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FREE AND FORCED VIBRATION OF A KINKED CANTILEVER BEAM

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ABSTRACT

In this paper we use the assumed modes method to derive an analytical model of a kinked cantilever beam of unit mass carrying a kink mass(m_k) and a tip mass(m_t). The model is used to study the free and forced vibration of such a beam. For the free vibration, we obtain the mode shape of the complete beam by solving an eight order polynomial whose coefficients are functions of the kink mass, kink angle and tip mass. A relationship of the form $f(m_k, m_t, \delta) = m_k + m_t(4 + \frac{10}{3}\cos\delta + \frac{2}{3}\cos^2\delta) =$ constant appears to give the same fundamental frequency for a given kink angle, δ, and different combinations of kink mass and tip mass.

To derive the dynamic equations of motion, the complete kinked beam mode shape is used in a Lagrangian formulation. The equations of motion are numerically integrated with a torque applied at the base and the tip response for various kink angles are presented. The results match those obtained from a traditional finite element formulation.

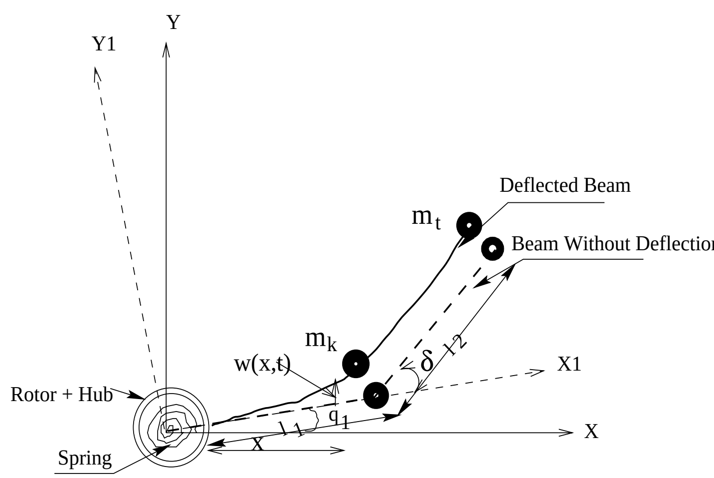

Figure 1. A Kinked Cantilever

INTRODUCTION

Many structures such as helicopter blades, gas turbine blades, robotic arms, satellite booms, and golf clubs can be modeled as rotating cantilever kinked beam. These structures are typically very long and have slender geometric dimensions. During motion, these flexible structures often exhibit mechanical vibrations. Hence a study of free and forced vibration of cantilevers attached to a rotating hub carrying discrete masses along their length is of interest in a variety of engineering applications. This classical problem has been approached at different levels of approximation ranging from the simplest discrete model of a massless beam with flexural rigidity to Timoshenko models which take into account shear deformation as well as rotary inertia (see, for example, (Centinkunt and Wen-Lung Yu, 1991; White and Heppler, 1996; Graff, 1975)). Large amount of results are available for straight beams with only a tip mass (see, for example, (Zhu and Mote, 1997; Laura et. al., 1974; Abramovich and Hamburger, 1991)) or a system of masses (Pan, 1965). Continuous beam formulation for a kinked cantilever carrying a tip mass and central kink mass using Euler - Bernoulli theory appears not to have been studied analytically. One of the reasons for this lacuna might be the explosive growth of numerical methods for vibration and modal analysis in the past few decades. Notwithstanding this situation, it is important to extend analytical methods to gain better insight for engineering design.

A kinked cantilever beam, as shown in figure 1, is a viable model of a two link flexible manipulator with m_k representing a motor and m_t the payload. In flexible manipulators with rotary joints, the joints permit free rotation of link during the motion of the payload, however, the rotations at the joints are stopped by control (actuator) torques once the payload reaches a desired destination. This maneuver typically induces vibrations in the flexible manipulator and hence suppression of unwanted vibration is an important problem in flexible manipulators (see, for example, (Centinkunt and Wen-Lung Yu, 1991; Zhu and Mote, 1997; Luo, 1972)).

Another example of a kinked cantilever situation arises in plastic bending under impact at the kink (Johnson, 1972). The resulting response after the kink formation is the free vibration of a kinked elastic beam.

In addition, attaching masses to reduce noise and vibration levels have been widely used for beams, plates and shells (Harris, 1991). Although the emphasis in vibration engineering is on reducing acoustic radiation, it is important to understand the dynamic stress levels during free or forced vibration. Hence, understanding free and forced vibration characteristics of a kinked cantilever carrying masses can help in evolving better active or passive control schemes in the case of flexible manipulators, better design procedures in impact problems or reducing acoustic radiation.

In this paper we deal with the free and forced vibration characteristics of a kinked cantilever beam using the assumed modes method. The mode shapes and frequencies of a kinked beam carrying a kink mass(m_k), a tip mass(m_t) and at various kink angles(δ) are obtained by solving an eighth degree polynomial. Non-dimensional parameters for the natural frequencies which are useful in the design are provided in the form of graphs for ready reference to the designer. A relationship in the form $f(m_k, m_t, \delta) = m_k + m_t(4 + \frac{10}{3}\cos\delta + \frac{2}{3}\cos^2\delta) =$ constant appears to give the same fundamental frequency for a given δ and different combinations of m_k and m_t. The dynamic response is obtained by first deriving the equations of motion with a Lagrangian formulation, using a single mode shape for the entire beam, and then numerically solving these equations.

