problem, we ensure an analytic solution for rigid-body motion of arbitrary degree-of-freedom greater than 3. The 3 eigen vectors corresponding to the non-zero eigenvalues can be computed by the standard method. However, we will also have at least $n-3$ principal twists in the nullspace of $\hat{J}$, and to compute them, we first have to find the vectors $\dot{\theta}_i$ in the null space of g, i.e., solve the equation

$$g\dot{\theta}_i = 0\ i = 1,\ldots,n-3 \tag{30}$$

where g is a $n \times n$ real matrix, whose rank is at the most 3. Hence we can solve for the eigenvectors by row-reducing g to get 3 independent equations, and choose the $n-3$ free variables in each of them suitably. Finally, the corresponding principal twist is obtained by the mapping $\hat{\mathcal{V}}_i = \hat{J}\dot{\theta}_i$. These twists have only their dual parts, and they span the space of pure translational motions of the rigid-body.

Principal Twists for Parallel and Hybrid Manipulators

The above analysis is readily applicable for serial manipulators. In parallel manipulators, closed loop mechanisms, and hybrid manipulators, in addition to the actuated joints, we have one or more passive joints. A parallel device with m passive variables has m independent constraint equations denoted by

$$\eta(\theta, \phi) = 0 \tag{31}$$

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where η is a m-vector, θ, and ϕ, are $n-$ and $m-$ vectors denoting the actuated and passive variables respectively. Differentiating this equation with respect to time and rearranging [12], we get

$$J_{\eta\theta}\dot{\theta} + J_{\eta\phi}\dot{\phi} = 0 \tag{32}$$

At a non-singular configuration, $J_{\eta\phi}$ is invertible, and we can obtain the *passive joint rates* as

$$\dot{\phi} = -J_{\eta\phi}^{-1} J_{\eta\theta}\dot{\theta} \tag{33}$$

In general, equation (5) may be written in terms of the corresponding Jacobians as

$$\hat{\mathcal{V}} = J_{\omega\theta}\dot{\theta} + J_{\omega\phi}\dot{\phi} + \varepsilon(J_{v\theta}\dot{\theta} + J_{v\phi}\dot{\phi}) \tag{34}$$

Eliminating $\dot{\phi}$ using equation(32)

$$\begin{aligned} \hat{\mathcal{V}} &= (J_{\omega\theta} - J_{\omega\phi}J_{\eta\phi}^{-1}J_{\eta\theta})\dot{\theta} + \varepsilon(J_{v\theta} - J_{v\phi}J_{\eta\phi}^{-1}J_{\eta\theta})\dot{\theta} \\ &= \hat{J}_{\text{eq}}\dot{\theta} \end{aligned} \tag{35}$$

where from the dual Jacobian can be written as $\hat{J}_{\text{eq}} = (J_{\omega\theta} - J_{\omega\phi}J_{\eta\phi}^{-1}J_{\eta\theta}) + \varepsilon(J_{v\theta} - J_{v\phi}J_{\eta\phi}^{-1}J_{\eta\theta})$, and its columns may be considered as *equivalent input screws*. Once $\hat{J}_{eq}$ is formed, the rest of the analysis can proceed as shown earlier.

1 Analysis of Singularities

In the previous section, we have developed analytical expressions for principal twists. For analysis of singularities, we can readily use the above approach and obtain the principal singular directions. In this section, we discuss both the loss and gain kinds of singularities, while noting that the former type is possible only in purely serial manipulators, and the later in purely parallel manipulators, while hybrid manipulators having serially actuated branches connected in parallel can show both types of singularities [13].

