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A New Approach for Kinematic Resolution of Redundancy

Abstract

This paper deals with the trajectories of points embedded in a moving rigid body being guided by a linkage having extra or "redundant" motion parameters. A new method for the kinematic use of redundancy is presented. The method is based on the concept of a metric and differs from the typically used pseudo-inverse formulation. It is shown how the redundancy can be used to alter first-order properties such as the shape of the velocity ellipse (or the ellipsoid) and a scalar measure of transmission ratio or effectiveness. It is shown how such local use of the redundancy leads to some global results such as determining the alterable regions and the boundaries of the trajectories. An example of a planar 3R manipulator illustrates these new techniques.

1. Introduction

With respect to a generic set of tasks, redundant devices, such as manipulators, fingers, and wrists, are devices that have more than the minimum independent controllable motion parameters required for the set of tasks. Redundancy is expected to aid in avoiding obstacles and singularities usually present in the workspace (Paul and Stevenson 1983) and to generally improve the workspace.

Some redundant systems have been built; the most prominent of them are (a) a seven-degrees-of-freedom, tendon-driven, torque-controlled robot (Takase, Inoue, and Sato 1974), (b) the seven-degrees-of-freedom articulated arm UJIBOT used by Yoshikawa for obstacle avoidance (Yoshikawa 1984), (c) the eight-degrees-of-freedom, redundant, sheep-shearing robot (Trevelyan, Kovesi, and Ong 1984), and (d) Yoshikawa's four-joint wrist prototype (Yoshikawa 1985) designed to overcome the problem of wrist singularity. In addition, Hollerbach (1985a) has proposed a seven-degrees-of-freedom arm to eliminate singularities. Most of the research on redundancy is theoretical; it deals with the use of the extra freedoms, which is referred to as the resolution of redundancy. To date there have been two classes of methods developed for the resolution of the redundancy. One is based on the *optimization* of certain relevant criteria (such as minimizing time, energy, or joint motions), and the other requires determining the *pseudo-inverse* of certain matrices. We present a new method for the kinematic resolution of the redundancy, which is not based on either of these ideas. Our approach is based on altering the local first-order properties.

In this paper, we primarily deal with the problem of positioning a point using a redundant mechanical linkage. In Section 1.1, we define redundant motion and resolution of redundancy. Since our method is intended as a variant to the pseudo-inverse methods, we review the existing pseudo-inverse formulations in Section 1.2 and point out their shortcomings. Since our method is based on altering local first-order properties, in Section 2 we develop the local first-order properties of trajectories of points embedded in a moving rigid body undergoing a nonredundant motion. In Section 3 we deal with the redundant motion of a point and show how the local properties can be altered. Then we show how the local analysis yields global results. In Section 4 we present an example, based on a planar 3R manipulator, to illustrate the techniques and the results.

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1.1. Redundant Motion and Resolution of Redundancy

In the most general terms, the motion of a rigid body in a fixed reference three-space $\mathbf{R}^3$ may be described by a mapping of the form

$$\Psi:(\theta_1, \ldots, \theta_m) \rightarrow (\mathscr{E}). \quad (1)$$

In the above mapping, $(\theta_1, \ldots, \theta_m)$ are the m independent motion parameters, $(\mathscr{E})$ represents the scalars locating and orienting the rigid body, and Ψ represents the mapping functions that take points in the motion parameter space to positions and orientations of the rigid body in $\mathbf{R}^3$. In the case of a manipulator the independent motion parameters are the rotations and translations at the joints, $\mathscr{E}$ is usually three Cartesian coordinates of a point on the end-effector, or three independent scalars determining the orientation of the end-effector, or six independent scalars determining the position and the orientation of the end-effector, and Ψ depends on the geometry of the manipulator (i.e., link lengths, offsets, twist angles, and the type of joints).

We have a redundant motion of a point in $\mathbf{R}^3$ if the number of independent motion parameters is greater than three, and we have a redundant motion of a rigid body if the number of independent motion parameters is greater than six. In general, we have a redundant motion of an element $(\mathscr{E})$ if the number of independent motion parameters is greater than the number of independent scalars required to specify the position of the element in the space of its motion.