This paper is organized as follows: Section 2 presents the mathematical formulation of expressions for the frequencies and mode shapes of a kinked beam. Section 3 describes, in brief, the derivation of equations of motion for the kinked cantilever beam by using the Lagrangian formulation in conjunction with the assumed modes method. Section 4 presents and discusses the numerical results obtained from analytical models, and in section 5, we present the conclusions of this paper.

THE MODELING OF A KINKED CANTILEVER BEAM

Figure 1 shows a kinked cantilever attached to a rotating hub, vibrating in the X-Y plane. Any arbitrary material point along the beam is located by x. Denoting the elastic displacement of the point with reference to the neutral axis at time t by $w(x,t)$, one can write the free vibration equation of the beam by using the Euler-Bernoulli beam theory as

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0. \tag{1}$$

where EI is the flexural rigidity, ρ is the mass density and A is the cross sectional area.

The above partial differential equation can be solved by the well known technique of separation of variables. For a kinked cantilever of total length $L = l_1 + l_2$, we consider a solution of the form $w_i(x,t) = X_i(x)T(t)$ where $i = 1, 2$, denotes the two halves of the beam. The mode shapes X_1 and X_2 for the two halves of the beam are of the form

$$\begin{aligned} X_1 &= C_1\cos(Kx) + C_2\sin(Kx) \\ &\quad + C_3\cosh(Kx) + C_4\sinh(Kx) \\ X_2 &= C_5\cos(Kx) + C_6\sin(Kx) \\ &\quad + C_7\cosh(Kx) + C_8\sinh(Kx) \end{aligned} \tag{2}$$

where $K^4 = \frac{\omega_j^2\rho A}{EI}$. and ω_j is jth natural frequency.

The boundary conditions to determine the constants C_j are as follows (Reddy et. al., 1999):

At the hub end,

$$\begin{aligned} w_1 &= 0 \\ K_{s1}w_1' &= EIw_1'' - Jh_1\ddot{w}_1' \end{aligned} \tag{3}$$

where K_{s1} = spring stiffness and Jh_1 is hub inertia.
The bending displacement continuity at the kink stipulates

$$w_1\cos\delta = w_2 \tag{4}$$

The shear force balance at the kink, taking an effective mass of $m_k + (m_t + 1/2)\sin\delta$, gives

$$EI(w_1''' - w_2'''\cos\delta) = [m_k + (m_t + 1/2)\sin\delta]\ddot{w}_1 \tag{5}$$

At the kink, the continuity of slope and bending moment requires

$$w_1' = w_2' \tag{6}$$

$$w_1'' = w_2''$$

Finally, at the free end

$$w_2'' = 0$$
$$EIw_2''' = m_t \ddot{w}_2. \tag{7}$$

In the above equations $()'$ and $\ddot{()}$ denote derivatives with respect to x and t respectively. Thus, there are two boundary conditions at both the free end and the fixed end, and four conditions at the kink giving a total of 8 equations for 8 unknown coefficients C_j. Substitution of assumed solutions (2) in the boundary conditions lead to the eigen equation

$$F(KL)[C_1.....C_8]^T = 0. \tag{8}$$

where $F(KL)$ is an 8x8 matrix whose elements are given in Appendix 1. For non-trivial solutions, $\det(F) = 0$ gives the equation for the natural frequencies as a function of m_k, m_t and δ. The roots of this equation give positive values of KL which are used to obtain the frequencies and the coefficients C_j. The eigenvalue $y_j = KL$ is related to the frequency ω_j by

$$\omega_j = \left(\frac{y_j}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \tag{9}$$

The equations (8) and (9) were solved numerically for various sets of values of m_k, m_t and δ and these results are presented and discussed in detail in section 4. A frequency factor, p_j, is helpful in presenting the results and is defined as the ratio of the frequency of a kinked beam for a given (m_k, m_t, δ) to the frequency of a straight beam with no attached masses. For mode 1 (j=1) and mode 2 (j=2), p_j is given by

$$p_j = \frac{\omega_j(m_k, m_t, \delta)}{\omega_j(0,0,0)} \tag{10}$$

DYNAMIC EQUATIONS OF MOTION

The dynamic equations of motion for the kinked cantilever beam shown in figure 1 can be obtained by using the Lagrangian formulation. The elastic displacement $w(x,t)$ of a material point of the beam is discretized by the assumed modes method as

$$w_i(x,t) = \sum_{j=1}^{n} X_i(x)_j T(t) \quad \text{i} = 1 \text{ for } 1^{st} \text{ half of the beam}$$
$$= 2 \text{ for } 2^{nd} \text{ half of the beam}$$

where n is the number of modes retained in the expansion. Then next step in this process is to select a suitable set of coordinates. The approach used selects one rigid body coordinate associated with the joint rotation, and flexible transverse displacement from a set of axes attached to the joint. This is depicted in figure 1. Then a position vector R to every point of the kinked beam can be constructed:

$$R = [x, w_i(x,t)]^T \tag{11}$$

where $i = 1, 2$ are for two halves of the beam.

