Robustness Analysis of a Simple and Augmented Proportional Plus Derivative Controller in Trajectory Following Robots Using the Floquet Theory

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Abstract

This paper deals with the issue of robustness in control of robots using the proportional plus derivative (PD) controller and the augmented PD controller. In literature, a variety of PD and model based controllers for multi-link serial manipulator have been claimed to be asymptotically stable for trajectory tracking, in the sense of Lyapunov, as long as the controller gains are positive. In this paper, we first establish that for simple PD controllers, the criteria of positive controller gains is insufficient to establish asymptotic stability, and secondly that for the augmented PD controller the criteria of positive controller gains is valid only when there is no uncertainty in the model parameters. We show both these results for a simple planar two-degree-of-freedom robot with two rotary (R) joints, following a desired periodic trajectory, using the Floquet theory. We provide numerical simulation results which conclusively demonstrate the same.

Keywords: 2R planar robot, Nonlinear dynamics, Chaotic motion, Asymptotic Stability, Floquet Theory

1 Introduction

An industrial robot is expected to perform accurate trajectory following often in a repetitive manner. Typically trajectory following is achieved by advanced controllers and analysis and design of such controllers are an important area of study [1]. A feedback controlled system, represented by $\dot{\Theta} = f(\Theta, u)$ where $u(t)$ is the control input, is said to be stable if, for any $R > 0$, there exists $r > 0$, such that if $||\Theta(0) - \Theta_d(0)|| = ||e(0)|| < r$, then $||e(t)|| < R$ for all $t \geq 0$. The feedback controlled system is said to be asymptotically stable if it is stable and if in addition, $||e(0)|| < r$ implies that $||e(t)|| \longrightarrow 0$ as $t \longrightarrow \infty$. For a stable tracking controller, the error $e(t)$ must remain bounded, although it may not go to zero. In the case of an asymptotically stable controller, the error $e(t)$ goes to zero as time tends to infinity, i.e., the system trajectory tracks the desired trajectory asymptotically. Research has been done in the stability of nonlinear dynamical systems [2, 3, 4, 5]. In the case of a robot with joints, asymptotic stability implies that the joints of the robot, $\Theta(t)$, can track a desired trajectory, $\Theta_d(t)$, and as $t \to \infty$, $\Theta_d(t) - \Theta(t) \to \mathbf{0}$. In most robotic applications, trajectory tracking and asymptotic stability is required. In the literature on stability of robot

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controllers, extensive research has been done to design and implement controllers and asymptotic stability of controllers has been demonstrated by numerical simulation [6, 7, 8, 9, 10, 11, 12, 13, 14].

In this paper, we focus on two well-known controller used in robots. These are the *simple* PD control scheme and the *augmented* PD control law (see p. 194 in reference [15]) which uses a dynamic model of the robot. The PD control law is said to be asymptotic stable for set point control [1, 15]. Also, the PD control law (see [16]) is said to achieve exponential convergence for trajectory tracking for all positive PD controller gains, provided that the desired trajectory, its velocity and acceleration are all bounded. This claim is problematic because there exist numerical studies conducted on PD control of a planar two-link (RR) robot, where researchers (see, for example [17]) have demonstrated that for particular values of positive controller gains, the equations of motions of the RR robot exhibit chaos - i.e., implying that for those values of controller gains, the PD controller is not asymptotically stable. For the augmented PD control, the reference [15] provides a proof of asymptotic stability for trajectory tracking provided that the controller gains are positive. However, the authors themselves state that the controller is asymptotically stable only if accurate models of the inertial parameters are available. In practice this is a strong requirement since modeling errors are always present and the robustness of such a control law with respect to parameter uncertainities need to be investigated.

In this paper, we analyze the stability of both the *simple* PD controller and the *augmented* PD controller using Floquet theory. The ranges of controller gains where the robot is asymptotically stable are obtained using numerical simulations and it is shown that for some values of positive controller gains, the robot is *not* asymptotically stable. The results obtained using Floquet theory are an improvement over of the approach using the method of multiple scales [18] since there is no need to approximate the trigonometric terms using the Taylor's series approximations as required in the method of multiple scales. For the simplest possible planar two link (RR) robot, we demonstrate the following:

1. The criteria of positive controller gains claimed in Reference [16] is not a sufficient criteria for asymptotic stability of the *simple* PD controller – i.e., the controller gains need to be finitely large for the controller to be asymptotically stable.
2. The *augmented* PD control law in Reference [15] (page 194) for the planar RR robot is not robust – i.e., even for small errors in estimating the inertial parameters, the *augmented* PD controller does not show asymptotic stability for certain values of positive controller gains.

