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A DUAL ELLIPSE IS A CYLINDROID

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ABSTRACT

In this paper, we take a relook at two-degree-of-freedom instantaneous rigid body kinematics in terms of dual numbers and vectors, and show that a dual ellipse is a cylindroid. The instantaneous angular and linear velocities of a rigid body is expressed as a dual velocity vector, and the inner product of two dual vectors, as a dual number, is used. We show that the tip of a dual velocity vector lies on a dual ellipse, and the maximum and minimum magnitude of the dual velocity vector, for a unit speed motion, can be obtained as eigenvalues of a positive definite, symmetric matrix whose elements are the dual numbers from the inner products. From the real and dual parts of the equation of the dual ellipse, we derive the equation of a cylindroid(Ball,1900).

INTRODUCTION

One of the classical result in instantaneous kinematics of rigid bodies is that for a general system of two screws, the lines along the instantaneous screws axis lie on a cubic ruled surface called the cylindroid, and there is a pitch associated with each ruling of the cylindroid(Ball,1900; Hunt, 1978; Bottema and Roth, 1979; Roth, 1984). In this paper, we take a relook at the instantaneous two- and three-degree-of-freedom kinematics of a rigid body in terms of dual numbers and vectors and notions from differential geometry. Dual numbers, first introduced by Clifford(Clifford, 1873), have been used extensively in kinematics(Yang, 1969; Veldkamp, 1976; Pennock and Yang, 1985; McCarthy, 1986), and we represent the linear and angular velocity of a rigid body moving in three dimensional space, $\Re^3$, as a dual vector. We use a definition of an inner product of two dual vectors as a *dual number* and this allows us to obtain the maximum and minimum of the dual velocity vector as eigenvalues of a *positive definite and symmetric* dual matrix. We show that for a unit speed motion constraint, the tip of the dual velocity vector lies on a dual ellipse. Next, from the dual and real parts of the parametric equations of the dual ellipse, we show that the dual ellipse is equivalent to the cylindroid with a pitch associated with ruling on the cylindroid. Furthermore, we show that the invariant determinant of the positive definite symmetric matrix is equivalent to the pitch of the screw lying along the central axis of the cylindroid. For three-degree-of-freedom motion, we show that the tip of the dual velocity vector lies on a dual elliposid which is equivalent to a line congruence with a pitch associated with each line. These are the main results of this paper.

MATHEMATICAL PRELIMINARIES

A dual number, $\hat{a}$, has the form $a + \epsilon a_0$ where a and a_0 are real numbers and $\epsilon^2 = \epsilon^3 ... = 0$. A dual vector, $\hat{\mathbf{A}}$, has the form $\mathbf{a} + \epsilon \mathbf{a}_0$ where $\mathbf{a}$ and $\mathbf{a}_0$ are real vectors in $\mathbf{R}^3$. The inner product of two dual vectors $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ can be defined as(Brand, 1947)

$$< \hat{\mathbf{A}}, \hat{\mathbf{B}} >= \mathbf{a} \cdot \mathbf{b} + \epsilon(\mathbf{a} \cdot \mathbf{b}_0 + \mathbf{b} \cdot \mathbf{a}_0) \tag{1}$$

It may be noted that the above inner product is invariant to the choice of the origin of the coordinate system used

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to describe $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$. The inner product is also different from the inner product defined in (von Mises, 1924)(or see the English translation (von Mises, 1996)) where the inner product has been defined as $(\mathbf{a} \cdot \mathbf{b}_0 + \mathbf{b} \cdot \mathbf{a}_0)$.

A line in $\Re^3$ can be described as a dual vector as

$$\hat{\mathcal{L}} = \mathbf{Q} + \epsilon \mathbf{Q}_0 \tag{2}$$

where $\mathbf{Q}$ denotes the direction of the line, and $\mathbf{Q}_0 = \mathbf{r} \times \mathbf{Q}$ is the moment of the line with $\mathbf{r}$ as the position vector of any point on the line from an origin. There are 4 independent parameters in $\mathbf{Q}$ and $\mathbf{Q}_0$ since $|\mathbf{Q}| = 1$ and $\mathbf{Q} \cdot \mathbf{Q}_0 = 0$.