By differentiating the position vectors, we can obtain the velocity. From the velocity one can find the kinetic energy of the vibrating beam. The total kinetic energy of the flexible system is due to the motion of the links and joints, and kinetic energy due to kink and tip masses.

$$T_{b_1} = \frac{1}{2}\int_0^{l_1} \rho A \left[\frac{dR}{dt}\right]^T . \left[\frac{dR}{dt}\right] dx$$

(T_{b_1} = K.E due to first half of the beam)

$$T_{b_2} = \frac{1}{2}\int_{l_1}^{L} \rho A \left[\frac{dR}{dt}\right]^T . \left[\frac{dR}{dt}\right] dx$$

(T_{b_2} = K.E due to second half of the beam)

$$T_j = \frac{1}{2}\Omega^T [Jh_1]\Omega \qquad \text{(K.E due to joint)}$$
$$T_{m_k} = \frac{1}{2} m_k \left[\frac{dR}{dt}\right]^T . \left[\frac{dR}{dt}\right] \qquad \text{(K.E due to kink mass)}$$
$$T_{m_t} = \frac{1}{2} m_t \left[\frac{dR}{dt}\right]^T . \left[\frac{dR}{dt}\right] \qquad \text{(K.E due to tip mass)}$$

The total kinetic energy of the system is given by

$$T = T_{b_1} + T_{b_2} + T_j + T_{m_k} + T_{m_t} \tag{12}$$

The potential energy of the kinked beam is given by

$$V_{b_1} = \frac{1}{2}\int_0^{l_1} EI\left[\frac{d^2 w_1}{dx^2}\right]^2 dx$$

(P.E due to first half of the beam)

$$V_{b_2} = \frac{1}{2}\int_{l_1}^{L} EI\left[\frac{d^2 w_2}{dx^2}\right]^2 dx$$

(P.E due to second half of the beam)

The total potential energy is the sum of V_{b_1} and V_{b_2}.
The Lagrange's equations of motion for the kinked beam in terms of the generalized flexible variables and the rigid-body variable are derived from the equation

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q})}{\partial q} = \tau \qquad (13)$$

The equation of motion can be written in compact form as

$$[M]\ddot{q} + h(q,\dot{q}) + [K]q = Q \qquad (14)$$

These equations are given in Appendix 2.

We can make the following observation from the equations of motion:

- Since a single mode shape is used for the entire beam, the number of equations is smaller than the case if two mode shapes were used for the two halves. The number of equations is also less than the FEM formulation used in modeling of flexible robots, where the two halves of the beam are considered separately with one or more elements (Rex and Ghosal, 1995). This may be useful for model based control of flexible robot where the size of the model and minimizing the number of computations are important.

RESULTS AND DISCUSSION

Free Vibration

In this section, we first present results related to the natural frequencies of a kinked cantilever undergoing free vibration.

Frequency Analysis To obtain frequency factors, equation (8) was solved numerically using the software package Matlab (The MathWorks, Inc) and KL was obtained for various values of m_k, m_t and δ. Once KL is known, the frequency ω_j and the frequency factor p_j is obtained from equations (9) and (10) respectively. It may be noted that the FEM and analytical results in Table 1 are in good agreement. The FEM results were obtained from NISA.

As the values of attached masses increase, the frequency factors vary as seen from figures 2 and 3. As the kink angle increase the frequency factors increase in mode 1 and decrease in mode 2 for a given combination of m_k and m_t. It may be noted that the fundamental frequency increases roughly 3 times as the kink angle increases from 0 to π. For a 90° kink, the increase in fundamental frequency is about 30% from its value for a straight beam. Thus, the kink effect becomes more pronounced after 90°.

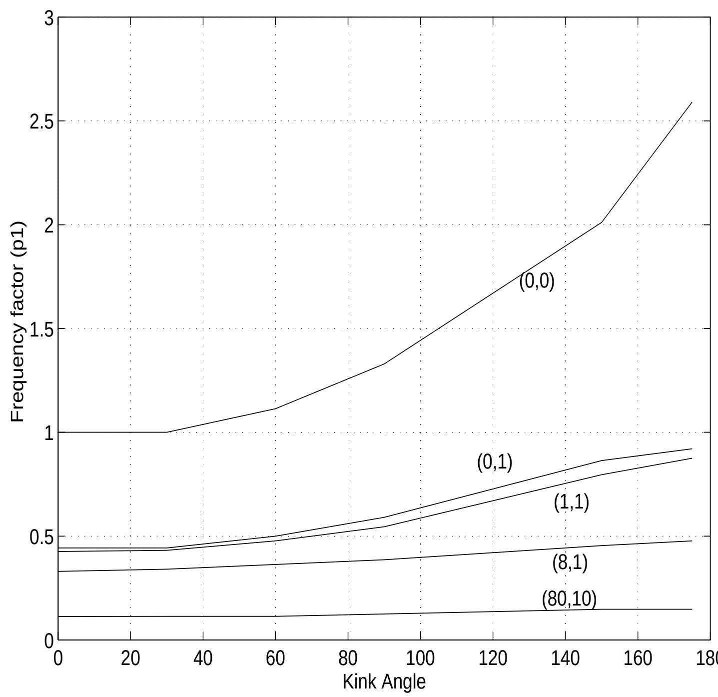

Figure 2. Mode 1 frequency factor vs Kink angle (δ) for different (m_k, m_t)