This paper is organized as follows: in section 2 we present the dynamic equations of motion of a planar two-link RR robot manipulator following a desired periodic trajectory under a simple PD and a augmented PD control laws. In section 3 we describe briefly how Floquet theory can be applied to analyze the stability of the simple and augmented PD controllers. In section 4, we present numerical simulation results and observations based on the numerical results. Finally, in section 6 we present our conclusions.

2 Modeling of the RR planar robot

Fig. 1 shows the schematic of a two-degree-of-freedom robot consisting of two rotary (R) joints actuated by two actuators which can generate torques Γ_1 and Γ_2, acting on links 1 and 2 respectively. The equations of motion of the planar two-link RR robot are available in standard textbooks on

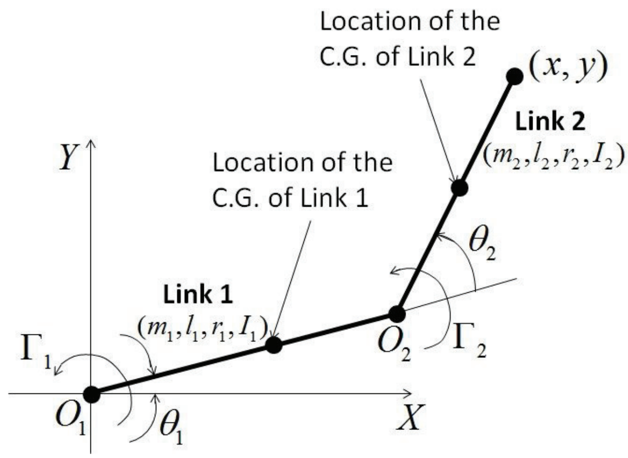


Figure 1: A RR planar robot

robotics (see, for example, [19]) and are a set of two non-linear ordinary differential equations (ODEs) of the form

$$\begin{bmatrix} \alpha_1 + \alpha_2 \cos(\theta_2) & \alpha_3 + \alpha_2 \cos(\theta_2) \\ \alpha_3 + \alpha_2 \cos(\theta_2) & \alpha_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -\alpha_2 \sin(\theta_2)\dot{\theta}_2 & -\alpha_2 \sin(\theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ \alpha_2 \sin(\theta_2)\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

$$\text{where } \alpha_1 = m_1 r_1^2 + I_1 + m_2 r_2^2 + I_2 + m_2 l_1^2, \alpha_2 = m_2 l_1 r_2, \alpha_3 = m_2 r_2^2 + I_2 \quad (1)$$

where m_j, l_j, I_j, r_j ($j = 1, 2$) are the masses, lengths, inertia and position of center of mass of link j respectively, and (Γ_1, Γ_2) are the joint torques. We consider the case of the planar two-link robot executing a periodic desired trajectory, $\Theta_d(t) = [\theta_{1_d}, \theta_{2_d}]^T = [A_f \sin(\Omega t),\ A_f \sin(\Omega t)]^T$ where A_f is the amplitude and Ω is the frequency of the desired motion. To trace the desired trajectory, a nominal torque needs to be applied at the joints and is given as

$$\Gamma_{nom} = \begin{bmatrix} \alpha_1 + \alpha_2 \cos(\theta_{2_d}) & \alpha_3 + \alpha_2 \cos(\theta_{2_d}) \\ \alpha_3 + \alpha_2 \cos(\theta_{2_d}) & \alpha_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_{1_d} \\ \ddot{\theta}_{2_d} \end{bmatrix} + \begin{bmatrix} -\alpha_2 \sin(\theta_{2_d})\dot{\theta}_{2_d} & -\alpha_2 \sin(\theta_{2_d})(\dot{\theta}_{1_d} + \dot{\theta}_{2_d}) \\ \alpha_2 \sin(\theta_{2_d})\dot{\theta}_{1_d} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_{1_d} \\ \dot{\theta}_{2_d} \end{bmatrix} \quad (2)$$