Another way to define redundancy is to look at the inverse map $\Phi:(\mathscr{E}) \rightarrow (\theta_1, \ldots, \theta_m)$. For a redundant motion, for any $(\mathscr{E})$ (specified by, say, n scalars with $m > n$) there are infinitely many $(\theta_1, \ldots, \theta_m)$. If $(\mathscr{E})$ is of dimension n, the values of $(\theta_1, \ldots, \theta_m)$ lie in a space of dimension $m - n$.

The problem of resolution of redundancy can be expressed as follows: how does one obtain a *unique* $(\theta_1, \ldots, \theta_m)$ for a given $(\mathscr{E})$ when one has to choose among the infinite solutions? This involves finding "uses" for the redundancy.

In general, the problem of finding a $(\theta_1, \ldots, \theta_m)$ for a given $(\mathscr{E})$ is difficult because the functions Ψ are typically very nonlinear. Instead, most researchers have tried to resolve the redundancy at the level of velocity and have used the Jacobian map at a point. The Jacobian map may be written as

$$J(\Psi):(\dot{\theta}_1, \ldots, \dot{\theta}_m) \rightarrow (\dot{\mathscr{E}}). \quad (2)$$

In the above equation $J(\Psi)$ is an $n \times m$ matrix; the ith column of $J(\Psi)$ is the vector $\partial\Psi/\partial\theta_i$, and $\dot{\theta}_i$ and $\dot{\mathscr{E}}$ are the rates of change of θ_i and $\mathscr{E}$ (with respect to time); respectively. The problem now reduces to finding or choosing $(\dot{\theta}_1, \ldots, \dot{\theta}_m)$ for a given $(\dot{\mathscr{E}})$ and determining $(\theta_1, \ldots, \theta_m)$ by integration.

1.2. Pseudo-Inverse-Based Resolution Schemes

To obtain a $(\dot{\theta}_1, \ldots, \dot{\theta}_m)$, most researchers use the Moore-Penrose generalized inverse, also called the pseudo-inverse of $J(\Psi)$ (Rao and Mitra 1971). The most popular scheme is as follows.

Letting $(\dot{\theta}_1, \ldots, \dot{\theta}_m)$ be represented by the column vector $\dot{\Theta}$ and dropping the modifier (Ψ) for convenience, we have

$$\dot{\mathscr{E}} = J\dot{\Theta}. \quad (3)$$

If we solve for $\dot{\Theta}$, we obtain

$$\dot{\Theta} = J^{+}\dot{\mathscr{E}} + (I - J^{+}J)\dot{\mathscr{F}}, \quad (4)$$

where

$$J^{+} = J^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}. \quad (5)$$

In the above equations J^{+} is the Moore-Penrose generalized inverse, also called the pseudo-inverse of J. $(I - J^{+}J)\dot{\mathscr{F}}$ is an arbitrary vector from the null-space of J. The pseudo-inverse solution, without the null-space term, has the attractive least squares property; i.e., for a given $\dot{\mathscr{E}}$, the computed $\dot{\Theta}$ is such that the quantity $\dot{\Theta}^{\mathrm{T}}\dot{\Theta}$ is the minimum obtainable. In addition, researchers have attempted to use the second term of Eq. (4) for various other purposes. Liegeois (1977) developed a general formulation for satisfying position-dependent scalar performance criteria and applied it to the avoidance of joint limits; Yoshikawa (1984) maximized manipulability defined as $[\det(JJ^T)]^{1/2}$; Klein and Huang (1983) and Baillieul, Hollerbach, and Brockett (1984) have made repetitive motion "conservative"; Yoshikawa (1985), Klein (1985), Hanafusa, Yoshikawa, and Nakamura (1981), and Nakamura and Hanafusa (1985) have used the null space vector for obstacle avoidance. Khatib (1983; 1986) used the generalized inverse at the acceleration level for control, and Hollerbach (1985b) incorporated dynamics and examined the effect of redundancy resolution on joint torques.