For a given kink angle it is possible to find an infinite number of m_k and m_t combinations that give the same fundamental frequency. These combinations appear to fit the equation $m_k + m_t(4 + \frac{10}{3}\cos\delta + \frac{2}{3}\cos^2\delta) =$ constant. This result was obtained after trial and error. It may be noted that the expression inside the bracket, $(4 + \frac{10}{3}\cos\delta + \frac{2}{3}\cos^2\delta)$, is equal to 8 when $\delta = 0$. It should, however, be noted that the above fit is not accurate for low values of m_k and m_t,

Table 1. Frequency factors for kinked beam

			MODE 1	MODE 1	MODE 2	MODE 2
m_k	m_t	δ	ANALYTICAL	FEM	ANALYTICAL	FEM
0	0	0^o	1.0000	1.0000	1.0000	1.0000
0	1	-	0.4430	0.4449	0.7375	0.7539
1	1	-	0.4256	0.4276	0.4297	0.4437
0	0	90^o	1.3295	1.3159	0.5789	0.5808
0	1	-	0.5909	0.5859	0.2922	0.2942
1	1	-	0.5455	0.5504	0.2450	0.2489
0	0	175^o	2.5909	2.6777	0.4483	0.4620
0	1	-	0.9205	0.9303	0.3721	0.4469
1	1	-	0.8750	0.8811	0.2105	0.2244

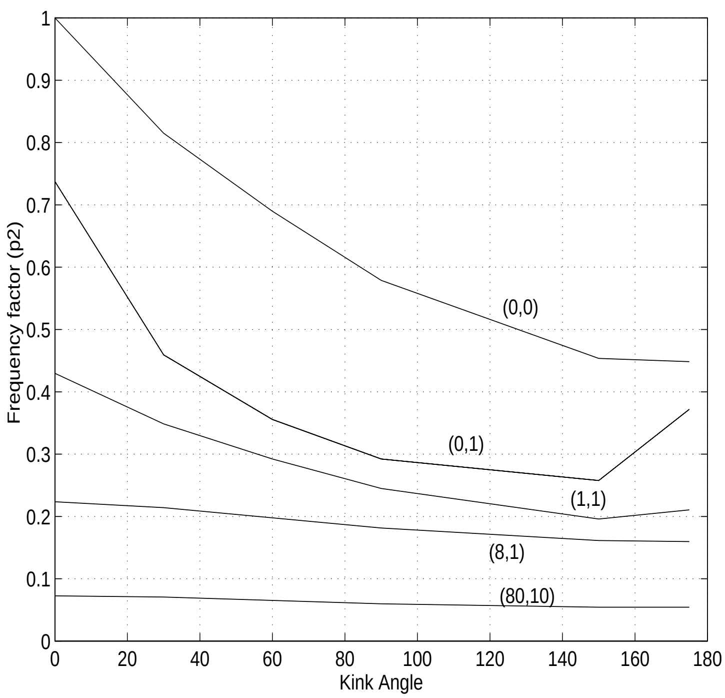

Figure 3. Mode 2 frequency factor vs Kink angle (δ) for different (m_k, m_t)

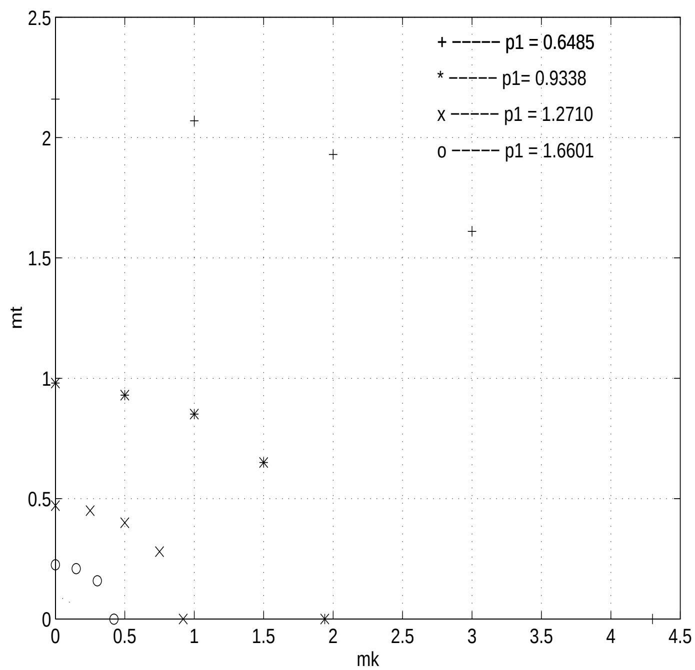

Figure 4. Iso-frequency chart for a 175° kink

or for high kink angles. For a hairpin like kinked beam, e.g. $\delta = 175°$, the loci of iso-frequency points are curved as shown in figure 4. For smaller kink angles the loci are nearly straight as shown in figure 5.

