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DIFFERENTIAL GEOMETRIC ANALYSIS OF SINGULARITIES OF POINT TRAJECTORIES OF SERIAL AND PARALLEL MANIPULATORS

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ABSTRACT

In this paper, we present a differential-geometric analysis of singularities of point trajectories of two and three-degree-of-freedom serial and parallel manipulators. At non-singular configurations, the first order local properties are characterized by the metric coefficients, and, geometrically, by the shape and size of a velocity ellipse and ellipsoid for two and three-degree-of-freedom motions respectively. At singular configurations, the definition of a metric is no longer valid and the velocity ellipsoid degenerates to an ellipse, a line or a point, and the area or the volume of the velocity ellipse or ellipsoid becomes zero. The second and higher order properties, such as curvature, are also not defined at a singularity. In this paper, we use the rate of change of the area or volume to characterize the singularities of the point trajectory. For parallel manipulators, singularities may lead to either loss or gain of one or more degrees-of-freedom. For loss of degree of freedom, the ellipsoid degenerates to an ellipse, a line, or a point as in serial manipulators. For a gain of degree-of-freedom the singularities can be pictured as growth to lines, ellipses, and ellipsoids. The method presented gives a clear geometric picture as to the possible directions and magnitude of motion at a singularity and the local geometry near a singularity. The theoretical results are illustrated with the help of a general spatial 2R manipulator and a three-degree-of-freedom RPSSPR-SPR parallel manipulator.

INTRODUCTION

Evaluation of singularities plays an important role in several aspects of robotics including design, trajectory planning, and control. Much of the past research in the area of singularities of manipulators have been related to the study of manipulator configurations resulting in singularities (see, for example, (Wang and Waldron, 1987; Litvin et. al., 1990; Hunt, 1986; Martinez et. al., 1994)), enumeration and classification of kinematic structure of manipulators and mechanisms with singular configurations(see, for example, (Lipkin and Pohl, 1991; Karger, 1995; Karger, 1996; Sugimoto et. al., 1982; Gosselin and Angeles, 1990; Litvin et. al., 1986; Merlet, 1991)), novel designs of manipulators and wrists, including use of redundancy, that would exclude singularities from the useful portion of the workspace(see, for example, (Stanisic and Duta, 1990; Tchnon and Matuszok, 1995; Shamir, 1990)), analysis of singular sets for serial manipulators(see, for example, (Karger, 1996; Tchnon and Muszynski, 1997)) and planning of trajectories at singularities(see, for example, (Chevallereau, 1996; Lloyd, 1996; Nenchev et. al., 1996; Necchev and Uchiyama, 1996)). In practice, one is confronted with commercial manipulator geometries that usually do have singularities in their workspace. One therefore has to develop a better understanding of the geometric nature of singularities to develop path planning algorithms which can avoid singularities or recover from a singularity once it is encountered. There have been some studies in this area (see, for example, (Martinez et. al., 1994; Sardis et. al., 1992)). There has also been some analysis of singularities of point trajectories and their bifurcations(Kieffer, 1992; Kieffer, 1994). This paper is related to these later works in that it deals with analysis of singularities of point trajectories. The paper differs from these works in that a) it develops a geometric method for local characterization of sin-

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gularities which is not restricted to one-degree-of-freedom motions, and b) the method is applied to both serial and parallel manipulators. The main idea of this paper is based on the concept of a *metric* on a manifold and the associated concepts of a velocity ellipsoid or ellipse(Ghosal and Roth, 1987), whose size and shape characterizes the local first order properties of non-singular point trajectories. At singular positions, the definition of the metric is no longer valid and the velocity ellipsoid degenerates to an ellipse, line or a point. For second and higher order properties, we consider the rate of change of the volume of the ellipsoid since the familiar concepts of curvature etc. are no defined at a singularity. We extend the concept of the velocity ellipsoid and the rate of change of volume to parallel manipulators. The results of this paper, in addition, to their theoretical interests in kinematics of manipulators, have applications in trajectory planning and control.

The paper is organized as follows: In section 2, we briefly present the differential-geometric concepts of a metric and the associated velocity ellipse and ellipsoid and then discuss its usefulness for differential analysis of point trajectories traced out by non-redundant, serial and parallel manipulators. In section 3, we discuss singularities of point trajectories traced out by two and three-degree-of-freedom serial and parallel manipulators by considering the rate of change of the volume of the velocity ellipsoid. In section 4, we illustrate our theory with the help of a general spatial 2R and a three-degree-of-freedom RPSSPR-SPR parallel manipulator. Finally, in section 5, we present the conclusions.

MATHEMATICAL FORMULATION

The trajectory traced by a point in a moving rigid body can be expressed as a set of equations giving the coordinates of the point in the terms of the n independent motion parameters. Assuming that the coordinates of the point are the Cartesian coordinates, (x, y, z), and the n independent motion parameters are denoted by $\theta_i,\ i = 1, 2, ..., n$, the set of equations can be written in a symbolic form as

$$(x, y, z)^T = \boldsymbol{\psi}(\theta_1, ..., \theta_n) \tag{1}$$

In the case of a manipulator, the vector function $\boldsymbol{\psi}$ depends on the point chosen on the end-effector, the geometry and structure of the manipulator and its dimensions. The function $\boldsymbol{\psi}$ and can be thought of as a mapping which takes points in the motion parameter space, $(\theta_1, ..., \theta_n)$, to points in the 3D (Euclidean) space of the motion. These equations are the familiar *direct kinematics* equations for a manipulator.

In the case of serial manipulators with n degrees of freedom, the n motion parameters are the rotations or translations at the joints and are independently actuated. In the case of parallel manipulators and closed-loop mechanisms, not all the n motion parameters are actuated and m of them may be passive. In such a case the degree of freedom of the parallel manipulator or the closed-loop mechanism is $(n - m)$, and in addition to the above equations, we have m independent constraint equations of the form

$$\boldsymbol{\eta}(\theta_1, ..., \theta_n) = \mathbf{0} \tag{2}$$

where $\boldsymbol{\eta}(.)$ denotes the m constraint functions $\eta_i(.), i = 1, 2, .., m$.

In this paper, we restrict ourselves to non-redundant manipulators, i.e., $n \leq 3$ for serial manipulators and $(n-m) \leq 3$ for parallel manipulators and closed-loop mechanisms.

Differential kinematics of serial manipulators at non-singular points

In the case of serial manipulators, the velocity at any point, $\mathbf{p}$, on the point trajectory can be written as

$$\mathbf{v} = \sum_{i=1}^{n} \boldsymbol{\psi}_i \dot{\theta}_i \tag{3}$$

where $\dot{\theta}_i$ is the time derivative of θ_i and $\boldsymbol{\psi}_i$ is the first partial derivative of $\boldsymbol{\psi}$ with respect to θ_i or $\partial\boldsymbol{\psi}/\partial\theta_i$. The partial derivatives are evaluated at $\mathbf{p}$.

The above equation can also be written in terms of the matrix of first partial derivatives or the Jacobian matrix as

$$\mathbf{v} = [J(\boldsymbol{\psi})]_{\mathbf{p}} \dot{\boldsymbol{\theta}} \tag{4}$$

where $\boldsymbol{\theta}$ is the vector $(\theta_1, ..., \theta_n)^T$ and $[J(\boldsymbol{\psi})]_{\mathbf{p}}$ is the Jacobian matrix of $\boldsymbol{\psi}$ evaluated at $\mathbf{p}$. By varying $\dot{\boldsymbol{\theta}}$, we can get any arbitrary velocity $\mathbf{v}$ at $\mathbf{p}$. It is more instructive to look at the variation of $\mathbf{v}$ with a normalizing constraint of the form $\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}} = k^2$. For $k = 1$, we have a *unit speed motion* and by varying k one can get all possible velocities, $\mathbf{v}$, at the point $\mathbf{p}$ under consideration.