As seen in equation (2), the nominal torque is obtained by replacing θ_1 by θ_{1_d} and θ_2 by θ_{2_d} in equation (1) and this is the torque required to trace the desired trajectory in the absence of external disturbances and parameter mismatch. Due to external disturbances, mismatch in parameters and initial conditions, an error between the desired and actual trajectory $e(t)$ given by $(\Theta_d(t) - \Theta(t))$ is generated and a feedback part is required to drive this error to zero. A *simple* PD control scheme is given by

$$\Gamma_{pd} = [K_V]\dot{e} + [K_P]e \quad (3)$$

where $[K_P]$ and $[K_V]$ are positive definite and diagonal matrices of the controller gains K_p and K_v (assumed in this work to be same for both actuators). The *augmented* PD control law [15] incorporates the dynamic model of the robot and the additional feedback torque required to drive $e(t)$ to zero is chosen as

$$\Gamma_{aug} = [M(\Theta)]\ddot{\Theta}_d + [C(\Theta, \dot{\Theta})]\dot{\Theta}_d + [K_V]\dot{e} + [K_P]e \quad (4)$$

where $M(\Theta)$ denotes the mass matrix and $C(\Theta, \dot{\Theta})$ represents the Coriolis/Centripetal terms as defined in equation (1). It is shown in reference [15], that the above augmented PD control law is asymptotically stable for trajectory tracking, provided there are no errors in estimating the model parameters. Since modeling errors are inherent in practice, the robustness of this controller needs to be investigated. To do so, we assume that the estimated model parameters differ from the actual model parameters by a mismatch parameter ε and are given by

$$\hat{m}_i = (1+\varepsilon)m_i, \hat{r}_i = (1+\varepsilon)r_i, \hat{I}_i = (1+\varepsilon)I_i, \forall\ i = 1, 2 \tag{5}$$

Hence matrices M and C in equation (4) can be replaced by $\hat{M}$ and $\hat{C}$, which denote the estimates of the mass matrix and Coriolis matrix, which in turn are defined in equation (1). The value of ϵ is assumed in this paper to be the same for $i = 1, 2$. Using the procedure given in reference [20], we non-dimensionalize the equation (1) for the simple and augmented PD controllers given by equations (3, 4). We have used the $(\dot{})$ symbol shown earlier in equations (1-4) to represent differentiation with respect to dimensional time. From now on we use the $(')$ symbol to represent differentiation with respect to non-dimensional time. We get for *simple* PD control and *augmented* PD control, respectively

$$[M(\Theta)]\Theta'' + [C(\Theta, \Theta']\Theta' = \Gamma_{nom} + [K_{P_{pd}}]\mathbf{e} + [K_{V_{pd}}]\mathbf{e}' \tag{6}$$

$$[M(\Theta)]\Theta'' + [C(\Theta, \Theta']\Theta' = \Gamma_{nom} + [\hat{M}(\Theta)]\Theta_d'' + [\hat{C}(\Theta, \Theta')]\Theta_d' + [K_{P_{aug}}]\mathbf{e} + [K_{V_{aug}}]\mathbf{e}' \tag{7}$$

3 Application of Floquet theory

In this section we show how Floquet Theory can be applied for analysing the problem of trajectory tracking in robots. For more details on general Floquet theory, the reader is referred to textbooks (see for example, chapter four of [21] and chapter three of [22]). Floquet theory deals with stability of periodic solutions in dynamical systems and we follow reference [23] (and references (13, 14) contained therein).

3.1 Stability of controllers for trajectory tracking in robot manipulators

A two-link serial robot manipulator moving on a horizontal plane (i.e., without the effects of gravity) is described by

$$[M(\Theta)]_{2\times2}\Theta'' + [C(\Theta, \Theta')]_{2\times2}\Theta' = [\Gamma]_{2\times1} = [\Gamma_{nom}]_{2\times1} + [\Gamma_{feedback}]_{2\times1} \tag{8}$$