A partial explanation for the equation $m_k + m_t(4 + \frac{10}{3}\cos\delta + \frac{2}{3}\cos^2\delta)$ = constant can be given from the discrete model of the continuous system.

Assuming that E, I and L are unity, the flexibility matrix of a discrete kinked beam is given by

$$F = \frac{1}{48}\begin{bmatrix} 2 & 3+2\cos\delta \\ 3+2\cos\delta & 8+6\cos\delta+2\cos^2\delta \end{bmatrix};$$

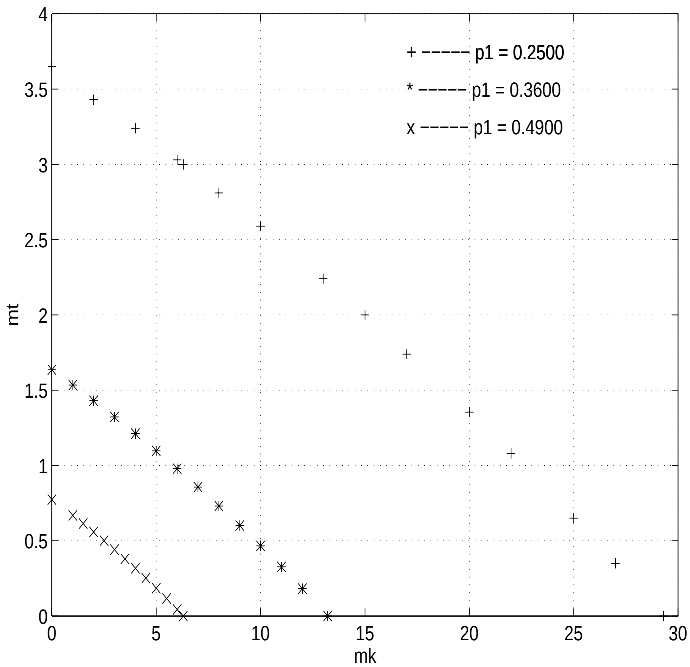

Figure 5. Iso-frequency chart for a $0°$ kink

The stiffness and mass matrices are given as

$$K_s = F^{-1} \tag{15}$$
$$M = \begin{bmatrix} m_{ek} & 0 \\ 0 & m_{et} \end{bmatrix}$$

where $m_{ek} = m_k + m_t \sin\delta$; $m_{et} = m_t$. Expanding $|K_s - M\omega^2| = 0$ gives the frequency equation

$$\omega^4 - \left(\frac{96}{7}\right)\left(\frac{1}{m_{et}} + \frac{4 + 3\cos\delta + \cos^2\delta}{m_{ek}}\right)\omega^2 + \frac{48^2}{7}\frac{1}{m_{ek}m_{et}} = 0 \tag{16}$$

From the above equation, we conclude that the product of $\omega_1^2\omega_2^2$ is invariant with respect to the product $m_{ek}m_{et}$. Defining a mass ratio $\alpha = m_{ek}/m_{et}$ the individual values of ω_1^2, ω_2^2 are

$$\omega_{1,2}^2 = \frac{48}{7m_{ek}}\left[(\alpha + 4 + 3\cos\delta + \cos^2\delta) \pm \sqrt{(\alpha + 4 + 3\cos\delta + \cos^2\delta)^2 - 7\alpha}\right] \tag{17}$$

The fundamental frequency is given by,

$$\omega_1^2 = \frac{48}{7m_{ek}}(\alpha + 4 + 3\cos\delta + \cos^2\delta) \left[1 - \sqrt{1 - \frac{7\alpha}{(\alpha + 4 + 3\cos\delta + \cos^2\delta)^2}}\right] \tag{18}$$

Assuming $\frac{7\alpha}{(\alpha+4+3\cos\delta+\cos^2\delta)^2} << 1$, we get

$$\omega_1^2 = \frac{24}{m_{ek} + m_{et}(4 + 3\cos\delta + \cos^2\delta)} \tag{19}$$

The above result implies that the fundamental frequency will not change if $m_{ek} + m_{et}(4 + 3\cos\delta + \cos^2\delta)$ is held constant. Recalling that $m_{ek} = m_k + m_t\sin\delta$ and $m_{et} = m_t$, the condition yields $m_k + m_t(4 + 3\cos\delta + \sin\delta + \cos^2\delta) = C$, a constant, for constant fundamental frequency.

Forced Vibration

For the study of forced vibration a sinusoidal torque $\sin(\omega * t)$ was applied at the hub and ω was varied. The kinked beam now undergoes both a rigid-body rotation and flexural vibratory motions. The beam is assumed to be uniform with parameter values $\rho A = 0.52334$ kg/m, $EI = 100$ N m^2, and $l_1 = l_2 = 0.5$ m. We present results for two cases of kink angles $0°$ and $90°$, and for two cases of kink and tip mass, namely $m_t = m_k = 0$ and $m_t = 0$, $m_k = 1$. The equations of motion were solved by Runge-Kutta method using MATLAB.