The inner product of two lines follows from equation (1) and we have

$$\begin{aligned} < \hat{\mathcal{L}}_1, \hat{\mathcal{L}}_2 > &= \mathbf{Q}_1 \cdot \mathbf{Q}_2 + \epsilon(\mathbf{Q}_1 \cdot \mathbf{Q}_{02} + \mathbf{Q}_2 \cdot \mathbf{Q}_{01}) \\ &= \cos\phi - \epsilon d \sin\phi \end{aligned} \tag{3}$$

where ϕ and d are the angle and the shortest distance respectively between the two lines.

A screw can be also described as a dual vector as

$$\hat{\mathcal{S}} = \mathbf{S} + \epsilon \mathbf{S}_0 \tag{4}$$

where, in terms of line coordinates, $\mathbf{S} = \mathbf{Q}$ and $\mathbf{S}_0 = \mathbf{Q}_0 + h\mathbf{Q}$. In the previous equation h is called the pitch of the screw and is the ratio of the translational displacement to the rotational displacement. A screw has 5 independent parameters, i.e., 4 associated with the line along the screw and a pitch.

The inner product of two screws follows from equations (1) and (3), and we have

$$\begin{aligned} < \hat{\mathcal{S}}_1, \hat{\mathcal{S}}_2 > &= \mathbf{S}_1 \cdot \mathbf{S}_2 + \epsilon(\mathbf{S}_1 \cdot \mathbf{S}_{02} + \mathbf{S}_2 \cdot \mathbf{S}_{01}) \\ &= \cos\phi + \epsilon((h_1 + h_2)\cos\phi - d\sin\phi) \end{aligned} \tag{5}$$

where h_1 and h_2 are the pitches associated with the two screws.

The inner product of a screw with itself, from equation (5), can be written as

$$< \hat{\mathcal{S}}_i, \hat{\mathcal{S}}_i >= 1 + \epsilon(2h_i) \tag{6}$$

The angular velocity, $\boldsymbol{\omega}$, and the linear velocity, $\mathbf{v}$, of a point on a rigid body, can be together considered as a dual vector of the form

$$\hat{\mathcal{V}} = \boldsymbol{\omega} + \epsilon \mathbf{v} \tag{7}$$

The quantity $\hat{\mathcal{V}}$ have also been called a twist and a motor, and can be thought of as a screw together with a magnitude. In terms of line coordinates, $\hat{\mathcal{V}}$ is given as

$$\hat{\mathcal{V}} = |\boldsymbol{\omega}|(\mathbf{Q} + \epsilon(\mathbf{Q}_0 + h\mathbf{Q})) \tag{8}$$

where $|\boldsymbol{\omega}|$ is the magnitude of the angular velocity vector.

MULTI-DEGREE-OF-FREEDOM MOTION OF A RIGID BODY

A general rigid body displacement can be expressed as a 4×4 matrix of homogeneous coordinates(Bottema and Roth, 1979) or as 3×3 dual orthogonal matrices(Yang, 1969). By using the properties of these matrices, and differentiating the matrix elements with respect to time, one can obtain expressions for left- and right-invariant velocities of the moving rigid body(see for example (Samuel et al., 1991)). For our purpose of studying instantaneous kinematics, we assume that the angular and linear velocity of the rigid body can be expressed, using dual vectors, as

$$\hat{\mathcal{V}} = \boldsymbol{\omega} + \epsilon \mathbf{v} = \sum_{i=1}^{n} \hat{\mathcal{S}}_i \dot{\theta}_i \tag{9}$$

where $\boldsymbol{\omega}$, $\mathbf{v}$ are the angular and linear velocity respectively, $\hat{\mathcal{S}}_i,\ i = 1, 2, .., n$ are n independent screws expressed using dual vectors, and $\dot{\theta}_i,\ i = 1, 2, ..., n$ are the time derivatives of the n motion parameters. The above equation can also be written in terms of a dual Jacobian matrix as

$$\hat{\mathcal{V}} = [\hat{J}]\dot{\boldsymbol{\theta}} \tag{10}$$

where $\dot{\boldsymbol{\theta}}$ is the vector $(\dot{\theta}_1, ..., \dot{\theta}_n)^T$ and the i'th column of $[\hat{J}]$ is the screw $\hat{\mathcal{S}}_i$.