The problem of trajectory tracking is to find a suitable control law Γ such that if we begin from $|\Theta(0) - \Theta_d(0)| \approx 0$, then $\Theta(t) \approx \Theta_d(t)$ as $t \to \infty$, i.e., the zero solution $\Theta_d(0)$ must be stable. Now we consider a feedback control law, namely the *simple* PD control law used for tracking problems, given by equation (3). To analyze the stability of zero solution, we consider a small perturbation about $\Theta_d(t)$ as

$$\Theta = \Theta_d + \mathbf{e} \Rightarrow \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}\theta_{d1}\\ \theta_{d2}\end{bmatrix} + \begin{bmatrix}e_1\\ e_2\end{bmatrix} \tag{9}$$

Substituting equations (2, 3, 9) into equation (8), we have

$$[M(\Theta_d + \mathbf{e})](\Theta_d'' + \mathbf{e}'') + [C(\Theta_d + \mathbf{e}, \Theta_d' + \mathbf{e}')](\Theta_d' + \mathbf{e}') = \Gamma_{nom} - [K_P]\mathbf{e} - [K_V]\mathbf{e}' \tag{10}$$

Equation (10) can be expanded as

$$
\begin{aligned}
&([M(\Theta_d)] + \begin{bmatrix} \frac{\partial M}{\partial \theta_1} & \frac{\partial M}{\partial \theta_2} \end{bmatrix}_{\Theta=\Theta_d} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix})(\Theta_d'' + \mathbf{e}'') \\
&+ \left(C(\Theta_d, \Theta_d') + \frac{\partial C}{\partial \theta_1} e_1 + \frac{\partial C}{\partial \theta_1'} e_1' + \frac{\partial C}{\partial \theta_2} e_2 + \frac{\partial C}{\partial \theta_2'} e_2' \right) (\Theta_d' + \mathbf{e}') = \Gamma_{nom} - [K_P]\mathbf{e} - [K_V]\mathbf{e}' \qquad (11)
\end{aligned}
$$

From equation (2), $\Gamma_{nom} = [M(\Theta_d)]\Theta_d'' + [C(\Theta_d, \Theta_d')]\Theta_d'$ and hence such terms in the above equation cancel out. After simplification and writing in state space form, we get

$$
\mathbf{y}' = [J_{pd}(\Theta_d, \Theta_d')] \;\; \mathbf{y} \qquad (12)
$$

where $\mathbf{y} = [\mathbf{e}, \ \mathbf{e}']^T$ and is of dimension $2n$, whereas $J_{pd}(\Theta_d, \Theta_d')$ is the Jacobian matrix of dimension $2n \times 2n$. It maybe noted that Θ_d and Θ_d' are functions of time t. The stability of the system given by equations (12) determine the evolution of $\mathbf{y}(t)$ and hence $\mathbf{e}(t)$ as $t \to \infty$. If $\mathbf{y}(t) = [\mathbf{e} \ \mathbf{e}']^T \to 0$ as $t \to \infty$ for particular values of positive controller gains, then $\Theta(t) \to \Theta_d(t)$ as $t \to \infty$ - implying asymptotic stability. On the other hand if $\mathbf{y}(t) \neq 0$ for $t \to \infty$, then $\Theta(t) \neq \Theta_d(t)$ as $t \to \infty$ for those values of positive controller gains. The stability of equation (12) can be analyzed using Floquet theory by computing the Floquet Multipliers [24] as follows.

1. Solve the following matrix differential equation over one period ($t = 0$ to $t = T$) using initial condition $X(0) = I$.

$$
\frac{d\mathbf{X}}{dt} = J(t)\mathbf{X} \qquad (13)
$$

   where $J(t)$ is the Jacobian matrix as described in equation (12). The matrix $X(t)$ is the fundamental solution matrix and the eigenvalues of $X(T)$ (monodromy matrix) are the Floquet multipliers (for details, see references [24]).

2. If all Floquet multipliers have modulus less than one, then the zero solution is asymptotically stable, i.e. some $\mathbf{e}(t) \to 0$ as $t \to \infty$.

3. If any Floquet multiplier has modulus greater than one, then the zero solution is unstable, i.e., $\mathbf{e}(t) \to \infty$ as $t \to \infty$.