1.3. Shortcomings of the Pseudo-Inverse Formulation

The pseudo-inverse solution, without the null-space term, is attractive because of its least squares property. In addition, the pseudo-inverse solution, including the null-space term, is sufficiently general that any differentiable trajectory can be realized by proper use of the null-space term (Baillieul, Hollerbach, and Brockett 1984). However, there are two major shortcomings.

1. The pseudo-inverse-based schemes, with or without the null-space term, are local numerical procedures, and it is very difficult to find any analytical results by using the pseudo-inverse-based schemes. For example, we know that for the pseudo-inverse solution (without the null-space term) $\dot{\Theta}^T\dot{\Theta}$ is minimized at every point, but it is very difficult to know a priori how large the minimum is. (The pseudo-inverse solution, without the null-space term, also has the shortcoming of yielding nonconservative motions.) In addition, the numerical procedure typically introduces errors.
2. The pseudo-inverse-based schemes do not operate at the position level; i.e., we cannot obtain $(\theta_1, \ldots, \theta_m)$ directly for a given position of the end-effector. $(\theta_1, \ldots, \theta_m)$ are obtained from $\dot{\Theta}$ (or $\ddot{\Theta}$) by integration, and we cannot easily find any local or global properties of the workspace of a redundant device.

Intuitively, we expect different (local) properties at different positions in the workspace, and different (local) properties at the same position in space of the motion when reached by different devices. We expect the local and global properties of the workspace of a redundant device to be different from those of a nonredundant device. The pseudo-inverse-based schemes do not easily yield such local or global properties of the workspace.

1.4. Schemes Based on Optimization

In addition to the pseudo-inverse-based schemes, some researchers have tried to "use" the redundancy for optimizing some motion criteria. Yahsi and Özgören (1984) used the criterion of minimal joint motion in their optimization procedure. Vukobratovic and Kircanski (1984), Fournier and Khalil (1977), and Renaud (1975) have used the criterion of minimum energy. Chang (1986) has developed a method to resolve the redundancy at the inverse kinematic level by using Lagrange multipliers and criteria for minimization. These approaches, like the pseudo-inverse-based schemes, are numerical, and they have the same shortcomings.

In Section 3 we present a new analytic method for redundancy resolution based on altering local and global properties. We first develop the concept of local first-order properties for a nonredundant motion in Section 2.

2. Nonredundant Motion of a Point

In this section we consider a point **p** in three-space moving by virtue of being rigidly attached to the free end of a two-degrees-of-freedom linkage. The point's trajectory is a surface. We will use techniques from differential geometry of surfaces and linear algebra (see, for example, Millman and Parker 1977 and Strang 1976) to develop the first-order motion properties of interest. In Section 3 we show how the local properties can be altered for redundant motions, and how we can obtain local and global properties of the workspace of a redundant device.

2.1. Velocity Distribution and Transmission Ratio

Consider a point $\mathbf{p}(x, y, z)$ moving with two degrees of freedom in $\mathbf{R}^3$. The equation of its trajectory surface may be written as

$$(x, y, z) = \Psi(\theta_1, \theta_2). \tag{6}$$

Any specific point $(\theta_1, \theta_2)_0$ in the (θ_1, θ_2) space maps to a point $\mathbf{p}_0(x, y, z)$ in $\mathbf{R}^3$. The velocity at $\mathbf{p}_0$ in $\mathbf{R}^3$ is given by

$$\mathbf{v}_{\mathbf{p}_0} = J(\Psi)_{\mathbf{p}_0}\dot{\Theta} = \Psi_1\dot{\theta}_1 + \Psi_2\dot{\theta}_2, \tag{7}$$

where $J(\Psi)_{\mathbf{p}_0}$ is the 3×2 Jacobian matrix at $\mathbf{p}_0$ and $\Psi_i = (\partial\Psi/\partial\theta_i)_0$, $i = 1, 2$; i.e., the partial derivatives are evaluated at $(\theta_1, \theta_2)_0$.