Using the inner product between two screws (see equations (5) and (6)), we can write

$$< \hat{\mathcal{V}}, \hat{\mathcal{V}} >= \dot{\boldsymbol{\theta}}^T [\hat{g}] \dot{\boldsymbol{\theta}} \tag{11}$$

where the matrix elements $\hat{g}_{ij}$ are the inner products $< \hat{\mathcal{S}}_i, \hat{\mathcal{S}}_j >,\ i, j = 1, 2, .., n$. The elements of the matrix $[\hat{g}]$ are dual numbers and the matrix $[\hat{g}]$ is symmetric and positive definite. In the study of point trajectories(Ghosal and Roth, 1987) or differential geometry of curves and surfaces, one defines similar inner products which are, however, *real* numbers for point trajectories. The inner products (in language of differential geometry) define a **metric** in the tangent space(Millman and Parker, 1977) and allows one to define distance and angle in the tangent space.

We make the following observations, analogous to point trajectories, from the definition of $[\hat{g}]$ and $[\hat{J}]$:

- If $[\hat{g}]$ is non-singular(i.e, $\det[\hat{g}] \neq 0$), then we can write

$$\hat{\mathcal{V}}^T([\hat{J}][\hat{g}]^{-1})([\hat{J}][\hat{g}]^{-1})^T\hat{\mathcal{V}} = \dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} \tag{12}$$

  The matrix $([\hat{J}][\hat{g}]^{-1})([\hat{J}][\hat{g}]^{-1})^T$ is symmetric and positive definite, and for a constraint of the form $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1$, the tip of the dual velocity vector $\hat{\mathcal{V}}$ lies on a dual ellipsoid. For two-degree-of-freedom motions the tip of the dual vector $\hat{\mathcal{V}}$ lies on a dual ellipse.
- The maximum and minimum values of $\hat{\mathcal{V}}^2$ subject to constraint $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1$ can be obtained by solving the eigenvalue problem

$$[\hat{g}]\dot{\boldsymbol{\theta}} - \hat{\lambda}\dot{\boldsymbol{\theta}} = 0 \tag{13}$$

  For a two-degree-of-freedom motion the elements of the matrix $[\hat{g}]$, in terms of the pitches(h_1, h_2), the angle, ϕ, and the distance, d, between the two screws, $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, are

$$\begin{aligned} \hat{g}_{11} &= 1 + \epsilon(2h_1) \\ \hat{g}_{12} = \hat{g}_{21} &= \cos\phi + \epsilon((h_1 + h_2)\cos\phi - d\sin\phi) \\ \hat{g}_{22} &= 1 + \epsilon(2h_2) \end{aligned} \tag{14}$$

  The eigenvalues are given by

$$\begin{aligned} \hat{\lambda}_1 &= 2\cos^2\phi/2(1 + \epsilon((h_1 + h_2) - d\tan(\phi/2)) \\ \hat{\lambda}_2 &= 2\sin^2\phi/2(1 + \epsilon((h_1 + h_2) + d\cot(\phi/2)) \end{aligned} \tag{15}$$