Similar to the PD control scheme, we can also determine the equations for the *augmented* PD control law described in section 2. We get

$$
\mathbf{y}' = [J_{aug}(t)]\mathbf{y} \qquad (14)
$$

where the terms in $J_{aug}(t)$ can be calculated in a manner similar to that given above. Equations (12) and (14) are the basis of the stability analysis of a planar two-link (RR) robot following a desired periodic trajectory using Floquet theory.

3.2 Limits of Floquet Theory

We end this section by summarizing some of the limitations of the application of Floquet theory **solely** with respect to stability of control laws for trajectory tracking (as described in section 3.1). This section however, does not present all possible limitations of Floquet theory and the limitations (if any) of the Floquet theory approach for other applications are beyond the scope of this work.

Table 1: Parameters of the RR planar robot

Parameter	Link1	Link2
Mass (kg)	20.15	8.25
Length (m)	0.5	0.4
Center of gravity (m)	0.18	0.26
Inertia ($\mathrm{kg-m^2}$)	6.3	1.64

1. The stability analysis using Floquet theory [25] requires that the matrices $[J_{pd}(t)]$ and $[J_{aug}(t)]$ are time-periodic. Since these matrices are only functions of desired periodic $\Theta_d(t)$ or it's derivatives, the time-periodic nature of the matrices is satisfied.

2. Floquet theory only determines local stability, i.e., stability about the particular chosen periodic orbit and the results derived in this paper are valid for the particular desired orbit $\Theta_d(t)$. To obtain global stability, all orbits $\Theta_d(t)$ must be asymptotically stable and we need to apply Floquet theory to all periodic orbits. However, in this paper we are only interested in demonstrating that the criteria of positive controller gains for the simple and augmented PD control laws for *any bounded* $\Theta_d(t)$, is not true. Hence if we are able to show that there exists a bounded $\Theta_d(t)$, with bounded velocity and acceleration, for which at some values of positive controller gains the simple and augmented PD control laws give rise to Floquet multipliers greater than 1.0, then we would have established the main objective of this paper.

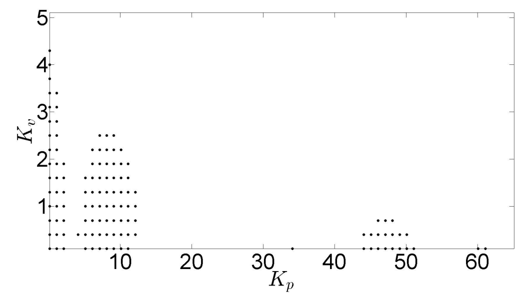
3. Floquet theory is inherently a linearized stability analysis, i.e., it only reveals the consequences of applying a small perturbation to an equilibrium or a periodic solution or some reference solution (see page 164 in reference [26]). This analysis assumes that the error between the perturbed solution and the reference solution, beyond the initial condition, either grows exponentially with time (unstable), or decays exponentially with time (asymptotic stability) or it remains constant (Lyapunov stability). In some nonlinear chaotic systems, with sensitivity to initial conditions, the error between the reference and perturbed solution can remain bounded and behave in an unpredictable manner visiting all points in a sub-space of the full state space. In the numerical simulations presented in section 4, we obtain many instances where the error between the desired and perturbed trajectories do not diverge exponentially, remaining bounded and unpredictable for certain values of controller gains. For an industrial robot executing a repetitive task, this unpredictability defeats the main goal of desired asymptotic stability, and we use the phrase *unstable* in these situations.

In Sec. 4, the results obtained from numerical simulations conducted on the planar RR robot equations driven by the *simple* and *augmented* PD controllers are presented.

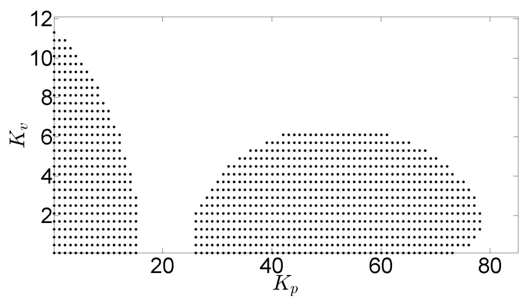
4 Results obtained from Numerical Simulations conducted on the planar RR robot equations

In this section, we present the numerical simulation results for the planar RR serial robot. To perform the numerical study, we choose the physical parameters of a robot used in reference [17]. These are as given in Table 1.