The dot product of the velocity with itself can be written as

$$\mathbf{v} \cdot \mathbf{v} = \dot{\boldsymbol{\theta}}^T [g] \dot{\boldsymbol{\theta}} \tag{5}$$

where the matrix elements g_{ij} are the dot products $(\boldsymbol{\psi}_i \cdot \boldsymbol{\psi}_j), i, j = 1, 2, .., n$. The matrix $[g]$, equal to $[J(\boldsymbol{\psi})]^T [J(\boldsymbol{\psi})]$, is symmetric and positive definite and its elements(in the language of differential geometry) define a **metric** in the tangent space(Millman and Parker, 1977). We make the following observations from the definition of $[g]$ and equation (5):

- If $[g]$ is non-singular(i.e, $\det[g] \neq 0$), then we can write

$$\mathbf{v}^T([J][g]^{-1})([J][g]^{-1})^T\mathbf{v} = \dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} \tag{6}$$

  The matrix $([J][g]^{-1})([J][g]^{-1})^T$ is symmetric and positive definite, and for a constraint of the form $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1$, the tip of the velocity vector $\mathbf{v}$ lies on an ellipsoid. For two-degree-of-freedom motions the tip of the vector $\mathbf{v}$ lies on an ellipse in the tangent plane.
- The maximum and minimum values of $\mathbf{v}^2$ subject to constraint $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = 1$ can be obtained by solving

$$\partial\mathbf{v}^{*2}/\partial\dot{\theta}_1 = \partial\mathbf{v}^{*2}/\partial\dot{\theta}_2 = 0 \tag{7}$$

  where $\mathbf{v}^{*2}$ is given as

$$\mathbf{v}^{*2} = \dot{\boldsymbol{\theta}}^T[g]\dot{\boldsymbol{\theta}} - \lambda(\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} - 1) \tag{8}$$

  The above reduces to solving the eigenvalue problem

$$[g]\dot{\boldsymbol{\theta}} - \lambda\dot{\boldsymbol{\theta}} = 0 \tag{9}$$

  The maximum and minimum $|\mathbf{v}|$ in terms of the maximum and minimum eigenvalues of $[g]$, λ_{max} and λ_{min}, are given as

$$|\mathbf{v}|_{max} = \sqrt{\lambda_{max}}$$
$$|\mathbf{v}|_{min} = \sqrt{\lambda_{min}} \tag{10}$$

  The directions of the maximum and minimum velocities are related to the eigenvectors of $[g]$ and are along the vectors $[J(\boldsymbol{\psi})]\dot{\boldsymbol{\theta}}_i$, $i = 1, 2, 3$ where $\boldsymbol{\theta}_i$ is the eigenvectors corresponding to eigenvalue λ_i.
  The maximum, minimum, and intermediate values of $|\mathbf{v}|$ are along the three principal axes of the ellipsoid and determine the shape of the ellipsoid(for an ellipse there are only a maximum and a minimum). If the normalization $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = k^2$ is used then the maximum and minimum values are scaled by k but the shape of the velocity ellipsoid(or ellipse) doesn't change.
- The volume (area in case of ellipse) is proportional to $\sqrt{\det[g]}$. For an ellipse the area is $k\pi\sqrt{\det[g]}$ and for an ellipsoid, the volume is given by $k\ (2\pi/3)\sqrt{\det[g]}$. It may be noted that $\det[g]$ is equal to the product of the eigenvalues.
- Yoshikawa(Yoshikawa, 1985) introduced an useful manipulability measure $\sqrt{\det([J][J]^T)}$ which has been used extensively by several researchers for resolution of redundancy(Nakamura, 1991). Several authors have also used the singular values of $[J]$ to analyze the first order properties. However, $\det[g]$ and $\det([J][J]^T)$, (and the square root of eigenvalues of $[g]$ and the singular values of $[J]$) have significant differences. We list some of them below.
  1) The elements of $[g]$ and $\det[g]$ define distance, angle and elemental area(or volume) on a manifold whereas the matrix $[J][J]^T$ comes from a least squares type of solutions to a system of linear equations[2]. In this paper, our approach is from a differential-geometric perspective and not from linear algebra, and the focus of the paper is on singularities where the definition of a metric on a manifold breaks down and $\det[g]$ equals zero.
  2) As shown later, the quantity $\det[g]$ naturally occurs when we consider second and higher-order properties of a manifold such as the Gaussian curvature. It is not clear how the the manipulability measure can be used to study second and higher properties of a manifold.
  3) The elements of $[g]$ and $\det[g]$ are well-defined for all non-redundant manipulators and mechanisms. The manipulability measure is more suited for redundant manipulators since $\det([J][J]^T)$ is zero for a non-redundant spatial 2R manipulator. It may be noted that $\det([J]^T[J])$ is always zero for a redundant manipulator.

We next discuss parallel manipulators and closed-loop mechanisms.

Differential kinematics of parallel manipulators at non-singular points

As mentioned before, in the case of a parallel manipulator or closed-loop mechanism not all the n joints are actuated and there are m constraint equations of the form (2). We denote the $(n-m)$ actuated joints by the vector $\boldsymbol{\theta}$ and the m passive joints by the vector $\boldsymbol{\phi}$. The velocity

[2]The solution to a set of linear equations(Golub and Van Loan, 1989) $Ax = b$, $A \in \Re^{m\times n}$, $x \in \Re^n$, $b \in \Re^m$, is given as $x = (A^TA)^{-1}A^Tb$ when $m \geq n$ and $x = A^T(AA^T)^{-1}b$ when $m \leq n$. $(A^TA)^{-1}A^T$ and $A^T(AA^T)^{-1}$ are called the pseudo-inverse of A.

vector, $\mathbf{v}$, at any point $\mathbf{p}$ on the point trajectory traced by a parallel manipulator or a closed-loop mechanism can be written as

$$\mathbf{v} = [J]\dot{\boldsymbol{\theta}} + [J^*]\dot{\boldsymbol{\phi}} \tag{11}$$

where the columns of $[J]$ are the partial derivatives of $\boldsymbol{\psi}$ with respect to the $n-m$ actuated joint variables θ_i and the columns of $[J^*]$ are the partial derivatives of $\boldsymbol{\psi}$ with respect to the m passive variables ϕ_i. The dimensions of $[J]$ and $[J^*]$ are $(dim(\mathbf{v}) \times (n-m))$ and $(dim(\mathbf{v}) \times m)$ respectively.

By differentiating the m constraints equations (2), we get

$$\mathbf{0} = \sum_{i}^{m} \boldsymbol{\eta}_i \dot{\theta}_i \tag{12}$$

where $\boldsymbol{\eta}_i$ is the partial derivative $\partial\boldsymbol{\eta}/\partial\theta_i$. Again assuming that the first $(n-m)$ θ_i's are actuated and the rest are passive, we can rearrange these m equations in the form

$$\mathbf{0} = [K]\dot{\boldsymbol{\theta}} + [K^*]\dot{\boldsymbol{\phi}} \tag{13}$$

where the columns of $[K]$ are the first $(n-m)$ $\boldsymbol{\eta}_i$'s and the columns of $[K^*]$ are the last m $\boldsymbol{\eta}_i$'s. It may be noted that $[K^*]$ is always a square matrix of dimension $m \times m$.

Assuming that $\det[K^*] \neq 0$, we can solve for $\dot{\boldsymbol{\phi}}$ from equation (13) as

$$\dot{\boldsymbol{\phi}} = -[K^*]^{-1}[K]\dot{\boldsymbol{\theta}} \tag{14}$$

and on substituting in equation (11), we get

$$\mathbf{v} = ([J] - [J^*][K^*]^{-1}[K])\dot{\boldsymbol{\theta}} \tag{15}$$

where $\boldsymbol{\theta}$ is the vector of $n-m$ actuated joint variables.

Equation (15) is similar to equation (3) for serial manipulators and we can define a metric for a parallel manipulator, $[g^*]$, as

$$[g^*] = ([J] - [J^*][K^*]^{-1}[K])^T([J] - [J^*][K^*]^{-1}[K]) \tag{16}$$

The metric $[g^*]$ is symmetric and positive definite[3] and we can again state that for a normalization constraint of the form $\dot{\boldsymbol{\theta}}^T\dot{\boldsymbol{\theta}} = k^2$, the tip of the velocity vector lies on an ellipsoid(ellipse). The shape and size of the ellipsoid(ellipse) is again determined by the eigenvalues of the matrix $[g^*]$.

[3] $[g^*]$ is clearly symmetric since it is of the form $[A]^T[A]$. It is also positive definite provided that $\det[K^*] \neq 0$ and $([J] - [J^*][K^*]^{-1}[K])$ is non-singular.