Loss Type of Singularity The loss kind of singularity is said to occur when the manipulator end-effector fails to twist about certain screws in spite of full actuation. This results in the loss of one or more degrees-of freedom of the end-effector [12]. In our formulation, we treat the rotational degrees-of-freedom as decoupled from purely translational degrees-of-freedom, and hence the loss may occur in either of the following three ways:

1. Loss of rotational degree-of-freedom: The manipulator end-effector has $1,2$ or 3 rotational degrees-of-freedom depending upon the number of non-zero eigenvalues $\hat{g}$ has at a non-singular configuration. If at a singular configuration, m additional eigenvalues vanish[5], then we say that the manipulator has lost m rotational degrees-of-freedom. It may be noted that the corresponding pitch also vanishes, and hence the corresponding twist can reduce to a pure translation in the nullspace of $\hat{J}$ at that configuration. We look at the possiblities on a case by case basis.

   One-degree-of-freedom In this case, the principal screw reduces to a null vector, $0+\varepsilon 0$, unless the origninal degree-of-freedom was translational (as in a P-joint), in which case there is no loss of rotational degree-of-freedom possible.

   Two-degrees-of-freedom From the set of equations (16), it can be seen that only one of the $\hat{\lambda}$s ($\hat{\lambda}_2$ in particular) can vanish, under the condition $\sin^2\phi_{12} = 0$. The expression for $\hat{\lambda}_2$ in equation (17) is invalidated, as it was derived under the condition $\hat{\lambda}_i \neq 0$. The other eigen value can be obtained from equation(16) as $\hat{\lambda}_1 = 2(1+\varepsilon(h_1+h_2))$. The two principal twists in equation (19) collapse to $\hat{\mathcal{V}}_1^* = \frac{1}{\sqrt{2}}(\hat{\$}_1 + \hat{\$}_2)$ which gives the resultant rotational degree-of-freedom in this case, and $\hat{\mathcal{V}}_2^* = \frac{1}{\sqrt{2}}(\hat{\$}_1 - \hat{\$}_2)$, now forms the left nullspace of $\hat{J}$, signifying a translatory motion.

   Three-degrees-of-freedom In this case, there may be loss of one or two angular degrees-of-freedom, the conditions of the same are found from equation (21) as $c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1 = 0$ and $c_{12}^2 + c_{23}^2 + c_{31}^2 - 2c_{12}c_{23}c_{31} - 1 = 0 = (3 - c_{12}^2 - c_{23}^2 - c_{31}^2)$ respectively. As in the case of two-degrees-of-freedom rigid-body motion, the non-zero roots may be computed from equations from equations(21,24), the first of which reduces to a quadratic and a linear equation in λ in the two cases respectively. The eigenvectors of g can be computed symbolically, and therefrom the principal twists in the columnspace and null space of $\hat{J}$ can be obtained using equation (27) and equation (27) respectively. It may be noted here that the loss of one or two rotational degree-of-freedom results in those many principal twists being pushed from the column space into the left nullspace of $\hat{J}$, which has interesting consequences when degree-of-freedom is greater than 3.

   ***n*-degrees-of-freedom** $n > 3$ The treatment in this case follows exactly the case of three-degrees-of-freedom. We need to consider equation (28) instead of equation (21), and the conditions for loss of one or two rotational degree-of-freedom are $a_{n-3} = 0$, and $a_{n-3} = 0 = a_{n-2}$ respectively.

2. Loss of a translational degree-of-freedom: The number of pure translational degrees-of-freedom equal the number of

[5] m can be either 1 or 2. All the three eigenvalues can vanish only for a purely Cartesian manipulator, whose analysis can be done much more conveniently by looking at its linear velocity distribution in $\Re^3$.

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linearly independent pure dual vectors in the left null space of $\hat{J}$. These vectors span the space of pure translational velocities of the rigid-body and writing their dual parts as the columns of a $3 \times m$ real matrix $\mathcal{B}$, m being the number of such vectors, their distribution can be found from the eigen system of $\mathcal{B}\mathcal{B}^T$. The rank of this matrix determines the number of independent pure translations possible at the end-effector. It may be noted that the degeneracy of rotational motion, as described above, leads to the addition of a column to $\mathcal{B}$, but since the rank of $\mathcal{B}$ is limited to 3, the degeneracy of rotational motion does not lead to an additional translational degree-of-freedom if rank of $\mathcal{B}$ is already 3.