Tip Displacement: To obtain the tip displacement, ω was chosen as 10 rad/sec. The amplitude of the tip displacement for $0°$ and $90°$ kink angles is shown in figures 6 - 9. It is noticed that when there are no masses at the tip and kink, the tip displacement in the case of $90°$ kink angle is less compared to $0°$ kink angle. The possible reason can be attributed to the reduction in the effective length of the kinked cantilever in the case of $90°$. When there is a mass at the kink, the displacement at the tip increases in both the cases, straight beam and $90°$ kink.

Frequency Response: We also obtained the maximum amplitude of the tip as the forcing frequency ω was varied from 45 rad/sec to 70 rad/sec with zero kink and tip mass,

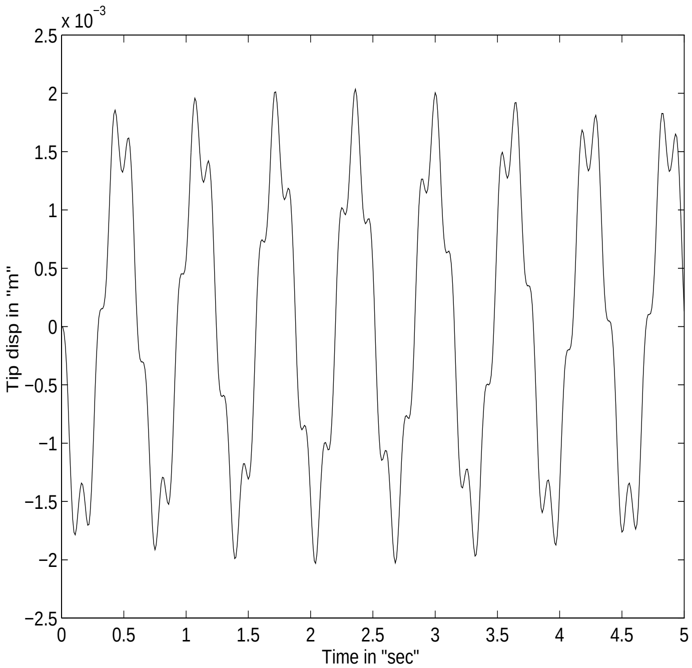

Figure 6. Tip response with m_k, m_t and δ =0

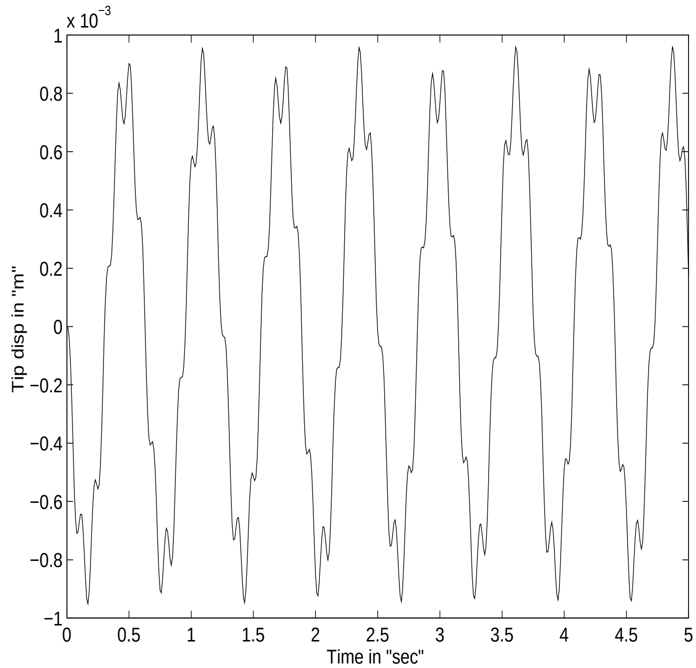

Figure 7. Tip response with m_k, m_t =0 and δ = 90

and for kink angles 0° and 90°. The maximum amplitude versus frequency is shown in figure 10. It can be noticed that the maximum amplitude is obtained at ω approximately equal to 48.6 rad/sec for the zero kink angle beam. It may be recalled that the natural frequency of the free vibration is 48.92 rad/sec. For the 90° kinked beam the maximum tip displacement was obtained at 64.8 rad/sec and also it should be noticed that the natural frequency of free vibration of the same beam is 64.79 rad/sec.

CONCLUSION

This paper deals with the free and forced vibration characteristics of a kinked cantilever beam carrying discrete masses. The mode shapes and frequencies of the kinked cantilever beam with attached masses were obtained from the solution of an eighth degree polynomial. The fundamental frequency appear to satisfy a simple relation, namely $m_k + m_t(4 + \frac{10}{3}\cos\delta + \frac{2}{3}\cos^2\delta) = \text{constant}$. We have presented a partial explanation for this relation using a lumped approximations of the kinked beam.