Higher order properties at non-singular points

The velocity vector, $\mathbf{v}$, derived from the first derivative of the mapping function or the elements of the matrix $[g]$ (or $[g^*]$) determine the first order properties of the point trajectories. For the second order properties we consider the acceleration vector given in terms of the first and second partial derivatives of $\boldsymbol{\psi}$ as

$$\mathbf{a} = \sum_{i=1}^{n} \boldsymbol{\psi}_i \ddot{\theta}_i + \sum_{i,j=1}^{n} \boldsymbol{\psi}_{ij} \dot{\theta}_i \dot{\theta}_j \tag{17}$$

When the the number of independent θ_i is two and the point trajectory is in $\Re^3$, we can define a normal vector $\mathbf{n}$ at the non-singular point, $\mathbf{p}$, as

$$\mathbf{n} = \frac{\boldsymbol{\psi}_1 \times \boldsymbol{\psi}_2}{|\boldsymbol{\psi}_1 \times \boldsymbol{\psi}_2|} \tag{18}$$

where the partial derivatives are evaluated at $\mathbf{p}$. It may be noted that the denominator is the same as $\sqrt{\det[g]}$ and is non-zero at a non-singular point. The normal component of acceleration, a_n, is given by

$$a_n = \sum_{i,j=1}^{2} L_{ij} \dot{\theta}_i \dot{\theta}_j \tag{19}$$

where the L_{ij}'s are the dot products $\boldsymbol{\psi}_{ij} \cdot \mathbf{n}, i, j = 1, 2$.

The tangential components are given as

$$a_{t_k} = \ddot{\theta}_k + \sum_{i,j=1}^{2} \Gamma_{ij}^k \dot{\theta}_i \dot{\theta}_j \quad k = 1, 2 \tag{20}$$

where the six Γ_{ij}^k are known as *Christoffel symbols*(Millman and Parker, 1977) and are given as

$$\Gamma_{ij}^k = \sum_{l=1}^{2} (\boldsymbol{\psi}_{ij} \cdot \boldsymbol{\psi}_l) g^{lk} \quad i, j, k = 1, 2 \tag{21}$$

where g^{lk} is the (l, k) element of $[g]^{-1}$. It may be noted that the six *Christoffel symbols* can be expressed as partial derivatives of the metric coefficients g_{ij}'s(or g_{ij}^*'s for parallel manipulators)(Millman and Parker, 1977).

The second order properties are completely determined by the elements g_{ij}, L_{ij}, and Γ_{ij}^k. The local geometry of the surface is determined by the Gaussian curvature given by

$$K = \frac{\det[L]}{\det[g]} \tag{22}$$

and K can also be derived only in terms of the metric coefficients, g_{ij} and their first partial derivatives(g^*_{ij} and its first partial derivatives for parallel manipulators)(Millman and Parker, 1977). It is also well known from differential geometry, that a surface is locally *flat* if all the L_{ij}'s are zero ($K = 0$), a surface is locally *parabolic* if $K = 0$ but all L_{ij}'s are not zero, a surface is locally *elliptic* if $K > 0$, and a surface is locally *hyperbolic* if $K < 0$.

The six Christoffel symbols determine the nature of the curves *on* the surface and a curve is said to be a *geodesic* if the geodesic curvature is zero. It can be shown (Millman and Parker, 1977) that the geodesic curvature is zero if the tangential acceleration is zero, and if $\theta_1(t)$, $\theta_2(t)$ satisfies the non-linear differential equations

$$\ddot{\theta}_k + \sum_{i,j=1}^{2} \Gamma^k_{ij}\dot{\theta}_i\dot{\theta}_j = 0 \quad k = 1,2 \tag{23}$$

the point trajectory is a geodesic.

In case the point trajectory is a solid region in $\Re^3(n = 3)$, the tangent space is of the same dimension as the space of the motion and there is no notion of a normal vector. One can consider 2D sections (surfaces) of the solid region and compute the Gaussian curvature of each of the sections by computing the appropriate L_{ij}'s and the $\det[g]$'s. Another general approach is to compute the first and second partial derivatives of g_{ij}'s(g^*_{ij} in case of parallel manipulators) with respect to the motion parameters and define the Riemannian curvature tensor(Millman and Parker, 1977)

$$R_{ijkl} = (1/2)(\frac{\partial^2 g_{il}}{\partial\theta_j\partial\theta_k} + \frac{\partial^2 g_{jk}}{\partial\theta_i\partial\theta_l} - \frac{\partial^2 g_{ik}}{\partial\theta_j\partial\theta_l} - \frac{\partial^2 g_{jl}}{\partial\theta_i\partial\theta_k}) + g^{st}(\Gamma_{jsk}\Gamma_{itl} - \Gamma_{jsl}\Gamma_{itk}) \tag{24}$$

where g^{st} is the (s,t) element of $[g]^{-1}$ and

$$\Gamma_{jkl} = (1/2)(\frac{\partial g_{jk}}{\partial\theta_i} + \frac{\partial g_{ik}}{\partial\theta_j} - \frac{\partial g_{ij}}{\partial\theta_k}) \tag{25}$$

The above rank four tensor has the properties of curvature since one can show that if all the R_{ijkl} vanishes everywhere, then the volume of the velocity ellipsoid is constant everywhere similar to the case when the mapping ψ is linear and the surface is flat. The Gaussian curvature of a 2D subspace can also be computed as

$$K = \frac{R_{1212}}{\det[g]} \tag{26}$$

It may be noted that in the expression of the Gaussian curvature (and Riemannian curvature) and the Christoffel symbols, $\det[g]$($\det[g^*]$ for parallel manipulators and closed-loop mechanisms) appears in the denominator. If $\det[g]$ or $\det[g^*]$ is zero, then we have a *singularity* and at a singularity, the Christoffel symbols and the Gaussian or the Riemannian curvatures are not defined. The expressions have an indeterminate form 0/0, and hence, we cannot characterize the geometry of the point trajectory, at a singularity, using Γ^k_{ij}, K or R_{ijkl}. In the next section, we analyze the behavior of $\det[g]$, to characterize the singularities in two and three-degree-of-freedom motions.

SINGULARITY ANALYSIS

For a single degree-of-freedom motion, the singularities of the point trajectory(in this case a curve in $\Re^3$) are points in $\Re^3$. These have been classified as ordinary, bifurcations and isolated singularities(Kieffer, 1992; Kieffer, 1994). For two and three-degree-of-freedom motions, it is not attractive to study the singularities of the infinite number of curves which make up the surface or the solid region. In this paper, we propose to study the nature of the singularity by analyzing the behavior of $\det[g]$(or $\det[g^*]$ and $\det[K^*]$ for a parallel manipulator) near a singularity. There are two main intuitive reasons for this approach.

1) As discussed in section 2, the elements g_{ij} define a *metric* on the manifold and the quantity $\det[g]$ is related to the shape and size of the velocity ellipsoid. At a singularity the metric is not defined, $\det[g]$ is zero, and the ellipsoid degenerates to an ellipse, a line or a point, and the volume of the ellipsoid becomes zero.
2) For second order properties, in the expressions for the Christoffel symbols, the Gaussian and Riemannian curvature, $\det[g]$ appears in the denominator and at a singularity $\det[g]$ is zero. Hence, it is intuitive to consider the behavior of $\det[g]$ near the zero. Although the second order properties of a surface, such as curvature, is determined by both the numerator and denominator, clearly the denominator plays an important part near its zero.

Singularity analysis for serial manipulators

As mentioned above the condition for singularity in serial manipulators is $\det[g] = 0$. To study the behavior of $\det[g]$ near a singularity, we denote the values of $(\theta_1, .., \theta_n)^T$ which satisfy the singularity condition, $\det[g] = 0$, by $\boldsymbol{\theta}^*$, and expand det$[g]$ in a Taylor series about $\boldsymbol{\theta}^*$. We can write

$$\delta(\det[g]) = \det[g] + \sum_{i=1}^{n} \frac{\partial \det[g]}{\partial \theta_i} \delta\theta_i + (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \det[g]}{\partial \theta_i \, \partial \theta_j} \delta\theta_i \delta\theta_j + ... \quad (27)$$

where all the partial derivatives on the right-hand side are evaluated at $\boldsymbol{\theta}^*$.