  The determinant, $\det[\hat{g}]$, is given by

$$\det[\hat{g}] = \sin^2\phi(1 + \epsilon 2(h_1 + h_2 + d\cot\phi)) \tag{16}$$

  It may be noted that the eigenvalues are dual numbers and the $\det[\hat{g}]$ is zero if $\sin\phi = 0, n\pi$, i.e., the axis of the two screws are parallel. In such a case the two screws are not independent and we have a one-degree-of-freedom motion of the rigid body.
  The maximum and minimum $|\hat{\mathcal{V}}|$ are the square roots of the maximum and minimum eigenvalues of $[\hat{g}]$. For a two-degree-of-freedom motion, $\sqrt{\hat{\lambda}_1}$ and $\sqrt{\hat{\lambda}_2}$, are given as

$$\begin{aligned} \sqrt{\hat{\lambda}_1} &= \sqrt{2}\cos(\phi/2)(1 + 0.5\epsilon(h_1 + h_2 - d\tan(\phi/2))) \\ \sqrt{\hat{\lambda}_2} &= \sqrt{2}\sin(\phi/2)(1 + 0.5\epsilon(h_1 + h_2 + d\cot(\phi/2))) \end{aligned} \tag{17}$$

  The directions of the maximum and minimum velocities are related to the eigenvectors of $[\hat{g}]$ and are along the vectors $[\hat{J}]\dot{\boldsymbol{\theta}}_i$, $i = 1, 2, 3$ where $\dot{\boldsymbol{\theta}}_i$ is the eigenvectors corresponding to eigenvalue $\hat{\lambda}_i$.
  For a three-degree-of-freedom motion of the rigid body, the matrix $[\hat{g}]$ is 3×3 and it has three eigenvalues. The maximum, minimum, and intermediate values of $|\hat{\mathcal{V}}|$ are the square roots of the three eigenvalues and are along the three principal axes of the dual ellipsoid.
  It may be noted that the normalization $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = k^2$ scales the eigenvalues without changing the the shape of the dual velocity ellipse or ellipsoid.
- The dual area (volume in case of ellipsoid) is proportional to $\sqrt{\det[\hat{g}]}$. The dual area (volume in case of ellipsoid) is an invariant since it is proportional to $\det[\hat{g}]$. For a two-degree-of-freedom motion we have

$$\sqrt{\det[\hat{g}]} = \sin\phi(1 + \epsilon(h_1 + h_2 + d\cot\phi)) \tag{18}$$

In the next section, we give a geometrical interpretation of the dual ellipse and ellipsoid and their dual area and volume by considering the real and dual parts of the equations.

INTERPRETATION OF DUAL ELLIPSE AND ELLIPSOID

In the case of a point trajectory, the parametric equation of an ellipse in a plane are $(x,\ y) = (a\cos\theta,\ b\sin\theta)$, where a and b are the lengths of the major and minor axis. For a two-degree-of-freedom rigid-body motion, we use dual numbers and the parametric equation of a dual ellipse can be written as

$$\begin{aligned} \hat{X} &= \sqrt{\hat{\lambda}_1}\cos\theta \\ \hat{Y} &= \sqrt{\hat{\lambda}_2}\sin\theta \end{aligned} \tag{19}$$

where the local X and Y axis are chosen along eigenvectors(in this case lines) corresponding to the eigenvalues of $[\hat{g}]$. The Z axis is along a line perpendicular to lines along

X and Y axis(see figure 1). The quantities on the left-hand side of the equation (19), $\hat{X}$ and $\hat{Y}$ denote the tip of the dual vector $\hat{\mathcal{V}}$ given as in equation (8). In the chosen coordinate system, with (x, y, z') denoting the coordinates of any point on the line along $\hat{\mathcal{V}}$, we can write

$$\begin{aligned} \mathbf{Q} &= (\cos\theta, \sin\theta, 0)^T = (1/\sqrt{x^2+y^2})(x, y, 0)^T \\ \mathbf{Q}_0 &= (0,0,z')^T \times \mathbf{Q} = (-z'\sin\theta, z'\cos\theta, 0)^T \end{aligned} \tag{20}$$