3. A combination of the above two types.

Gain Type of Singularity A parallel devices gain one or more degrees-of-freedom in the configuration space when one of the constraint Jacobians, $J_{\eta\phi}$ in equation (32) loses rank, and the number of degree-of-freedom gain equals the nullity of $J_{\eta\phi}$ (see, for example [14]). The *gained* passive motions lie in the nullspase of $J_{\eta\phi}$, and may be obtained by solving the equation

$$J_{\eta\phi}\dot{\phi}_i = 0,\ i = 1, \ldots, \text{nullity}(J_{\eta\phi}) \tag{36}$$

The effect of this gain is that the manipulator end-effector can now twist about one or more screws even with all the actuators locked. These twists may be obtained by setting $\dot{\theta} = 0$ in equation (34), and substituting the solutions of (36) for $\dot{\phi}$:

$$\hat{\mathcal{V}}_i = J_{\omega\phi}\dot{\phi}_i + \varepsilon J_{v\phi}\dot{\phi}_i \tag{37}$$

We can obtain the *gained screws* $\hat{\$}_i$ by normalising $\hat{\mathcal{V}}_i$. Any *gained twist* may be written as $\hat{\mathcal{V}}_{\text{gain}} = \sum_{i=1}^{\text{nullity}(J_{\eta\phi})} c_i\hat{\$}_i,\ c_i \in \Re$. This equation is comparable with $\hat{\$} = \hat{J}\dot{\theta}$, and under a similar constraint, $\sum_{i=1}^{\text{nullity}(J_{\eta\phi})} c_i^2 = 1$, the principal twists of the system can be extracted, wich will give us the principal basis for the space of gained twists at a singularity.

In the following section, we obtain the principal twists and singular directions for two manipulators to illustrate our approach.

Illustrative Examples

We demonstrate the theoretical developments by a 3-degree-of-freedom parallel manipulator, and a 6-degree-of-freedom hybrid 3-fingered gripper.

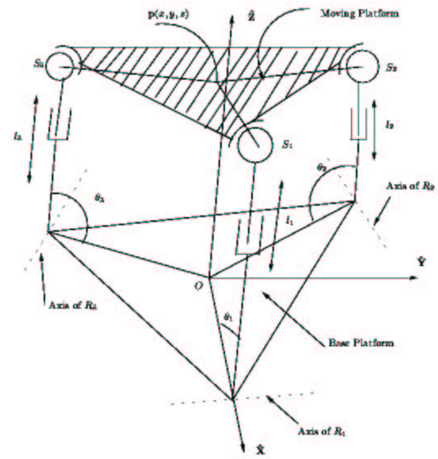

Figure 1. The RPSSPR-SPR parallel manipulator

A three-degree-of-freedom Parallel Manipulator

Figure 1 shows a three-loop, three-degree-of-freedom RPSSPR-SPR mechanism. The geometry chosen is same as in reference [6], which also presents the kinematic equations. The actuated variables are $\theta = (l_1, l_2, l_3)^T$, and the passive variables are $(\theta_1, \theta_2, \theta_3)^T$. The loop closure equations are obtained from the fact that the distance between the spherical joints, P_i, are constant and constraint equations, $\eta_k, k = 1,2,3$ are of the form

$$||P_i - P_j|| = a^2,\ i,j = 1,2,3, i \neq j \tag{38}$$

where a is the length of a side of the equilateral platform, which is assumed to be $\sqrt{3}/2$ units. The sides of the bottom triangle are $\sqrt{3}$ units each. The reference point on the moving platform is chosen as its centroid:

$$d = (x,y,z)^T = (1/3)(P_1 + P_2 + P_3) \tag{39}$$

In accordance with the theoretical development presented here, at a non-singular configuration defined by $l_1 = 0.5, l_2 = 1.0, l_3 = 2.0$, and corresponding passive variables $\theta_1 = 0.400$rad, $\theta_2 = 0.754$rad and $\theta_3 = 0.240$rad, the dual eigen values of $\hat{g}$ are are computed analytically, yielding the numerical values

$$\hat{\lambda}_1 = 19.62 + \varepsilon(-2.49)$$
$$\hat{\lambda}_2 = 1.17 + \varepsilon(-0.20)$$
$$\hat{\lambda}_3 = 0 + \varepsilon(0)$$

and the three principal pitches are given by

$$h_1^* = -0.06,\ h_2^* = -0.09,\ h_3^* = \infty$$

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Table 1. DH PARAMETERS OF THE jth FINGER