At a singularity the first term on the right-hand side, det$[g]$, is zero and hence we get up to the first order

$$\delta(\det[g]) = \sum_{i=1}^{n} \frac{\partial \det[g]}{\partial \theta_i} \delta\theta_i \quad (28)$$

The vector $(\frac{\partial \det[g]}{\partial \theta_1}, ..., \frac{\partial \det[g]}{\partial \theta_n})^T$ is the gradient of det$[g]$ and gives the direction of the maximum change of volume of the ellipsoid. If any of the partial derivatives, $\frac{\partial \det[g]}{\partial \theta_i}$ is zero, then we have a *second order* singularity in the θ_i direction. If the second or higher derivatives are zero, then we have *third or higher order* singularity.

Since det$[g]$ is the product of the eigenvalues, we can write

$$\frac{\partial \det[g]}{\partial \theta_i} = \lambda_1 \lambda_2 \frac{\partial \lambda_3}{\partial \theta_i} + \lambda_1 \lambda_3 \frac{\partial \lambda_2}{\partial \theta_i} + \lambda_2 \lambda_3 \frac{\partial \lambda_1}{\partial \theta_i} \quad (29)$$

At a singularity, we have the following cases:

- One eigenvalue is zero, say $\lambda_1 = 0$ and λ_2,λ_3 non-zero. In this case, the ellipsoid degenerates to an ellipse, and the velocity along the direction corresponding to the zero eigenvalue will be zero. From equation (29), we can see that the only nonzero term is $\lambda_2 \lambda_3 \frac{\partial \lambda_1}{\partial \theta_i}$. Hence $\frac{\partial \det[g]}{\partial \theta_i}$ greater than zero imply that the rate of change of velocity along the direction corresponding to the zero eigenvalue is positive. Likewise if $\frac{\partial \det[g]}{\partial \theta_i}$ is less than zero, the rate of change of velocity along the direction corresponding to zero eigenvalue is negative, and for $\frac{\partial \det[g]}{\partial \theta_i} = 0$, the rate of change of velocity is zero. In the last case we have a second order singularity.
- Two eigenvalues zero, say $\lambda_1 = \lambda_2 = 0$ and λ_3 non-zero. In such a case, the ellipsoid degenerates to a line, and the velocity in the plane spanned by the eigenvectors corresponding to zero eigenvalues will be zero. In such a case, we can observe from the definition of det$[g]$, that $\frac{\partial \det[g]}{\partial \theta_i} = \lambda_3 \frac{\partial \lambda_1 \lambda_2}{\partial \theta_i}$. The sign of $\frac{\partial \det[g]}{\partial \theta_i}$ determine whether the rate of change of velocity vector, in the plane spanned by the eigenvectors corresponding to zero eigenvalues, is positive, negative or zero. In the last case we have a second-order singularity.
- All three eigenvalues zero. In such a case the ellipsoid degenerates to a point, and no motion is possible in any direction.

It may be mentioned that researchers(Chevallereau, 1996; Lloyd, 1996) have pointed out that motion may be possible with non-zero acceleration. To analyse the relationship between the derivatives of det$[g]$ and the acceleration at a singularity, we write

$$\frac{\partial \det[g]}{\partial \theta_i} = 2g\Gamma^r_{ir} = \sum_{l=1}^{n} (\boldsymbol{\psi}_{ir} \cdot \boldsymbol{\psi}_l) G_{rl} \quad (30)$$

where G_{lr} is the co-factor of g_{lr} in det$[g]$ and Γ^r_{ir} are the Christoffel symbols(Millman and Parker, 1977). Hence, if any of the partial derivatives of det$[g]$ with respect to θ_i is zero, and the corresponding cofactor is not zero, then the dot product term $\sum_{l=1}^{n} (\boldsymbol{\psi}_{ir} \cdot \boldsymbol{\psi}_l)$ will be zero. A consequence of the dot-product term being zero, from equation (17), can be that the acceleration along certain directions is zero. This is illustrated in detail in the singularity analysis of a general 2R manipulator considered in a later section.

The matrix of second partial derivatives determine the second order properties at a singular point. If the matrix $\frac{\partial^2 \det[g]}{\partial \theta_i \, \partial \theta_j}$ evaluated at $\boldsymbol{\theta}^*$ has a positive determinant then the singularity is elliptic. If the matrix has a negative determinant, then the singularity is hyperbolic and if the determinant of the matrix is zero, then the singularity is flat or parabolic. In terms of the second derivatives of λ_i, the sign indicates whether λ_i is at a maximum, minimum or an inflexion point.

Singularity analysis for parallel manipulators

For a parallel manipulator, if the det$[K^*] \neq 0$, then we can analyze the singularity corresponding to det$[g^*] = 0$. We can replace $[g]$ by $[g^*]$ in the above analysis and can compute the first and higher partial derivatives of det$[g^*]$. When det$[K^*] = 0$, we have a singularity associated with the *gain* of one or more degree-of-freedom(Gosselin and Angeles, 1990). This can be seen readily from equation (13) as follows:

We assume that all the $(n - m)$ actuated joints are locked or $\dot{\boldsymbol{\theta}}$ is set to zero. If det$[K^*] \neq 0$, all the passive parameters $\dot{\boldsymbol{\phi}}$ become zero from equation (13) and as expected we get a structure. From linear algebra, we know that the homogeneous equation, $[K^*]\dot{\phi} = 0$, can have non-trivial solutions (not all $\dot{\phi}_i$ zero) when the matrix $[K^*]$ is singular or $\det[K^*] = 0$. This implies that the structure can have motion at the joint with non-zero $\dot{\phi}_i$ and gains one or more degrees-of-freedom at a singularity corresponding to matrix $[K^*]$ loosing rank.

A geometric picture of the singularity corresponding to the gain of degree of freedom is as follows:

With all the actuated joints locked($\dot{\boldsymbol{\theta}} = 0$), at non-singular positions, we get $\dot{\boldsymbol{\phi}} = 0$ from equation (13). Since $\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\phi}}$ are both zero, from equation (11), we get $\mathbf{v}^T\mathbf{v} = 0$. Hence *at a non-singular position* with actuated joints locked, we can think of the velocity distribution as an ellipsoid of zero size. At a singularity, the matrix $[K^*]$ looses rank. If the rank is $(m-1)$ then we can extract the eigenvector of $[K^*]$ corresponding to the zero eigenvalue. Let the eigenvector corresponding to the zero eigenvalue be $\dot{\boldsymbol{\phi}}_1$. Since, $c_1\dot{\boldsymbol{\phi}}$ is also an eigenvector with c_1 any scaling constant, from equation (11), we get

$$\mathbf{v} = c_1[J^*]\dot{\boldsymbol{\phi}}_1 \tag{31}$$

and there can be motion along the direction of $[J^*]\dot{\boldsymbol{\phi}}_1$. In this case, we can think of the zero velocity ellipsoid "growing" into a line. If the rank of the matrix $[K^*]$ is $(m-2)$, then with a similar reasoning we can get

$$\mathbf{v} = c_1[J^*]\dot{\boldsymbol{\phi}}_1 + c_2[J^*]\dot{\boldsymbol{\phi}}_2 \tag{32}$$

where $\dot{\boldsymbol{\phi}}_1$, $\dot{\boldsymbol{\phi}}_2$ are the two eigenvectors corresponding to the two zero eigenvalues of $[K^*]$ and c_1, c_2 are the two scaling constants. If we normalize $c_i, i = 1,2$, to be between -1 and $+1$ (or $c_1^2 + c_2^2 = 1$), then the tip of the velocity vector traces an ellipse[4]. If the rank of $[K^*]$ is $(m-3)$, then the tip of the velocity vector will lie on an ellipsoid. If the rank is less than $(m-3)$, then we have a situation similar to the redundant serial manipulator.

The ellipsoids or their degenerate forms associated with the gain of degree-of-freedom in parallel manipulators and closed-loop mechanisms can be analyzed by considering the matrix $[J^{*T}J^*]$. In particular, we can find the direction and magnitude of the maximum and minimum velocities from the eigenvalues of the matrix $[J^{*T}J^*]$.

The second order properties of the singularities associated with loss of degree of freedom, in case of parallel manipulators and closed-loop mechanisms, can be analyzed in a manner similar to that of a serial manipulator by considering $[g^*]$ instead of $[g]$. For loss of degree-of-freedom in parallel manipulators and closed loop mechanisms, we have to consider the Taylor series expansion of $\det[K^*]$.