The equations of motion for the kinked cantilever beam were derived by using the Lagrangian formulation of dynamics in conjunction with the assumed modes method. The number of equations are less since the entire beam is described by a single mode shape and this is significant from the point of view of less computational requirement in areas such as model based control of flexible manipulators. The numerical simulation results match those obtained using traditional methods where two mode shapes or finite element method is used.

APPENDIX 1

Elements of the 8×8 matrix $F(KL)$ in eqn (8)

$a(1,1) = 1; a(1,2) = 0; a(1,3) = 1; a(1,4) = 0; a(1,5) = a(1,6) = 0; a(1,7) = a(1,8) = 0; a(2,1) = 0; a(2,2) = 1; a(2,3) = 0; a(2,4) = 1; a(2,5) = 0; a(2,6) = 0; a(2,7) = 0; a(2,8) = 0; a(2,8) = 0; a(3,1) = \cos(0.5 * y_j) * \cos(\delta); a(3,2) = \sin(0.5 * y_j) * \cos(\delta); a(3,3) = \cosh(0.5 * y_j) * \cos(\delta); a(3,4) = \sinh(0.5 * y_j) * \cos(\delta); a(3,5) = -\cos(0.5*y_j); a(3,6) = -\sin(0.5*y_j); a(3,7) = -\cosh(0.5* y_j); a(3,8) = -\sinh(0.5 * y_j); a(4,1) = -\sin(0.5 * y_j); a(4,2) = \cos(0.5 * y_j); a(4,3) = \sinh(0.5 * y_j); a(4,4) = \cosh(0.5 * y_j); a(4,5) = \sin(0.5 * y_j); a(4,6) = -\cos(0.5 * y_j); a(4,7) = -\sinh(0.5 * y_j); a(4,8) = -\cosh(0.5 * y_j); a(5,1) = -\cos(0.5 * y_j); a(5,2) = -\sin(0.5 * y_j); a(5,3) = \cosh(0.5 * y_j); a(5,4) = \sinh(0.5 * y_j); a(5,5) = \cos(0.5 * y_j); a(5,6) = \sin(0.5 * y_j); a(5,7) = -\cosh(0.5 * y_j); a(5,8) = -\sinh(0.5 * y_j); a(6,1) = \sin(0.5 * y_j) + m_{ek} * y_j * \cos(0.5 * y_j); a(6,2) = -\cos(0.5 * y_j) + m_{ek} * y_j * \sin(0.5 * y_j); a(6,3) = \sinh(0.5 * y_j) + m_{ek} * y_j * \cosh(0.5 * y_j); a(6,4) = \cosh(0.5 * Y) + m_{ek} * y_j * \sinh(0.5 *$