The velocity $\mathbf{v}_{\mathbf{p}_0}$ is a 3×1 vector. However, it always lies in the tangent plane[1] at $\mathbf{p}_0$. In addition, as $\dot{\theta}_1$ and $\dot{\theta}_2$ are varied at $(\theta_1, \theta_2)_0$, the direction and magnitude of the velocity vector change, and without any constraint on $|\dot{\Theta}|$ the vector $\mathbf{v}_{\mathbf{p}_0}$ completely fills the tangent plane. It is instructive to look at the velocity vector subject to the constraint $\dot{\Theta}^T\dot{\Theta} = k^2$, where k is constant. For such a constraint, we can make the following general statements:[2]

1. The maximum and minimum magnitudes of the velocity vector are k times the square root of the eigenvalues of $[g]$, where $[g]$ is a 2×2 symmetric positive definite matrix with elements $g_{ij} = \Psi_i \cdot \Psi_j$, $i, j = 1, 2$.
2. As $\dot{\theta}_1$ and $\dot{\theta}_2$ are varied subject to the constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$, the tip of the velocity vector describes an ellipse in the tangent plane.
3. The shape of the ellipse is independent of k; it is determined by the ratios of the eigenvalues of $[g]$, and the area of the ellipse is given by $k^2\pi(\det[g])^{1/2}$.

In addition, the scalar quantity $(\det[g])^{1/2}$ may also be visualized as a measure of the "transmission ratio" or "effectiveness" at a point. If we denote the square of the geometric mean of $|\mathbf{v}|_{max}$ and $|\mathbf{v}|_{min}$ by $\bar{v}^2$, we can write

$$\bar{v}^2/k^2 = (\det[g])^{1/2}. \tag{8}$$

In the above equation, k^2 is a measure of the input effort and $\bar{v}^2$ is a measure of the output. Hence, $(\det[g])^{1/2}$ is a transmission ratio.[3] It is also a measure of effectiveness in that it is zero when the degrees of freedom are less than 2, say at the boundary or at a singularity; and when $(\det[g])^{1/2}$ has a maximum value, the freedoms are most effective in producing a large output velocity.

The elements of $[g]$ define a *metric*[4] on the surface. (The elements of $[g]$ are also called the coefficients of the *first fundamental form* of the surface.) An important property of the elements of $[g]$ is that they are differential invariants; i.e., they remain unaltered by rotation and translation of the reference frame. In the context of manipulators with the reference frame attached at the first joint, the elements of $[g]$ are independent of the rotation or translation at the first joint and are only functions of the rotations or translations at the other joints. This occurs because rotation or translation at the first joint is equivalent to rotation or translation of the reference frame.

1. The tangent plane to the trajectory surface at $\mathbf{p}_0$ is given by $(\mathbf{r} - \mathbf{p}_0) \cdot \mathbf{n} = 0$, where $\mathbf{r}$ locates a point in the plane and $\mathbf{n} = (\Psi_1 \times \Psi_2)/|\Psi_1 \times \Psi_2|$ is the normal to the surface at $\mathbf{p}_0$. From (7), $\mathbf{v}_{\mathbf{p}_0} \cdot \mathbf{n} = 0$, which proves the statement.

2. Brief proofs of the statements are given in the appendix. For more details see Ghosal (1986) and Ghosal and Roth (1986).

3. Our transmission ratio is different from the manipulability measure of Yoshikawa (1984; 1985). He defines manipulability for redundant motion as $(\det[JJ^T])^{1/2}$, whereas we use $(\det[J^TJ])^{1/2}$.

4. A metric defines distance and angle on the surface. For details see Millman and Parker (1977).

2.2. Kinematic Linearity

In general, the velocity distribution is an ellipse, because the eigenvalues are not equal. However, at particular positions and for particular dimensions of the physical device generating the motion, the eigenvalues may be equal and then the velocity distribution can be described by a circle. These positions have been called isotropic (Salisbury and Craig 1982). At such positions, all directions are equivalent as far as the velocity of the point is concerned. It can be seen that this happens at all positions if the function Ψ (Eq. (6)) is linear in the motion parameters. For manipulators and other nonlinear devices, if the eigenvalues of $[g]$ are equal at any position we will call the device *kinematically linear* up to the first order at that position.