In the next section, we look at two cases to illustrate the theory developed in the last two sections.

CASE STUDIES OF SERIAL AND PARALLEL MANIPULATORS

In this section, we illustrate the theory developed in the two previous section by means of two examples, namely a general two-degree-of-freedom serial manipulator and a three-degree-of-freedom parallel manipulator described in (Lee and Shah, 1988).

A general 2R manipulator

Figure 1 shows a two-degree-of-freedom manipulator with two revolute(R) joints of general geometry. The joint variables are θ_1 and θ_2 and the point trajectory is traced by the point $(x,y,z)^T$ in $\Re^3$. Hence the point trajectory is a surface in $\Re^3$. In terms of the link lengths, a_{ij}'s, link offsets, d_i's, and the twists α_{ij}, the mapping function $\boldsymbol{\psi}$ can be written as[5]

$$(x,y,z)^T = \boldsymbol{\psi}(\theta_1,\theta_2) = d_1\mathbf{S}_1 + a_{12}\mathbf{a}_{12} + d_2\mathbf{S}_2 + a_{23}\mathbf{a}_{23} \tag{33}$$

where

$$\begin{aligned}
\mathbf{S}_1 &= (0,\ 0,\ 1)^T \\
\mathbf{a}_{12} &= (c_1,\ s_1,\ 0)^T \\
\mathbf{S}_2 &= (s_1 s\alpha_{12},\ -c_1 s\alpha_{12},\ c\alpha_{12})^T \\
\mathbf{a}_{23} &= [(c_1c_2 - s_1s_2c\alpha_{12}),\ (s_1c_2 + c_1s_2c\alpha_{12}),\ s\alpha_{12}s_2]^T
\end{aligned} \tag{34}$$

The partial derivatives $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are given by

$$\begin{aligned}
\boldsymbol{\psi}_1 &= a_{12}(-s_1,c_1,0)^T + d_2(s\alpha_{12}c_1, s\alpha_{12}s_1, 0)^T \\
&\quad + a_{23}(-s_1c_2 - c_1s_2c\alpha_{12}, c_1c_2 - s_1s_2c\alpha_{12}, 0)^T \\
\boldsymbol{\psi}_2 &= a_{23}(-c_1s_2 - s_1c_2c\alpha_{12}, -s_1s_2 + c_1c_2c\alpha_{12}, s\alpha_{12}c_2)^T
\end{aligned} \tag{35}$$

The coefficients of the metric $[g]$ are given by

$$\begin{aligned}
g_{11} &= a_{12}^2 + d_2^2 s\alpha_{12}^2 + a_{23}^2(c_2^2 + s_2^2 c\alpha_{12}^2) + 2a_{23}a_{12}c_2 \\
&\quad - 2d_2 a_{23} c\alpha_{12} s\alpha_{12} s_2 \\
g_{12} &= a_{23}(a_{12}c\alpha_{12}c_2 - d_2 s\alpha_{12}s_2 + a_{23}c\alpha_{12}) \\
g_{22} &= a_{23}^2
\end{aligned} \tag{36}$$

[4] c_1 and c_2 are similar to $\dot{\theta}_1$ and $\dot{\theta}_2$ in the differential kinematics of serial manipulators, and as in section 2, we can easily prove that the tip of $\mathbf{v}$ lies on an ellipse.

[5] We will use the symbols c_i, s_i etc. to represent $\cos(\theta_i)$, $\sin(\theta_i)$ etc. respectively throughout the paper.

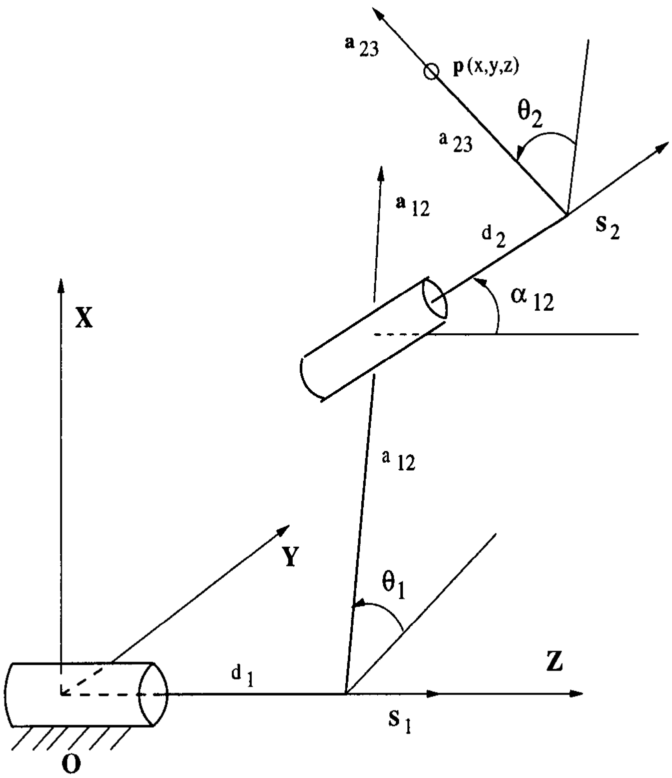

Figure 1. A schematic of a general 2R manipulator

and

$$\det[g] = g_{11}g_{22} - g_{12}^2 = a_{23}^2[(a_{12} + a_{23}c_2)^2 s\alpha_{12}^2 + (a_{12}c\alpha_{12}s_2 + d_2 s\alpha_{12}c_2)^2] \quad (37)$$

It may be noted that the g_{ij}'s and the determinant of $[g]$ are independent of θ_1. This is because the metric coefficients are always independent of translation and rotation of a coordinate system and the effect of θ_1 is equivalent to a rotation of the fixed coordinate system.

The Gaussian curvature is given as

$$K = \frac{a_{23}s\alpha_{12}^2 c_2 g_{11}g_{22}(a_{12} + a_{23}c_2) - (a_{23}s_{12}c_2 g_{12})^2}{(\det[g])^2} \quad (38)$$

At (θ_1, θ_2) given by $(0, 0)$ degrees, the velocity ellipse, in three sectional views and a 3D view, is shown in figure 2. We have assumed $\alpha_{12} = 45^\circ$, $a_{12} = d_2 = 1$ and $a_{23} = 1.5$. The maximum and minimum values of the magnitude of velocity for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ are $\sqrt{7.9776}$ and $\sqrt{1.0224}$ along the principal axis of the ellipse as shown in figure 2. The ellipse is in tangent plane with normal along $(0.9285, -0.2626, 0.2626)^T$ and the maximum and minimum velocities are along vectors $(-0.6417, -2.7143, -0.4456)^T$ and $(0.2971, 0.878, -0.9625)^T$ in the tangent plane. The Gaussian curvature at $(0, 0)$ degrees is 0.3092 implying that the point $(0, 0)$ is elliptic. At the point $(0, 120)$ degrees the Gaussian curvature is -0.5791 and the surface is hyperbolic. One can also obtain points where the surface is parabolic.

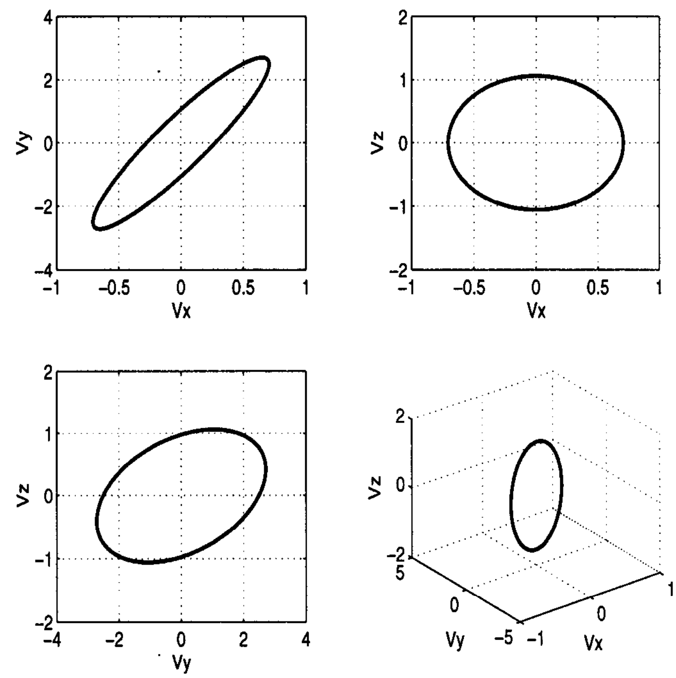

Figure 2. Velocity ellipse at a non-singular point $(\theta_1, \theta_2) = (0, 0)^\circ$