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_j
2	$\frac{\pi}{2}$	l_{j1}	0	ψ_j
3	0	l_{j2}	0	ϕ_j
4	0	l_{j3}	0	0

The principal twists, at this configuration, is given by

$$\hat{\mathcal{V}}_1 = (-1.71, -4.05, 0.54)^T + \varepsilon(0.61, 0.31, 1.91)^T$$
$$\hat{\mathcal{V}}_2 = (0.35, -0.01, 1.02)^T + \varepsilon(0.38, -0.34, -0.23)^T$$
$$\hat{\mathcal{V}}_3 = (0, 0, 0)^T + \varepsilon(0, 0, 1.22)^T$$

It may be noted, however, that one of the dual eigenvalues is always zero and the corresponding principal pitch h_3^* is infinite at all configurations. This fact can be resolved as follows: *the three degrees-of-freedom of the platform is partitioned into two angular degrees-of-freedom and one pure translatory motion*. Intuitively this is clear since the rotary joint axis in the base are in a plane and the top platform can be made to translate parallel to the **Z** axis without any angular motion by changing the leg lengths. However, one of the strengths of our approach is that we can analytically capture this *partitioning* of degrees of freedom.

Spatial 3-Fingered Gripper

We now analyze a 3-looped 6-degree-of-freedom hybrid spatial manipulator (see figure 2). The manipulator has 3-fingers, whose *DH parameters* (of the jth finger) are given in table 1.

The first two of the joints in each finger are actuated, and the last link is passive. Hence the active variable is given by $\theta = (\theta_1, \theta_2, \theta_3, \psi_1, \psi_2, \psi_3)^T$, and the passive variable given by $\phi = (\phi_1, \phi_2, \phi_3)^T$. The individual legs have the same architecture, and their link-lengths are taken such that $l_1 = 2l_2 = 4l_3 = 1$. The other architectural parameters are chosen as follows (see figure 2): $d = 1/2$, $h = \sqrt{3}/2, s = \sqrt{3}/2$. The third finger is rotated about the Y axis through an angle of $\pi/4$. The constraint equations are formed in a manner similar to that of the previous example, i.e., $\eta_k, k = 1,2,3$ has the form:

$$||P_i - P_j|| = s^2, i, j = 1,2,3, i \neq j \qquad (40)$$

These equations are solved after reducing them to a univariate polynomial using Sylvester's dialectic method.

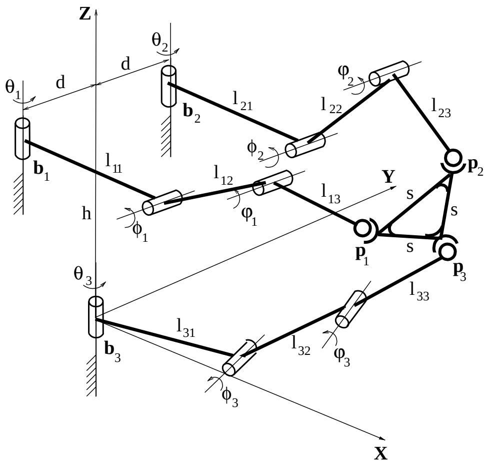

Figure 2. The Spatial 3-fingered Gripper

Non-Singular Configuration A typical non-singular configuration is obtained at $\theta = (0.2, 0.1, 0.3, -1., -1.2, 1)^T$,for which ϕ is solved as $(0.3679, 1.4548, 0.8831)^T$. The dual eigen values of $\hat{g}$ are computed as

$$\hat{\lambda}_1 = 0.0322 + \varepsilon(0.0881)$$
$$\hat{\lambda}_2 = 2.1000 + \varepsilon(5.4494)$$
$$\hat{\lambda}_3 = 1496.4500 + \varepsilon(1070.4100)$$

and the three principal pitches are given by

$$h_1^* = 1.3675,\ h_2^* = 1.2974,\ h_3^* = 0.3576\ h_4^* = \ h_5^* = \ h_6^* = \infty$$