In conclusion, since the elements of $[g]$ are functions of (θ_1, θ_2) (in the case of manipulators, the g_{ij}'s are independent of θ_1, the translation or rotation at the first joint), they will be different at different points on the surface, and since they depend on Ψ they will also be different if the mechanism geometry is different. The elements of $[g]$, its eigenvalues, and the quantity $(\det[g])^{1/2}$ determine the first-order properties of the nonredundant motion of the point, and these are the quantities of interest. In the next section, we will show the relevance of these quantities and the use of the concept of *kinematic linearity* for redundant motions.

3. Redundant Motion of a Point

As mentioned before, we have a redundant motion of a point if the number of motion parameters is greater than 3 for motion in three-space (or greater than 2 for motion in a plane). In this section, we deal with redundant motion of a point in three-space.

Mathematically, the motion of a point in $\mathbf{R}^3$ can be described by a mapping of the form

$$\Psi : (\theta_1, \theta_2, \ldots, \theta_m) \rightarrow (x, y, z). \tag{9}$$

In the above equation, θ_i, $i = 1, \ldots, m$, are the m parameters of the motion, and (x, y, z) are the coordinates of a point in $\mathbf{R}^3$. Thus we have $m - 3$ "extra" or redundant parameters. The velocity of the point at a generic position, $\mathbf{p}_0$, along a trajectory curve $\Psi(\theta_1(t), \theta_2(t), \ldots, \theta_m(t))$ is given by

$$\mathbf{v}_{\mathbf{p}_0} = \sum_{i=1}^{m} \Psi_i \dot{\theta}_i, \tag{10}$$

where $\Psi_i i = 1, 2, \ldots, m$, are the partial derivatives of Ψ with respect to θ_i, $i = 1, 2, \ldots, m$, evaluated at the values of θ_i that correspond to $\mathbf{p}_0$.

Since the motion is in three-space, at most three of the m $\Psi_i\dot{\theta}_i$'s are independent. We can write the $(m - 3)$ $\Psi_i\dot{\theta}_i$'s as a linear combination of the three independent ones. We will assume that the three independent terms are $\Psi_1\dot{\theta}_1$, $\Psi_2\dot{\theta}_2$, and $\Psi_3\dot{\theta}_3$. Thus for $\Psi_4\dot{\theta}_4$ we have

$$\Psi_4\dot{\theta}_4 = \alpha_{11}\Psi_1\dot{\theta}_1 + \alpha_{12}\Psi_2\dot{\theta}_2 + \alpha_{13}\Psi_3\dot{\theta}_3. \tag{11}$$

In general, we have $m - 3$ equations of the form

$$\Psi_{j+3}\dot{\theta}_{j+3} = \sum_{i=1}^{3} \alpha_{ji}\, \Psi_i\dot{\theta}_i, \qquad j = 1, \ldots, m-3, \tag{12}$$

where α_{ji} are the elements of an $(m - 3) \times 3$ matrix of scalars. We get $(m - 3) \times 3$ equations by taking the dot product of each of the above $m - 3$ equations with Ψ_i, $i = 1, 2, 3$. Solving for the α_{ji}'s in terms of known quantities[5] and the $\dot{\theta}$'s, we have

$$\alpha_{ji} = \frac{\det[A_i]\dot{\theta}_{j+3}}{\det[g]\dot{\theta}_i}, \qquad i = 1, 2, 3, \quad j = 1, 2, \ldots, m-3, \tag{13}$$

where $[g]$ is the matrix with elements g_{ij} given by the dot products $\Psi_i \cdot \Psi_j$, $i, j = 1, 2, 3$, and $[A_i]$ is the matrix $[g]$ with the jth element in the ith column replaced by $(\Psi_i \cdot \Psi_{j+3})$ for $i = 1, 2, 3$. The α_{ji}'s may be determined in terms of $\det[g]$, $\det[A_i]$, and $\dot{\theta}_i$ as long as $\det[g] \neq 0$ and $\det[A_i] \neq 0$. We have assumed that none of the $\dot{\theta}_i$'s are zero, because if any of them are

5. In Eq. (13) $[g]$ and $[A_i]$ are known once the position of the point $\mathbf{p}_0$, is known.

zero we have less than m motion parameters. In the region where $\det[g] = 0$ and $\det[A_i] = 0$, the α_{ji} are indeterminate. We discuss the issue of indeterminate α_{ji} in Section 3.2.