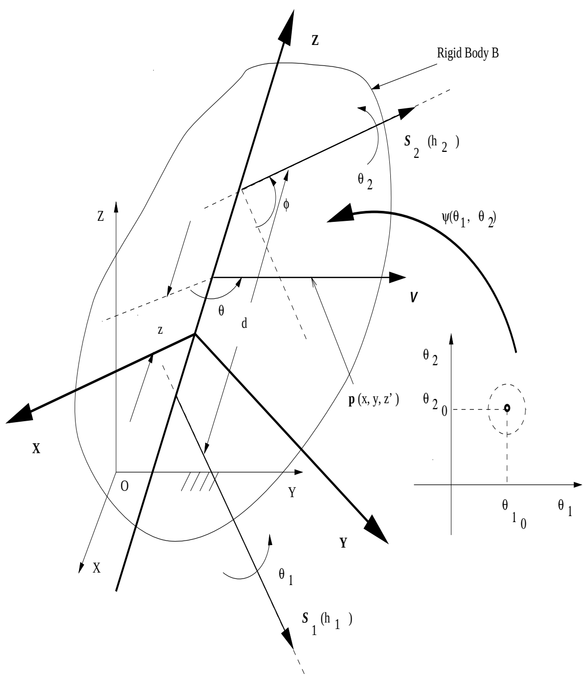

Figure 1. Two-degree-of-freedom motion of a rigid body

Hence, we get

$$\begin{aligned} \hat{X} &= |\boldsymbol{\omega}_x|(\cos\theta + \epsilon(h\cos\theta - z'\sin\theta)) \\ \hat{Y} &= |\boldsymbol{\omega}_y|(\sin\theta + \epsilon(h\sin\theta + z'\cos\theta)) \end{aligned} \tag{21}$$

The square root of the eigenvalues of $[\hat{g}]$ are given in equation (17) and denoting the dual parts by h_1^* and h_2^*[2], we can write

$$\begin{aligned} \sqrt{\hat{\lambda}_1} &= \sqrt{2}\cos(\phi/2)(1+\epsilon h_1^*) \\ \sqrt{\hat{\lambda}_2} &= \sqrt{2}\sin(\phi/2)(1+\epsilon h_2^*) \end{aligned} \tag{22}$$

[2] h_1^* and h_2^* are the maximum and minimum or principal pitches.

Substituting equations (21) and (22) in equation (19), and separating the real and dual part, we get

$$\begin{aligned} |\boldsymbol{\omega}_x|\cos\theta &= \sqrt{2}\cos(\phi/2)\cos\theta \\ |\boldsymbol{\omega}_y|\sin\theta &= \sqrt{2}\sin(\phi/2)\sin\theta \\ h_1^* &= h - z'\tan\theta \\ h_2^* &= h + z'\cot\theta \end{aligned} \tag{23}$$

The last two equations give

$$\begin{aligned} z' &= -\frac{1}{2}(h_1^* - h_2^*)\sin 2\theta \\ h &= \frac{1}{2}(h_1^* + h_2^*) + \frac{1}{2}(h_1^* - h_2^*)\cos 2\theta \end{aligned} \tag{24}$$

It may be noted that the second equation in (24) is similar to the expression of the pitch of a line lying on a cylindroid(see, for example,(Hunt, 1978), p. 97). The equation of the cylindroid can be recovered from the first two equations in (23), and by using the first equation in (24). We get

$$|\boldsymbol{\omega}_x||\boldsymbol{\omega}_y|\sin\theta\cos\theta = |\boldsymbol{\omega}_x||\boldsymbol{\omega}_y|(\frac{-z'}{h_1^* - h_2^*}) = \sin\phi(\frac{xy}{x^2+y^2}) \tag{25}$$

which can be rewritten as

$$(\frac{|\boldsymbol{\omega}_x||\boldsymbol{\omega}_y|}{\sin\phi})z'(x^2+y^2) + (h_1^* - h_2^*)xy = 0 \tag{26}$$

If we rescale the Z coordinate such that $z = \frac{|\boldsymbol{\omega}_x||\boldsymbol{\omega}_y|}{\sin\phi}z'$, we have the classical equation of a cylindroid(see, for example, (Hunt, 1978))

$$z(x^2+y^2) + (h_1^* - h_2^*)xy = 0 \tag{27}$$

It may be noted that if $\phi = 0, \pi$, or either or both the magnitudes, $|\boldsymbol{\omega}_x|$ and $|\boldsymbol{\omega}_y|$, are zero, we no longer have a cylindroid. This is intuitively consistent since, in these cases, we no longer have a two-degree-of-freedom motion.