The principal twists in the column space of $\hat{J}$, at this configuration, is given by

$$\hat{\mathcal{V}}_1 = (14.5577, 35.2770, 6.2392)^T + \varepsilon(15.1630, 7.4271, 8.2884)^T$$
$$\hat{\mathcal{V}}_2 = (-1.3407, 0.5223, 0.1723)^T + \varepsilon(-2.0408, -0.0341, 0.0269)^T$$
$$\hat{\mathcal{V}}_3 = (0.0089, -0.0352, 0.1759)^T + \varepsilon(-0.3566, -0.1899, 0.2307)^T$$

and the three pure dual principal twists are

$$\hat{\mathcal{V}}_4 = (0, 0, 0)^T + \varepsilon(-0.0934, 0.7672, 0.1079)^T$$
$$\hat{\mathcal{V}}_5 = (0, 0, 0)^T + \varepsilon(-0.0263, -0.4937, 0.0605)^T$$
$$\hat{\mathcal{V}}_6 = (0, 0, 0)^T + \varepsilon(0.1847, -0.0674, 0.2720)^T$$

Singular Configuration: Loss We look at the singular configuration where all the three fingers are stretched

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out [13], i.e., $\phi_i = 0$. The configuration is defined by $\theta = (0.0500, -0.0500, 0, -1.0998, -1.0998, 1.0026)^T$, $\phi = (0,0,0))^T$. We expect a loss of three degrees-of-freedom since all three fingers are in singular configuration, as we find that the pure dual principal twists vanish identically, signifying the loss of three translational degrees of freedom. The other three principal twists are given as

$$\hat{\mathcal{V}}_1 = (-1.7888, -27.7299, 0.0514)^T + \varepsilon(12.0096, -0.0062, -8.6253)^T$$
$$\hat{\mathcal{V}}_2 = (12.2507, -0.7909, -0.3526)^T + \varepsilon(1.7537, 0.0443, -1.2595)^T$$
$$\hat{\mathcal{V}}_3 = (0.0001, 0, 0.0050)^T + \varepsilon(-0.0048, -0.7853, -0.0035)^T$$

Singular Configuration: Gain The gain condition we use here is that one of the passive links lie in the plane of the moving platform. We derive such a configuration at $\theta = (0.0554, -0.0544, -0.8119, -0.8199, 0, 1.5708)^T$, $\phi = (-1.3300, -1.3300, 0.7854)^T$. There is a gain of a single degree-of-freedom, and the corresponding gained passive motion in the nullspace of $J_{\eta\phi}$ is obtained as $(0,0,1)^T$, indicating that ϕ_3 has an instantaneous motion even with actuators locked. The gained twist is essentially the 3rd column of $J_{\omega\phi} + \varepsilon J_{v\phi}$, whose analytical expression is of the form $(0, \omega_y, 0)^T + \varepsilon(v_x, 0, v_z)^T$. In particular, at the chosen architecture and configuration, the gained twist is $(0, 1/3, 0)^T + \varepsilon(-1/12, 0, 0)^T$.

Conclusion

In this paper, we have presented a dual-number based analytical approach for computation of the principal twists and singular directions of a manipulator end-effector for general multi-degree-of-freedom rigid-body motionU̇sing the dual vectors to represent twists, we pose the problem as an eigen-problem, and provide the solution of the same. The eigen-problem allows us to obtain compact analytical expressions for principal twists and also allows us to partition degrees of freedom in manipulators. We demonstrate that the eigen-problem allows us to recover the key results of classical screw theory, and also apply our formulations on a parallel and a hybrid manipulator to determine their principal twists and singularities. We hope that the concept of dual-valued metric and degree-of-freedompartitioning will be useful for manipulator design and analysis.