The numerical simulations were performed in MATLAB $R2014a$ using the in-built $ode15s$ solver. The relative and absolute tolerances were kept at 10^{-9} and 10^{-9}, respectively. The results were checked for smaller values of tolerances and convergence was observed. The following procedure for numerical simulation was adopted.



(a) $\Omega = 2$ rad/s



(b) $\Omega = 5$ rad/s

Figure 2: Maps showing instability in (K_p, K_v) space for the *simple* PD controller for different Ω for $A_f = \pi$ rad

4.1 Comments on the instability maps for the simple PD controller

For the *simple* PD controller, we integrate equation (12) from $t = 0$ to $t = T$ $(T = 2\pi)$ for various values of controller gains K_p and K_v and compute the largest floquet multiplier $\rho[X(T)]$ at those values (in the manner described in section 3.1 for equation 13). We plot those values of K_p and K_v where one of the floquet multipliers are greater than one (implying lack of asymptotic stability) in a 2D map (**instability map**) in (K_p, K_v) space, for varying values of desired amplitude A_f, desired frequency Ω. We have computed the instability maps for a large set of values of A_f $(\pi/10 \leq A_f \leq 2\pi)$ rad, but showed the instability maps for one case, namely for $A_f = \pi$ rad at $\Omega = (2,\ 5) rad/s$. Figure 2 shows the ranges of controller gains at which the *simple* PD controller is unstable (marked in black). We can make the following comments from the figure and several other numerical simulations conducted (not presented here).

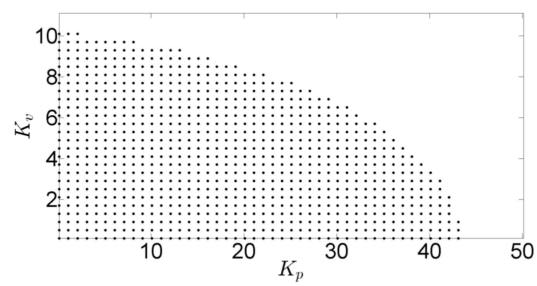
1. The ranges of controller gains at which the *simple* PD controller is unstable vary with change in amplitude A_f. For lower values of A_f, the controller is unstable for very low values of $(K_p,\ K_v)$. However, as A_f increases, the regions of instability in $(K_p,\ K_v)$ space also increase.

2. As the value of A_f is increased from $\pi/2$ to π rad, even for small values of Ω (for example $\Omega = 2$ rad/s), there were positive controller gains for which the controller was unstable.

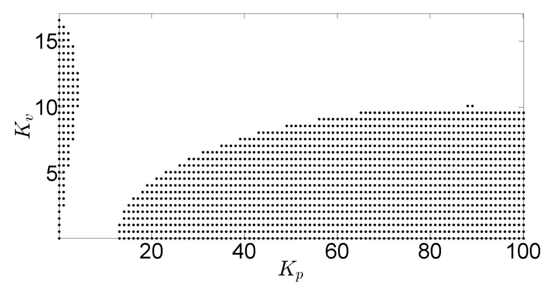
3. For very low values of Ω $(\Omega \leq 0.1$ rad/s$)$, the controller showed stability for all positive controller gains even for high values of A_f (for example, $A_f = 4\pi$ rad).

4. Ranges of instability in $(K_p,\ K_v)$ space also increases with increase in Ω. In Figure 2, for $\Omega = 2$ rad/s, the instability domain was $0.1 \leq K_p \leq 10$, $43 \leq K_p \leq 50$ and $0.1 \leq K_v \leq 5$. But as Ω increases to 5 rad/s, the instability domain was $0.1 \leq K_p \leq 15$, $25 \leq K_p \leq 80$ and $0.1 \leq K_v \leq 12$.