At a singular point, $\det[g] = 0$ and this implies

$$a_{12} + a_{23}c_2 = 0 \text{ or } s\alpha_{12} = 0$$
$$c_{12}c\alpha_{12}s_2 + d_2 s\alpha_{12}c_2 = 0 \quad (39)$$

If $s\alpha_{12} = 0$, then the manipulator is planar and the singularities can occur only if $\theta_2 = 0, \pi$. If $\alpha_{12} \neq 0$, then the singularities can occur when

$$\tan^2\alpha_{12} = \frac{a_{23}^2 - a_{12}^2}{{d_2}^2}$$
$$\tan(\theta_2/2) = \frac{d_2 s\alpha_{12}}{a_{23} - a_{12}} \quad (40)$$

The above equation implies that the general 2R serial manipulator can have singularities only for special values of link lengths, offsets and twist angles, and if one has a geometry satisfying the first equation in (40), the singularities lie along a curve with the value of θ_2 given by the second equation in the set of equations (40).

For the values of $a_{12} = d_2 = 1.0$ and $a_{23} = 1.5$, the singularities occur for $\alpha_{12} = \pm 48.1897^{\circ}$ and $\theta_2 = \pm 131.8103^{\circ}$. The velocity ellipse for such values degenerates to a straight line along the unit vector $(-0.7454, -0.4444, -0.4969)^T$. Figure 3 shows the three orthographic views of this line and also a 3D view of the degenerate velocity "ellipse". The maximum velocity for $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$ is 1.5 along the direction of the straight line. By use of equations (40), one can calculate the values of α_{12} and θ_2 for any other set of values of a_{12}, d_2 and a_{23} and get plots as in figure 3.

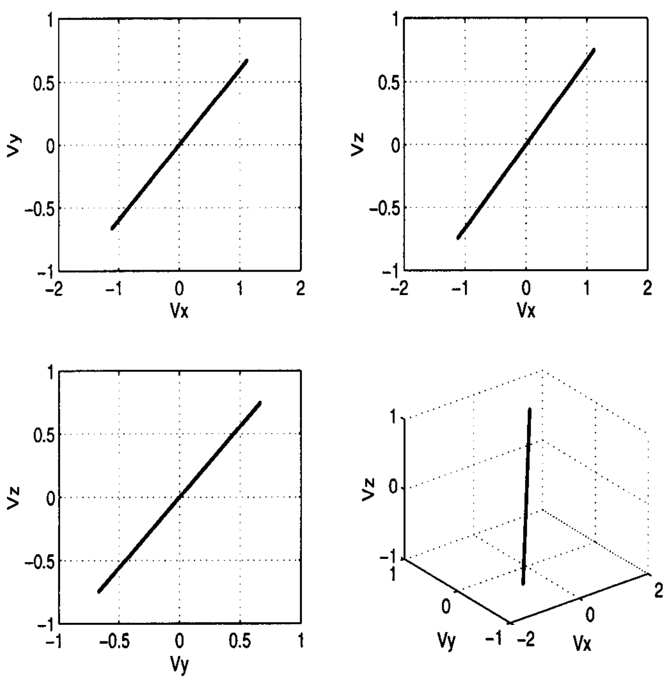

Figure 3. Degenerate velocity "ellipse" at a singular point

On substituting α_{12} and θ_2 from equations (40) in expressions for g_{11} and g_{12}, we can show that for the general 2R manipulator $g_{11} = g_{12} = 0$ at a singularity. Since det$[g]$ is independent of θ_1 all its partial derivatives with respect to θ_1 is zero. In addition, since g_{22} is constant, all the partial derivatives of g_{22} with respect to θ_2 are zero. The partial derivative of det$[g]$ with respect to θ_2 is given by

$$\frac{\partial \det[g]}{\partial \theta_2} = g_{11}\frac{\partial g_{22}}{\partial \theta_2} + g_{22}\frac{\partial g_{11}}{\partial \theta_2} - 2g_{12}\frac{\partial g_{12}}{\partial \theta_2} \quad (41)$$

At the singularity $\frac{\partial g_{11}}{\partial \theta_2} = 0$ and hence the whole of the right-hand side is zero. From equation (30), we obtain that

$$\sum_{l=1}^{2}(\boldsymbol{\psi}_{ir} \cdot \boldsymbol{\psi}_l)G_{rl} = 0, \quad i, r = 1, 2 \quad (42)$$

Referring to equation (17) and using the above equation, we get at a singularity,

$$\mathbf{a} \cdot \boldsymbol{\psi}_1 = 0$$
$$\mathbf{a} \cdot \boldsymbol{\psi}_1 = g_{22}\ddot{\theta}_2 \quad (43)$$

The above equation implies that acceleration is only possible along the $\boldsymbol{\psi}_2$ direction at a singularity.

On computing the second partial derivatives of det$[g]$, we find that the only non-zero term is

$$\frac{\partial^2 \det[g]}{\partial \theta_2^2} = g_{22}\frac{\partial^2 g_{11}}{\partial \theta_2^2} = 2a_{23}^4(s_2^2 + c_2^2 c\alpha_{12}^2) \quad (44)$$

The above implies that the singularity in a 2R manipulator is of second order. In addition, the determinant of the matrix of second partial derivatives of det$[g]$ is zero implying the singularities are parabolic.

It may be noted that in the planar case, $\alpha_{12} = 0$ and $\theta_2 = 0, \pi$, again the first partial derivatives of det$[g]$ is zero. In the second partial derivatives of det$[g]$, the only non-zero term is

$$\frac{\partial^2 \det[g]}{\partial \theta_2^2} = 2a_{23}^2 a_{12}^2 \quad (45)$$

Hence the matrix of second partial derivatives has a zero determinant implying the singularities are parabolic.

A RPSSPR-SPR parallel manipulator

In reference (Lee and Shah, 1988), the three-loop, three-degree-of-freedom RPSSPR-SPR mechanism of figure 4 has been proposed as a "parallel" wrist. The authors have discussed the direct and inverse kinematics but they have not dealt with its singularities. In this subsection, we use the theory developed in section 3, to determine the geometry of the solid region near a singularity for this parallel manipulator.

The geometry chosen is same as in (Lee and Shah, 1988) where the revolute joints axes are assumed to be co-planar and are perpendicular to the medians passing through the

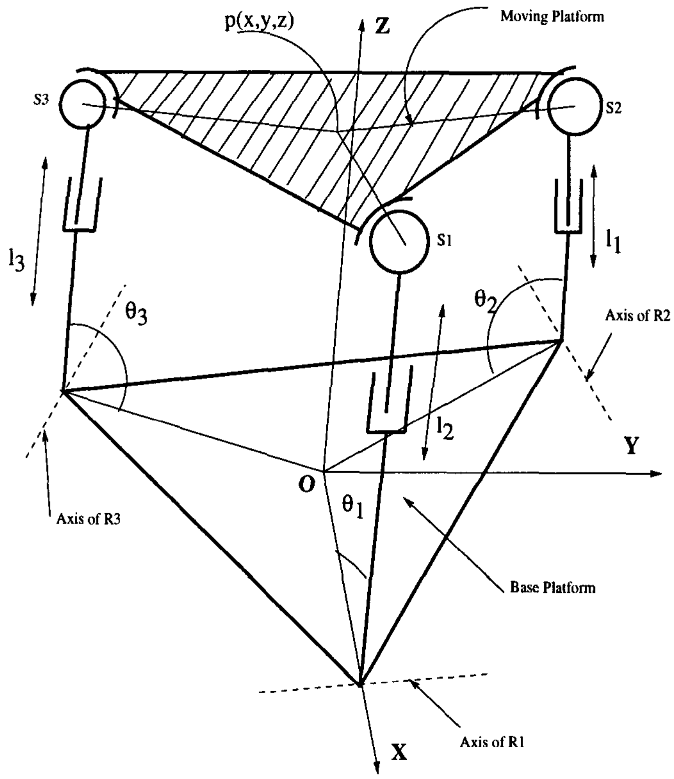

Figure 4. The RPSSPR-SPR parallel manipulator

respective vertices. Assuming that the length of the medians in the base equilateral triangle are unity, we can obtain the coordinates of the centre of the three spherical joints in the fixed coordinate system $\{0\}$. These are given by

$$\begin{aligned} \mathbf{S}_1 &= [(1-l_1c_1),\ 0,\ l_1s_1]^T \\ \mathbf{S}_2 &= [-0.5(1-l_2c_2),\ \sqrt{3}/2(1-l_2c_2),\ l_2s_2]^T \\ \mathbf{S}_3 &= [-0.5(1-l_3c_3),\ -\sqrt{3}/2(1-l_3c_3),\ l_3s_3]^T \end{aligned} \quad (46)$$

where $\theta_i,\ i=1,2,3$ are rotations at the three passive rotary joints and $l_i,\ i=1,2,3$ are the translations at the actuated prismatic joints.