REFERENCES

[1] Murray, R. M., Li, Z., and Sastry, S. S., 1994. *A mathematical introduction to robotic manipulation*. CRC Press, Boca Raton.

[2] Ball, R. S., 1900. *A treatise on the theory of screws*. Cambridge University Press, Cambridge.

[3] Hunt, K. H., 1978. *Kinematic geometry of mechanisms*. Clarendon Press, Oxford.

[4] Fang, Y., and Huang, Z., 1998. "Analytical identification of the principal screws of the third order screw system". *Mechanisms and Machine Theory,* **33** [], pp. 987–992.

[5] McCarthy, J. M., 1986. "Dual orthogonal matrices in manipulator kinematics". *The International Journal of Robotics Research,* **5** [], pp. 45–51.

[6] Ghosal, A., and Ravani, B., 2001. "A differential-geometric analysis of singularities of point trajectories of serial and parallel manipulators". *Trans. of ASME, Journal of Mechanical Design,* **123** [], pp. 80–89.

[7] Wolfram, S., 1999. *The Mathematica Book*. Cambridge University Press, Cambridge.

[8] Veldkamp, G. R., 1976. "On the use of dual numbers, vectors, and matrices in instantaneous kinematics". *Mechanism and Machine Theory,* **11** [], pp. 141–156.

[9] Chevallier, D. P., 1991. "Lie algebras, modules, dual quarternions and algebraic methods in kinematics". *Mechanisms and Machine Theory,* **26** [], pp. 613–627.

[10] Brand, L., 1947. *Vector and tensor analysis*. John Wiley & Sons, New York.

[11] Hao, K., 1998. "Duak number method, rank of a screw system and generation of lie sub-algebras". *Mechanisms and Machine Theory,* **33** [], pp. 1063–1084.

[12] Gosselin, C., and Angeles, J., 1990. "Singularity analysis of closed loop kinematic chains". *IEEE Journal of Robotics and Automation,* **6** [], pp. 281–290.

[13] Hunt, K., Samuel, A., and McAree, P., 1991. "Special configurations of multi-finger multi-freedom grippers- a kinematic study". *International Journal of Robotics Research,* **10** [], pp. 123–134.

[14] Bandyopadhyay, S., and Ghosal, A., Accepted for publication. "Analysis of configuration space singularities of closed-loop mechanisms and parallel manipulators". *Mechanisms and Machine Theory* [].

[15] Artin, M., 1990. *Algebra*. Prentice-Hall Inc., Englewood Cliffs.

Appendix A: Dual Eigen Problem

The general eigen problem of a square dual matrix $\hat{A} = A + \varepsilon A_0$, $A, A_0 \in \Re^{n\times n}$, may be written as

$$\hat{A}\hat{x} = \hat{\lambda}\hat{x} \tag{41}$$

However, here we give the solution for only case encountered in the paper, i.e., $\hat{A}$ is symmetric, and $\hat{x} = x$ is real. Concentrating on the real part of equation (41), we get

$$(A - \lambda I)x = 0 \tag{42}$$

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which corresponds to the eigen problem of A, the real part of $\hat{A}$. We can compute λ, x from it using the usual techniques. To compute λ_0, the dual part of $\hat{\lambda}$, we equate the determinant $\det(\hat{A}-\hat{\lambda}I)$ to 0, and obtain the dual characteristic polynomial of the form:

$$\sum_{r=0}^{n} \hat{a}_r \hat{\lambda}^r = \sum_{r=0}^{n} (a_r + \varepsilon a_{r0})(\lambda^r + \varepsilon r \lambda^{r-1}\lambda_0) = 0 \qquad (43)$$

Equating the real and dual parts of the above equation to zero separately, we get

$$\sum_{r=0}^{n} a_r \lambda^r = 0$$
$$\sum_{r=1}^{n} a_r r \lambda^{r-1}\lambda_0 + \sum_{r=0}^{n} a_{r0}\lambda^r = 0, \quad \hat{a}_n = 1 \qquad (44)$$