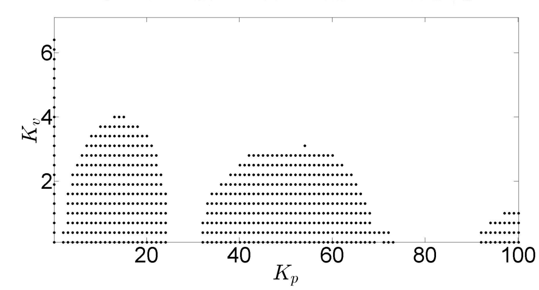
5. It must also be pointed out that range of all K_p, K_v values in the plot is kept less than the values for critical damping. The critical damping of a RR planar robot is given by $K_v = 2\sqrt{K_p}$. Hence, if we consider K_p values from 0 to 100, then critical damping is given by $K_v = 20$ and we consider K_v values from 0 to 20. The values of K_p and K_v outside this range (overdamped case) were not considered in numerical simulation, as our object is simply to show that there exist certain values of positive controller gains (underdamped or overdamped) for which the system is not asymptotic stable.



(a) $\Omega = 5$ rad/s and $A_f = \pi$ rad



(b) $\Omega = 5$ rad/s and $A_f = 2\pi$ rad



(c) $\Omega = 2$ rad/s and $A_f = 2\pi$ rad

Figure 3: Maps showing instability in (K_p, K_v) space for the *augmented* PD controller for ε=-0.3 at varying values of Ω and A_f

4.2 Comments on the instability maps for the augmented PD controller

For the *augmented* PD controller, we integrate equation (14) from $t = 0$ to $t = T$ ($T = 2\pi$) for various values of controller gains K_p and K_v, compute the largest floquet multiplier $\rho[X(T)]$ at those values and plot instability maps in (K_p, K_v) space, for varying values of desired amplitude A_f, desired frequency Ω and mismatch parameter ϵ. Shown below in Figure 3 are the instability maps for particular values of mismatch parameter ε while varying the amplitude A_f and frequency Ω of the desired motion. We computed the instability map for large set of values of A_f and Ω ($\pi/4$ rad $\leq A_f \leq 2\pi$ rad and 0.1 rad/s $\leq \Omega \leq$ 50 rad/s). However, we have shown the instability maps for only three particular cases, namely $A_f = \pi$ rad, $\Omega = 5$ rad/s, $A_f = 2\pi$ rad, $\Omega = 5$ rad/s and $A_f = 2\pi$ rad and $\Omega = 2$ rad/s. Figure 3 shows the ranges of controller gains at which the *augmented* PD controller is *not* asymptotically stable (marked in black). We can make the following comments from the figure and other numerical simulations (not presented here).

1. The ranges of controller gains for which the *augmented* PD controller is unstable varies with A_f and Ω. For example, we kept ($A_f = 2\pi$ rad) constant and varied Ω. For $(\Omega,\ \varepsilon) =$ (5 rad/s, -0.3), the range of instability was $0 \leq K_p \leq 5$, $13 \leq K_p \leq 100$ and $0.1 \leq K_v \leq 17$, whereas for $(\Omega,\ \varepsilon) =$ (2 rad/s, -0.3) the range of instability was $1 \leq K_p \leq 23$, $33 \leq K_p \leq 73$ and $92 \leq K_p \leq 100$ and $0.1 \leq K_v \leq 6.5$. Then we kept $\Omega = 5$ rad/s constant and varied A_f. For $(A_f,\ \varepsilon) = (\pi$ rad, $-0.3)$ the range of instability was $0.1 \leq K_p \leq 43$ and $0.1 \leq K_v \leq 10.1$. Similar differences can be observed for other values of ε when we vary A_f and Ω (not presented here). This shows that apart from the controller gains K_p and K_v, amplitude A_f and frequency Ω of the desired motion also determine the extent of instability of the controller.

2. For $\Omega \leq 0.1$ rad/s and $A_f \leq \pi/8$ rad, we did not find any regions of gains for which the controller was unstable and all Floquet multipliers were within the unit circle.

3. From the above instability maps, it appears that the range of controller gains for which the *augmented* PD controller is unstable, increases with an increase in A_f and an increase in Ω.

4. There was no instability for $\varepsilon > -0.03$. The range of controller gains for which the augmented PD controller is unstable increases with an increase in the underestimation between the actual and estimated model parameters, i.e., as the mismatch parameter ε reduces further below zero. For overestimation, i.e., $\varepsilon > 0$, the instability increases with increase in ε .The maps for overestimation are not presented here.

5. Even for very small mismatch parameter ($\varepsilon \leq -0.03$), we found instability at particular values of controller gains, given certain values of A_f and Ω. This implies that even for very small errors in estimation of the model parameters, the augmented PD controller does not show asymptotic stability. Hence, the augmented PD controller is not robust with respect to modeling errors.