The loop closure equations are obtained from the fact that the distance between the spherical joints are constant and are of the form

$$(\mathbf{S}_i - \mathbf{S}_j)\cdot(\mathbf{S}_i - \mathbf{S}_j) = k_{ij}^2, \quad i,j = 1,2,3, i \neq j \quad (47)$$

where k_{ij} is the distance between spherical joint i and spherical joint j respectively.

Differentiating the three constraint equations with respect to time, we get

$$\begin{aligned} \mathbf{0} = &\begin{pmatrix} 3l_1s_1 - l_1l_2s_1c_2 - 2l_1l_2c_1s_2 \\ 0 \\ 3l_1s_1 - l_1l_3s_1c_3 - 2l_1l_3c_1s_3 \end{pmatrix}\dot{\theta}_1 + \\ &\begin{pmatrix} 3l_2s_2 - l_1l_2c_1s_2 - 2l_1l_2s_1c_2 \\ 3l_2s_2 - l_2l_3s_2c_3 - 2l_2l_3c_2s_3 \\ 0 \end{pmatrix}\dot{\theta}_2 + \\ &\begin{pmatrix} 0 \\ 3l_3s_3 - l_2l_3c_2s_3 - 2l_2l_3s_2c_3 \\ 3l_3s_3 - l_1l_3c_1s_3 - 2l_1l_3s_1c_3 \end{pmatrix}\dot{\theta}_3 + \\ &\begin{pmatrix} 2l_1 - 3c_1 + l_2c_1c_2 - 2l_2s_1s_2 \\ 0 \\ 2l_1 - 3c_1 + l_3c_1c_3 - 2l_3s_1s_3 \end{pmatrix}\dot{l}_1 + \\ &\begin{pmatrix} 2l_2 - 3c_2 + l_1c_1c_2 - 2l_1s_1s_2 \\ 2l_2 - 3c_2 + l_3c_2c_3 - 2l_3s_2s_3 \\ 0 \end{pmatrix}\dot{l}_2 + \\ &\begin{pmatrix} 0 \\ 2l_3 - 3c_3 + l_2c_2c_3 - 2l_2s_2s_3 \\ 2l_3 - 3c_3 + l_1c_1c_3 - 2l_1s_1s_3 \end{pmatrix}\dot{l}_3 \end{aligned} \quad (48)$$

The above equation can be written in the form of equation (13) as

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = -[K^*]^{-1}[K]\begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \quad (49)$$

where the columns of $[K^*]$ and $[K]$ are coefficients of $\dot{\theta}_i,\ i=1,2,3$ and $\dot{l}_i,\ i=1,2,3$ respectively.

Assuming all the lengths k_{ij}'s are $\sqrt{3}/2$(the lengths of the medians of the top platform are 0.5 units each) the coordinates of the centroid of the moving platform are given as

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= (1/3)(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3) \\ &= \frac{1}{3}\left[\begin{pmatrix} 1-l_1c_1 \\ 0 \\ l_1s_1 \end{pmatrix} + \begin{pmatrix} (-1/2)(1-l_2c_2) \\ (\sqrt{3}/2)(1-l_2c_2) \\ l_2s_2 \end{pmatrix}\right] \\ &+ \left[\begin{pmatrix} (-1/2)(1-l_3c_3) \\ (-\sqrt{3}/2)(1-l_3c_3) \\ l_3s_3 \end{pmatrix}\right] \end{aligned} \quad (50)$$

and the velocity of the centre is given by

$$
\begin{aligned}
\mathbf{v} &= \frac{1}{3}\left[\begin{pmatrix} l_1 s_1 \\ 0 \\ l_1 c_1 \end{pmatrix}\dot{\theta}_1 + \begin{pmatrix} -0.5 l_2 s_2 \\ \frac{\sqrt{3}}{2} l_2 s_2 \\ l_2 c_2 \end{pmatrix}\dot{\theta}_2 + \begin{pmatrix} -0.5 l_3 s_3 \\ \frac{-\sqrt{3}}{2} l_3 s_3 \\ l_3 c_3 \end{pmatrix}\dot{\theta}_3\right] \\
&+ \frac{1}{3}\left[\begin{pmatrix} -c_1 \\ 0 \\ s_1 \end{pmatrix}\dot{l}_1 + \begin{pmatrix} 0.5 c_2 \\ \frac{-\sqrt{3}}{2} c_2 \\ s_2 \end{pmatrix}\dot{l}_2 + \begin{pmatrix} 0.5 c_3 \\ \frac{\sqrt{3}}{2} c_3 \\ s_3 \end{pmatrix}\dot{l}_3\right] \\
&= [J^*]\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} + [J]\begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \qquad (51)
\end{aligned}
$$

where $[J^*]$ and $[J]$ are 3×3 matrices obtained from the coefficients of $\dot{\theta}_i$, $i = 1, 2, 3$, and $\dot{l}_i$, $i = 1, 2, 3$ respectively. Using equation (49) in equation (51), we get

$$
\mathbf{v} = ([J] - [J^*][K^*]^{-1}[K])(\dot{l}_1, \dot{l}_2, \dot{l}_3)^T = \sum_{i=1}^{3} \alpha_i \dot{l}_i \qquad (52)
$$

The metric in this case is the matrix $([J] - [J^*][K^*]^{-1}[K])^T([J] - [J^*][K^*]^{-1}[K])$, and the three-degree-of-freedom RPSSPR-SPR parallel manipulator will *loose* one or more degree-of-freedom when

$$
\det[g^*] = ([J] - [J^*][K^*]^{-1}[K])^T([J] - [J^*][K^*]^{-1}[K]) = 0 \qquad (53)
$$

The above equation is a function of the passive variables θ_i, $i = 1, 2, 3$ and the three actuated variables l_i, $i = 1, 2, 3$. Equation (53) together with the loop closure equations (47) represent 4 equations in 6 unknowns and hence the singularities occur on a high-order *2D surface*. It is very difficult to derive analytical results for this case; we, therefore, present numerical results.

At a typical non-singular point given by $(l_1, l_2, l_3) = (0.5, 1.0, 2.0)$ meters, and the corresponding passive variables, $(\theta_1, \theta_2, \theta_3)$, given by $(0.4, 0.7535, 0.2402$ radians, the tip of the velocity vector will lie on the ellipsoid shown in figure 5. The maximum, intermediate, and minimum velocities along the principal axes of the ellipsoid are given by $0.3724, 0.3162, 0.2031$ m/sec respectively. The directions of the corresponding principal axes are $(0.9921, -0.0394, 0.1187)^T$, $(0.1166, 0.6338, -0.7646)^T$ and $(-0.0452, 0.7724, 0.6335)^T$ respectively.