Solution of the first of equations(44) gives, in general, n values of λ, and for each of these values, we can solve for the corresponding λ_0 *uniquely* from the second. In particular, λ_0 is given in terms of λ as

$$\lambda_0 = -\frac{\sum_{r=0}^{n} a_{r0}\lambda^r}{\sum_{r=1}^{n} a_r r \lambda^{r-1}} \qquad (\lambda \neq 0) \qquad (45)$$

We make a few important observations on the dual eigen values:
Zero roots: If $\lambda = 0$, λ_0 is also 0. This follows from the application of L'Hospital's rule to equation (45) at the limiting case $\lambda \to 0$.
Repeated roots: If the real part of the characteristic polynomial has a repeated root of order m, the corresponding repeated value of λ_0 has the expression

$$\lambda_0 = -\frac{\frac{d^{m-1}}{d\lambda^{m-1}}\left(\sum_{r=0}^{n} a_{r0}\lambda^r\right)}{\frac{d^{m-1}}{d\lambda^{m-1}}\left(\sum_{r=1}^{n} a_r r \lambda^{r-1}\right)} \quad \lambda \neq 0 \qquad (46)$$

It may be noted here that construction of the characteristic polynomial by expansion of the determinant of $(\hat{A}-\hat{\lambda}I)$ requires expensive symbolic computation. Alternatively, we can construct the polynomial by explicitly computing the invariants of $\hat{A}$, taking advantage of the principle of *permanence of identities*[6]. The formula required for computation of the invariants is obtained from the matrix-form of Newton's identities:

$$\hat{I}_k = \frac{(-1)^{k+1}}{k}\left(\mathrm{tr}\left(\hat{A}^k\right) + \sum_{i=1}^{k-1}(-1)^i \hat{I}_i \mathrm{tr}\left(\hat{A}^{k-i}\right)\right), \ \hat{I}_1 = \mathrm{tr}\hat{A}$$

where $k = 1,\ldots,n$. The coefficients, $\hat{a}_r$, are obtained from the dual invariants as $\hat{a}_r = (-1)^{n-r}\hat{I}_{n-r}, r = 1,\ldots,n-1$, while $\hat{a}_n = 1$ as characteristic polynomials are always monic.

Appendix B: Derivation of the Classical Screw-theory Results

Derivation of the Dual Ellipse, and Cylindroid

From equation (20), any admissible twist of the rigid-body can be written as a linear combination of the two principal twists as follows:

$$\hat{\mathcal{V}} = l_1 \hat{\mathcal{V}}_1^* + l_2 \hat{\mathcal{V}}_2^* \qquad (47)$$

where l_1, l_2 are two arbitrary real numbers. Under a unit speed constraint, $l_1^2 + l_2^2 = 1$, and we can introduce a real parameter $\theta \in [0, 2\pi]$, such that $l_1 = c_\theta$ and $l_2 = s_\theta$ We get finally

$$\hat{\mathcal{V}} = (||\omega_1^*||c_\theta, ||\omega_2^*||s_\theta, 0)^T + \varepsilon(||\omega_1^*||h_1^* c_\theta, ||\omega_2^*||h_2^* s_\theta, 0)^T \qquad (48)$$

The real part of equation equation (48) gives the parametric form of an ellipse which describes the distribution of the angular velocity in the local coordinate system, whose semi-major and semi-minor axes are given by $||\omega_1^*||, ||\omega_2^*||$. This motivates us to call the geometric quantity described by equation (48) a *dual ellipse*. The dual part of equation (48) yields more information about the distribution of screws. We can write the resultant screw axis as a linear combination of the principal screws: $\hat{\$} = (c_\theta, s_\theta, 0)^T + \varepsilon(h_1^* c_\theta, h_2^* s_\theta, 0)^T$ The pitch of the screw is obtained from equation (3) as $h = S \cdot S_0 = h_1^* \cos^2\theta + h_2^* \sin^2\theta$. The foot of the perpendicular from the origin to the axis of $\hat{\$}$ is obtained as