It is well known (see, for example, (Samuel et al., 1991), p. 461) that the Lie bracket of two twist, $\hat{\mathcal{V}}_a$ and $\hat{\mathcal{V}}_b$(with magnitudes $|\boldsymbol{\omega}_a|$, $|\boldsymbol{\omega}_b|$, pitches h_1 and h_2 and at an angle ϕ), is another twist lying along the central axis of the cylindroid whose pitch and magnitude are given as $h_1 + h_2 + d\cot\phi$, $|\boldsymbol{\omega}_a||\boldsymbol{\omega}_b|\sin\phi$ respectively. The dual area of the dual ellipse is proportional to $\sqrt{\det[\hat{g}]}$ given in equation (18). We can observe from equation (18) that the dual part of $\sqrt{\det[\hat{g}]}$ is identical to the pitch of this central screw and the real part is a scaled version of the magnitude. Hence the dual area of the dual ellipse is proportional to the pitch and magnitude of the twist lying along the central axis of the cylindroid.

The above results can be summarized as follows:

The dual ellipse traced by the tip of the dual velocity vector, of a rigid body undergoing two-degree-of-freedom motion, is geometrically a cylindroid. The dual area of the ellipse is proportional to the pitch and magnitude of the invariant twist along the central axis of the cylindroid.

For a three-degree-of-freedom rigid-body motion, the tip of the dual vector $\hat{\mathcal{V}}$ traces a dual ellipsoid. The parametric equation of a dual ellipsoid are

$$\begin{aligned} \hat{X} &= \sqrt{\hat{\lambda}_1}\cos\alpha\sin\beta \\ \hat{Y} &= \sqrt{\hat{\lambda}_2}\sin\alpha\sin\beta \\ \hat{Z} &= \sqrt{\hat{\lambda}_3}\cos\beta \end{aligned} \tag{28}$$

where the X, Y and Z axis are chosen along eigenvectors(in this case lines) corresponding to the three eigenvalues of the 3×3 dual matrix $[\hat{g}]$. For any line in this chosen coordinate system, we have

$$\begin{aligned} \mathbf{Q} &= (l,m,n)^T = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)^T \\ \mathbf{Q}_0 &= (x,y,z)^T \times \mathbf{Q} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} (l,m,n)^T \end{aligned} \tag{29}$$

where $(l,m,n)^T$ is a unit vector of direction cosines and $(x,y,z)^T$ is any point on the line.

Using equation (4) and the above equation, we get

$$\begin{aligned} \hat{X} &= |\boldsymbol{\omega}_x|(l + \epsilon(lh + yn - mz)) \\ \hat{Y} &= |\boldsymbol{\omega}_y|(m + \epsilon(mh + zl - nx)) \\ \hat{Z} &= |\boldsymbol{\omega}_z|(n + \epsilon(nh + xm - yl)) \end{aligned} \tag{30}$$

The eigenvalues of the 3×3 dual matrix $[\hat{g}]$ are the roots of a cubic equation and are real since $[\hat{g}]$ is symmetric. The square root of the eigenvalues can be written in a symbolic form as

$$\begin{aligned} \sqrt{\hat{\lambda}_1} &= A_1(1+\epsilon h_1^*) \\ \sqrt{\hat{\lambda}_2} &= A_2(1+\epsilon h_2^*) \\ \sqrt{\hat{\lambda}_3} &= A_3(1+\epsilon h_3^*) \end{aligned} \tag{31}$$

where $h_i^*, A_i,\ i = 1,2,3$, can be obtained from solving the characterestic cubic polynomial of $[\hat{g}]$.