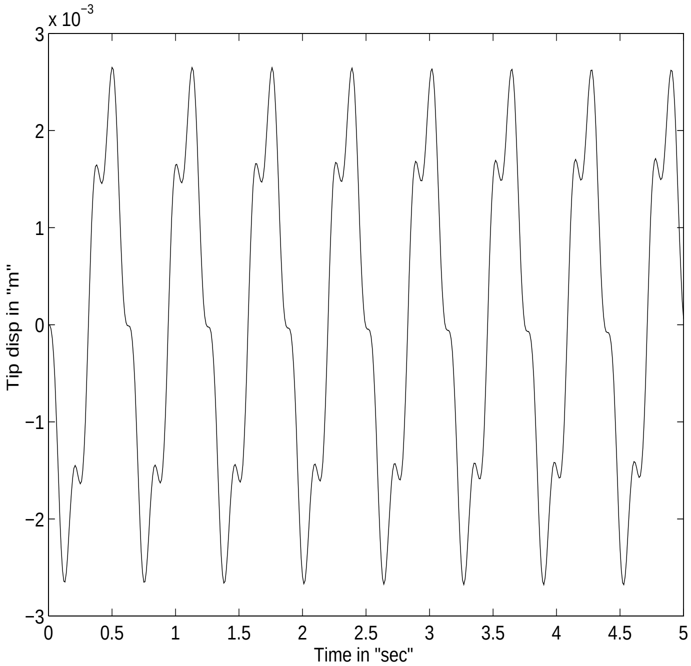

Figure 8. Tip response with m_k=1, m_t=0 and δ = 0

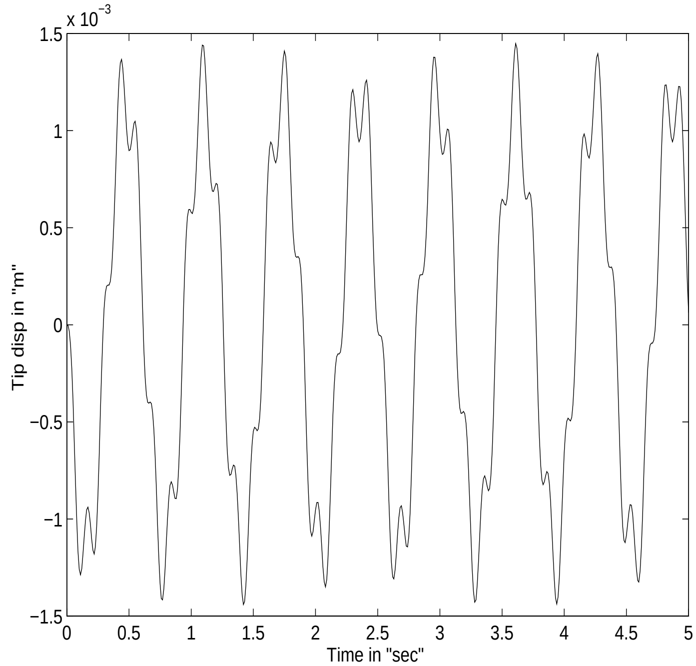

Figure 9. Tip response with m_k=1, m_t =0 and δ = 90

$y_j); a(6,5) = -\sin(0.5 * y_j) * \cos(\delta); a(6,6) = \cos(0.5 * y_j) * \cos(\delta); a(6,7) = -\sinh(0.5 * y_j) * \cos(\delta); a(6,8) = -\cosh(0.5 * y_j) * \cos(\delta); a(7,1) = 0; a(7,2) = 0; a(7,3) = 0; a(7,4) = 0; a(7,5) = \cos(y_j); a(7,6) = \sin(y_j); a(7,7) = -\cosh(y_j); a(7,8) = -\sinh(y_j); a(8,1) = 0; a(8,2) = 0; a(8,3) = 0; a(8,4) = 0; a(8,5) = \sin(y_j) + m_{et} * y_j * \cos(y_j); a(8,6) = -\cos(y_j) + m_{et} * y_j * \sin(y_j); a(8,7) = \sinh(y_j) + m_{et} * y_j * \cosh(y_j); a(8,8) = \cosh(y_j) + m_{et} * y_j * \sinh(y_j);$

APPENDIX 2

Elements of Mass Matrix

$M(1,1) = \frac{\rho A L^3}{3} + J_{h1} + \rho A T^2(t)[\int_0^{l_1} X_1^2(x)dx) + \int_{l_1}^{L} X_2^2(x)dx] + m_t L^2 + m_t[X_2^2(L)]T^2(t) + m_k l_1^2 + m_k X_1^2(l_1)T^2(t)$

$M(1,2) = \rho A[\int_0^{l_1} X_1(x)dx + \int_{l_1}^{L} X_2(x)dx] + m_t L2(X_2(L)) + m_k l_1 X_1(l_1)$

$M(2,1) = M(1,2)$

$M(2,2) = \rho A[\int_0^{l_1} X_1^2(x)dx + \int_{l_1}^{L} X_2^2(x)dx] + m_t[X_2^2(L)] + m_k X_1^2(l_1)$

Elements of Stiffness Matrix

$K(1,1) = 0;$

$K(1,2) = 0;$

$K(2,1) = 0;$

$K(2,2) = EI[\int_0^{l_1} X_1''^2(x)dx + \int_{l_1}^{L} X_2''^2(x)dx]$

Elements of h vector

$h(1,1) = 2\rho A\dot{\theta}_1 T(t)\dot{T}(t)[\int_0^{l_1} X_1^2(x)dx + \int_{l_1}^{L} X_2^2(x)dx] + 2m_t X_2^2(L)T(t)\dot{T}\dot{\theta}_1 + 2m_k X_1^2(l_1)T(t)\dot{T}\dot{\theta}_1$

$h(2,1) = -\rho A\dot{\theta}_1^2 T(t)[\int_0^{l_1} X_1^2(x)dx + \int_{l_1}^{L} X_2^2(x)dx] - m_t X_2^2(L)T(t)\dot{\theta}_1^2 - m_k X_1^2(l_1)T(t)\dot{\theta}_1^2$

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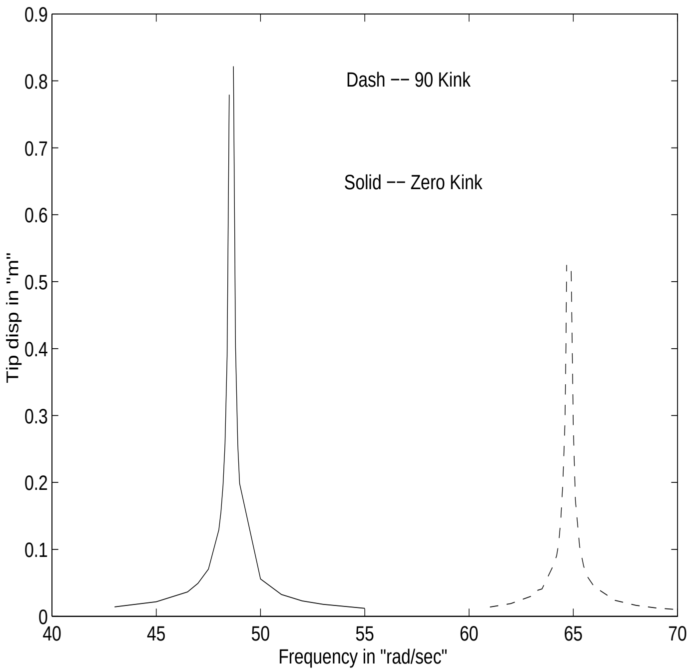

Figure 10. Frequency response with m_k=0, $m_t = 0$

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