$$r_0 = (0, 0, (h_2^* - h_1^*)s_\theta c_\theta)^T = (0, 0, z)^T \qquad (49)$$

The above equation shows that the axis of $\hat{\mathcal{V}}$ is perpendicular to the local Z axis, and is at a distance z from the origin, which varies as $\sin 2\theta$. Writing $\cos\theta = \frac{x}{x^2+y^2}$, $\sin\theta = \frac{y}{x^2+y^2}$ and rearranging, we get

$$z(x^2 + y^2) + (h_1^* - h_2^*)xy = 0 \qquad (50)$$

The above is the classical equation of the cylindroid (see, for example, [3]).

[6]The principle states that matrix identities continue to hold even when the elements are from an arbitrary ring (Δ in our case) instead of a field [15].

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Derivation of the Dual Ellipsoid and the Pitch Hyperboloid

Similar to the twodegree-of-freedom case, any admissible twist for three-degree-of-freedom rigid-body motion can be written as a linear combination the principal twists, and under a unit speed constraint, $l^2+m^2+n^2=1$, we get

$$\begin{aligned}\hat{\mathcal{V}} &= l\hat{\mathcal{V}}_1^* + m\hat{\mathcal{V}}_2^* + n\hat{\mathcal{V}}_3^* \\ &= (||\omega_1^*||l, ||\omega_2^*||m, ||\omega_3^*||n)^T \\ &+ \varepsilon(||\omega_1^*||h_1^* l, ||\omega_2^*||h_2^* m, ||\omega_3^*||h_3^* n)^T\end{aligned} \tag{51}$$

The real part of equation (51) gives the parametric form of an ellipsoid, which describes the distribution of the angular velocity in the local coordinate system. The semi-axes of the ellipse are given by $||\omega_1^*||, ||\omega_2^*||$ and $||\omega_2^*||$, and its orientation is given by the corresponding principal axes. We call the geometric entity described by equation (51) a *dual ellipsoid*, which is a generalization of the dual ellipse to 3-degree-of-freedom rigid-body motion. To explore the dual part of equation (51), we take a look at the distribution of screws, which may be written as

$$\hat{\$} = (l,m,n)^T + \varepsilon(h_1^* l, h_2^* m, h_3^* n)^T \tag{52}$$

The pitch of the screw is obtained from equation (3) as $h = S \cdot S_0 = h_1{}^* l^2 + h_2{}^* m^2 + h_3{}^* n^2$. The foot of the perpendicular from the origin to the axis of $\hat{\$}$ is obtained as

$$r_0 = ((h_3^* - h_2^*)mn, (h_1^* - h_3^*)ln, (h_2^* - h_1^*)lm)^T \tag{53}$$

By setting either of l,m or n to zero, we can obtain a cylindroid in each case, whose axis is parallel to the Z, Y and X axis respectively. The moment of the axis about the origin is $Q_0 = S_0 - hS$. If $(x,y,z)^T$ be a point on the screw axis, then we can write $(x,y,z)^T \times S = Q_0$. Expanding this equation and rearranging, we obtain

$$\begin{pmatrix} h-h_1^* & -z & y \\ z & h-h_2^* & -x \\ -y & x & h-h_3^* \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0 \tag{54}$$

For the above homogeneous equations to have a non-trivial solution, we must have the determinant of the matrix as zero. This condition yields

$$\begin{aligned}x^2(h-h_1^*) &+ y^2(h-h_2^*) + z^2(h-h_3^*) \\ &+ (h-h_1^*)(h-h_2^*)(h-h_3^*) = 0\end{aligned} \tag{55}$$

The above equation gives the pitch h associated with a line passing through any arbitrary point (x,y,z). Equation (55) describes a hyperboloid of one sheet and is identical to the one obtained by Ball [2] (also given in Hunt [3]) when considering a three screw-system.

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