Substituting equations (30), (31) in equation (28), and equating the dual parts, we get three equations

$$\begin{aligned} h_1^* l &= lh + yn - mz \\ h_2^* m &= mh + zl - nx \\ h_3^* n &= nh + xm - yl \end{aligned} \tag{32}$$

which can be written in a matrix form as

$$\begin{bmatrix} h-h_1^* & -z & y \\ z & h-h_2^* & -x \\ -y & x & h-h_3^* \end{bmatrix} (l,m,n)^T = 0 \tag{33}$$

For the above homogeneous equations to have a non-trivial solution, we must have the determinant of the matrix as zero. This condition yields

$$x^2(h-h_1^*)+y^2(h-h_2^*)+z^2(h-h_3^*)+(h-h_1^*)(h-h_2^*)(h-h_3^*) = 0 \tag{34}$$

The above equation gives the pitch h associated with a line passing through any arbitrary point (x,y,z). The above equation describes a hyperboloid of one sheet and is identical to the one obtained by Ball(Ball,1900) (also given in Hunt(Hunt, 1978)) when considering a three screw-system.

We can also eliminate h from equations (32), taking two at a time. We get

$$\begin{aligned} (h_1^* - h_2^*)lm - (yn - mz)m + (zl - nx)l &= 0 \\ (h_2^* - h_3^*)mn - (zl - nx)n + (xm - yl)m &= 0 \end{aligned} \tag{35}$$

The above two equations represent two quadratic line complexes and when they are satisfied, we automatically satisfy

$$(h_3^* - h_1^*)nl - (xm - yl)l + (yn - mz)n = 0 \tag{36}$$

From the above analysis, we recover the well known result(Hunt, 1978; Nayak and Roth, 1981) that the instantantenous screw axis of a general three-degree-of-freedom motion of a rigid body lie on a line congruence obtained as an intersection of three quadratic line complexes.

The determinant of $[\hat{g}]$ is given by

$$\begin{aligned} \det[\hat{g}] = 1 &+ 2\cos\phi_{12}\cos\phi_{23}\cos\phi_{31} - \cos^2\phi_{12} - \cos^2\phi_{23} \\ &- \cos^2\phi_{31} + \epsilon[2(h_1+h_2+h_3)(1 + \end{aligned}$$

$$2\cos\phi_{12}\cos\phi_{23}\cos\phi_{31} - \cos^2\phi_{12} - \cos^2\phi_{23} - \cos^2\phi_{31})$$
$$+2d_{12}\sin\phi_{12}(\cos\phi_{12} - \cos\phi_{23}\cos\phi_{31})$$
$$+2d_{23}\sin\phi_{23}(\cos\phi_{23} - \cos\phi_{12}\cos\phi_{31})$$
$$+2d_{31}\sin\phi_{31}(\cos\phi_{31} - \cos\phi_{12}\cos\phi_{23})] \quad (37)$$

where ϕ_{ij}, d_{ij} are the angles and distances between the lines along dual vectors $\hat{\mathcal{S}}_i$, $\hat{\mathcal{S}}_j$ respectively. Unlike the two-degree-of-freedom case, it is not clear what the dual volume of the dual ellipsoid, $\sqrt{\det[\hat{g}]}$, represent in $\Re^3$.

Finally, it may be noted that the real part of equation (30) yield

$$\begin{aligned} |\boldsymbol{\omega}_x| l &= A_1 l \\ |\boldsymbol{\omega}_y| m &= A_2 m \\ |\boldsymbol{\omega}_z| n &= A_3 n \end{aligned} \quad (38)$$

which imply, as in the two-degree-of-freedom case, that we need to scale the local coordinate axis X, Y and Z appropriately.

CONCLUSION

In this paper, the instantaneous kinematics of two- and three-degree-of-freedom rigid body motion has been studied in terms of dual numbers and vectors. The instantaneous linear and angular velocities of a rigid body are described by dual vectors, A inner product between two dual vectors as a dual number has been used and we have shown that the tip of the dual velocity vector lies on a dual ellipse or dual elliposid for two and three-degree-of-freedom motion respectively, and the maximum and minimum values of the dual velocity vectors are the eigenvalues of a positive, definite dual matrix of inner products. Furthermore, from the real and dual parts of the parametric equations, we have shown that the dual ellipse is equivalent to a cylindroid and the dual ellipsoid is equivalent to a line congruence with a pitch associated with each line on the cylindroid or the congruence.

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