From numerical solution of the constraint equations and the condition for loss of degree-of-freedom, we find that the leg lengths, (l_1, l_2, l_3), given by $(0.5, 1.0, 1.9710)$ meters and the corresponding passive variables, $(\theta_1, \theta_2, \theta_3)$, given by $(1.1691, 0.4781, 0.2355)$ radians is a singular point. The

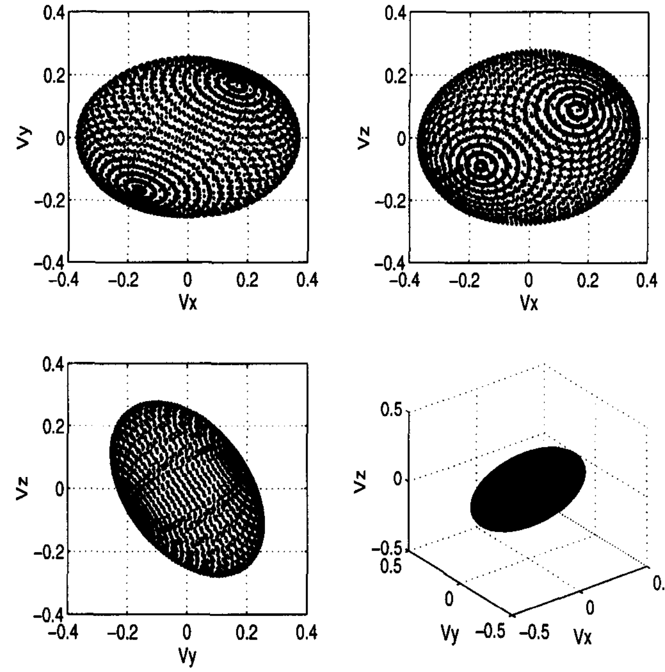

Figure 5. Velocity ellipsoid at a non-singular point

tip of the velocity ellipsoid no longer lies on an ellipsoid and the eigenvalues of the matrix $([J] - [J^*][K^*]^{-1}[K])$ are $(0.7647, 0, 2.2773)$ m/sec. At this singular point, the mechanism looses one degree-of-freedom and the velocity distribution is the ellipse shown in sectional views and as a 3D plot in figure 6. The centroid of the top platform can move along any direction in a plane spanned by the vectors corresponding to the two non-zero eigenvalues.

The RPSSPR-SPR parallel manipulator will *gain* one or more degrees-of-freedom when

$$
\begin{aligned}
&\det[K^*] = \\
&(3l_1 s_1 - l_1 l_2 s_1 c_2 - 2 l_1 l_2 c_1 s_2) \times (3 l_2 s_2 - l_2 l_3 s_2 c_3 - 2 l_2 l_3 c_2 s_3) \\
&\times (3 l_3 s_3 - l_1 l_3 c_1 s_3 - 2 l_1 l_3 s_1 c_3) \\
&+ (3 l_1 s_1 - l_1 l_3 s_1 c_3 - 2 l_1 l_3 c_1 s_3) \times (3 l_2 s_2 - l_1 l_2 c_1 s_2 - 2 l_1 l_2 s_1 c_2) \\
&\times (3 l_3 s_3 - l_2 l_3 c_2 s_3 - 2 l_2 l_3 s_2 c_3) = 0 \qquad (54)
\end{aligned}
$$

The above equation is a function of all the passive and active joint variables and again together with the loop closure equation (47) represent a set of 4 equations in 6 variables. Thus the singularities resulting in a *gain* of one or more degrees-of-freedom also lie on a *2D surface*. It is very difficult to get the analytical expressions for this surface and we present numerical results.

At leg-lengths, (l_1, l_2, l_3), given by $(0.575, 0.483, 0.544)$ meters respectively and the corresponding passive variables,

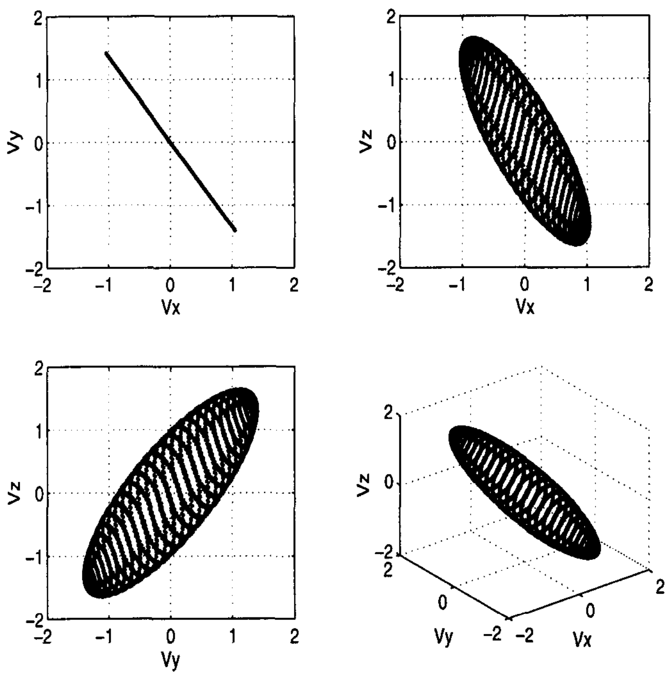

Figure 6. Velocity ellipse at a singular point

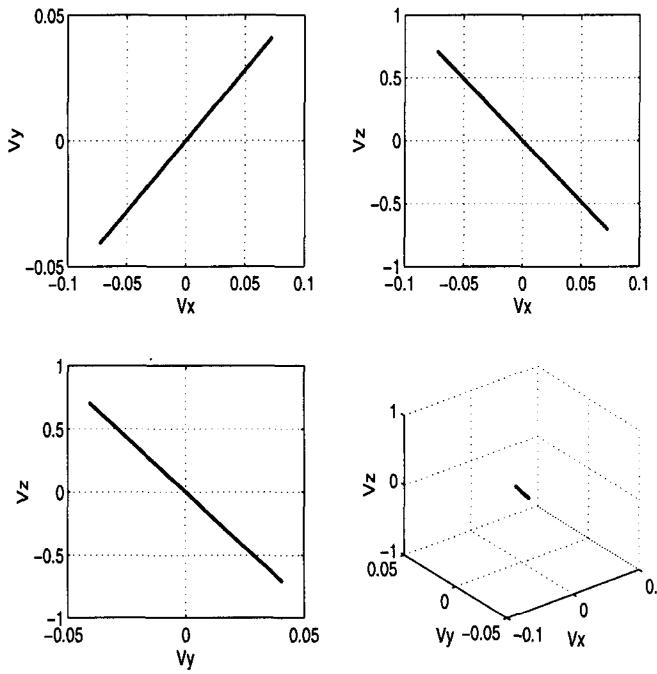

Figure 7. Velocity at a singular point

$(\theta_1, \theta_2, \theta_3)$, given by $(-0.3441, -0.0138, 0.2320)$ radians, $\det[K^*]$ is found to be very close to zero. The eigenvalues of $[K^*]$ are -0.5565, 0 and 0.4509 respectively and the three eigenvectors corresponding to the three eigenvalues are $(-0.8098, 0.3571, -0.4656)^T$, $(-0.3109, -0.8743, -0.3727)^T$ and $(-0.0877, -0.4781, -0.8739)^T$ respectively. Hence at this point, the mechanism gains one degree-of-freedom and the velocity of the centroid, with all actuated joints locked, is given as

$$\mathbf{v} = \begin{pmatrix} -0.0647 \\ 0 \\ 0.1804 \end{pmatrix} \dot{\theta}_1 + \begin{pmatrix} 0.0011 \\ -0.0019 \\ 0.1610 \end{pmatrix} \dot{\theta}_2 + \begin{pmatrix} -0.0208 \\ -0.0361 \\ 0.1763 \end{pmatrix} \dot{\theta}_3 \qquad (55)$$

where the eigenvector, $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T$, is given as $\alpha \times (-0.3109, -0.8743, -0.3727)^T$ with α arbitrary. It is clear that the velocity vector lies along a straight line and the mechanism has *gained* instantaneously a degree-of-freedom at this singular point. Figure 7 shows the velocity distribution at the singular point.

CONCLUSION

In this paper, we have presented a general geometric framework for differential analysis of point trajectories traced out by multi-degree-of-freedom serial and parallel, non-redundant manipulators. At non-singular points, the tip of the velocity vector of a point on the end-effector lies on an ellipsoid or ellipse. At singular configurations, the ellipsoid degenerates to an ellipse, a line or a point depending on the number of degrees-of-freedom lost at that point. For a parallel manipulator, at a gain of degree-of-freedom singularity, there is a growth to a line, an ellipse or an ellipsoid depending on the number of degrees-of-freedom gained. In both serial and parallel manipulators, the partial derivatives of shape and size of the ellipsoid or ellipse can be used to determine the geometry near the singularity. The developed theory was illustrated with the help of a serial and a parallel manipulator examples.

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