We numerically investigated the trajectory tracking of the two link planar RR robot, for simple and augmented PD control, for various values of K_p and K_v at different values of forcing frequency Ω and forcing amplitude A_f and the motion was observed to be chaotic, i.e. implying a lack of asymptotic stability. These numerical investigations match the results presented above, i.e., the ranges of instability presented above at particular values of controller gains K_p and K_v at various values of forcing frequency Ω and forcing amplitude A_f matched the ranges of chaos obtained by the numerical investigations. We present one result in the case of augmented PD control. From figure 3 (b), we pick controller gains $K_p = 50$ and $K_v = 1$ at $\Omega = 5$ rad/s and $A_f = 2\pi$ rad. Figure 4 presents the Poincaré section (details on Poincaré section can be found in [27]) of robot at the above mentioned values and it can be clearly seen that the motion is chaotic.

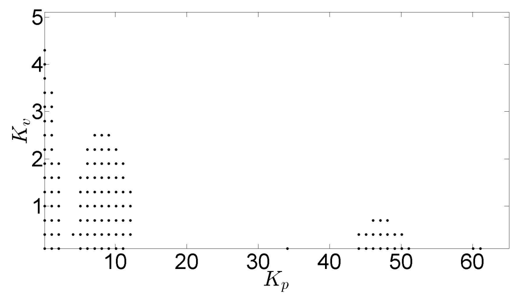


Figure 4: Poincaré map showing chaos for (K_p, K_v) = (50,1) for the *augmented* PD controller at $\Omega = 5$ rad/s and $A_f = 2\pi$ rad

5 Possible Areas of Future Research

We evaluated the asymptotic stability of the simple and augmented PD controllers for trajectory tracking using the method of Floquet Theory, which can be applied for other problems, two of which are given below.

1. **An alternative method to tuning of control parameters** - The most commonly used controller in industrial applications is the PID controller due to the simplicity of it's use. But tuning of controller gains is quite often time consuming. In this paper we showed maps of controller gains where the simple and augmented PD controllers of the 2R planar robot are unstable. The same way, ranges of controller gains where the controllers are asymptotically stable can also be calculated. This may not necessarily eliminate trial and error as the gains for stability still have to be numerically determined. However, with the approach presented in this work, the integration of the Floquet equations is only from $0 - 2\pi$, whereas to determine the asymptotic tracking stability by manual tuning of controller gains requires integration of the equations of the system for longer time periods. In that sense, Floquet theory is numerically faster. This could be explored in the future.

2. **Application of Floquet theory to underactuated systems** - Stability of underactuated systems is an area of active interest in the robotics community. Underactuated systems typically have lesser number of actuators than the degrees of freedom (DOF's), i.e. certain variables are passive. A control law could be devised to keep the passive variables tracking a periodic trajectory (since floquet theory is applicable only for periodic motions), and then another control law could be devised to control the active variables. Ranges of controller parameters for asymptotic stability could then be determined by using Floquet Theory. This could also be explored in the future.

6 Conclusions

In this paper, we have presented the stability analysis of a *simple* PD controller and the *augmented* PD controller with modeling errors, for trajectory tracking, using Floquet theory. We show successfully that the results in references [15, 16] are not sufficient to conclude asymptotic stability. We perturb the dynamic equations of a feedback controlled robot around a desired solution and compute the change in perturbation. We compute the Floquet multipliers for a range of gains K_p and K_v for varying values of amplitude A_f and frequency Ω of the desired motion and show that for some particular positive values of the gains, the Floquet multipliers are outside the unit circle, implying that controllers are unstable (or not asymptotically stable) at those values. This implies that the criteria of positive controller gains for trajectory tracking used in reference [16] is insufficient to conclude asymptotic stability in the simple PD controller. For the augmented PD controller (reference [15]), even small mismatches in model parameters will lead the controller to be unstable, implying that it is not robust with respect to modeling errors. We further showed that the unstable regions in $(K_p,\ K_v)$ space were dependent on A_f and Ω, meaning that controller gains $(K_p,\ K_v)$ are not the only parameters determining the stability of the controllers. Possible areas of future research are also presented.

7 Ethical Statement

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