3.1. Resolution of Redundancy

To compute the $\dot{\theta}_{j+3}$'s we need $m - 3$ more equations. These $m - 3$ equations can be obtained from setting constraints on the first-order properties, such as the velocity distribution and transmission ratio, at the point under consideration. We first develop the constraint equations for the velocity distribution.

Substituting (12) in (10) and dropping the subscript, we can write the velocity at a generic point as

$$\mathbf{v} = \sum_{i=1}^{3} \left(1 + \sum_{j=1}^{m-3} \alpha_{ji}\right) \Psi_i \dot{\theta}_i = [J']\dot{\Theta}, \tag{14}$$

where $\dot{\Theta}$ is the vector $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T$, and $[J']$ is a 3×3 matrix whose ith column is $(1 + \Sigma_{j=1}^{m-3} \alpha_{ji})\Psi_i$. The analysis of the nonredundant motion can be easily extended to show that for a normalization condition $\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 = k^2$ the tip of the velocity vector lies on the surface of an ellipsoid.[6] (We have assumed that all the $\dot{\theta}_i$ ($i = 1, 2, \ldots, m$) are finite.) The shape of the ellipsoid depends on the eigenvalues of $[g']$, where the elements of $[g']$ are

$$g'_{ij} = [J']^T[J'] = \sum_{p,k=1}^{m-3} [(1 + \alpha_{jk})(1 + \alpha_{ip})]\Psi_i \cdot \Psi_j. \tag{15}$$

We observe from Eq. (15) that the elements of $[g']$ are functions of the α_{ji}'s, which are in turn functions of the extra or redundant $(m - 3)$ $\dot{\theta}_i$'s. Hence, the shape and size of the ellipsoid can be changed by appropriate use of these redundant parameters. One special case is when the ellipsoid becomes a sphere; this gives a uniform or, so-called, spherical velocity distribution. (This is the three-dimensional analog of the planar circular velocity distribution described in the previous section.) At points where the velocity distribution is spherical, the device generating the motion is kinematically linear. To obtain the spherical velocity distribution, we make the eigenvalues of $[g']$ equal. The eigenvalues of $[g']$ are the roots of the characteristic cubic, which may be written as

$$\lambda^3 - a\lambda^2 + b\lambda - c = 0, \tag{16}$$

where

$$\begin{aligned} a &= g'_{11} + g'_{22} + g'_{33}, \\ b &= (g'_{11}g'_{22} - g'^2_{12}) + (g'_{22}g'_{33} - g'^2_{23}) \\ &\quad + (g'_{33}g'_{11} - g'^2_{13}), \\ c &= \det[g']. \end{aligned} \tag{17}$$

For the three eigenvalues to be equal, we require that

$$a^2 = 3b, \qquad a^3 = 27c. \tag{18}$$

The two equations in (18) are functions of the α_{ji}'s and the $\dot{\theta}_{j+3}$'s. Together with the normalization condition and Eq. (13), we can now find two of the $(m - 3)$ $\dot{\theta}_j$'s at the point under consideration. Equations (18) are also functions of the motion parameters θ_j, and, as we will see in an example in Section 4.1, they yield the regions in the workspace of a device where the eigenvalues can be made equal. These regions are called the *alterable regions.*

We can also get additional equations by considering a scalar measure of transmission ratio or effectiveness. In analogy to Eq. (8), a scalar measure in this case can be shown to be the square root of the determinant of $[g']$ and to be proportional to the volume of the velocity ellipsoid.[7] We denote this scalar measure by $\mathcal{V}$. If we take the derivative of $\mathcal{V}$ with respect to $\dot{\theta}_k$, it follows that for given $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$ extreme values of $\mathcal{V}$ occur when

$$\partial \det[g']/\partial\dot{\theta}_k = 0, \qquad k = 4, \ldots, m. \tag{19}$$

6. In Eq. (I.3) of the appendix, instead of $[g]$ we have a 3×3 symmetric matrix $[g']$, and the matrix $[A]$ is of rank 3; i.e., the velocity vector lies in three-space.

7. The volume of the ellipsoid is given by $(2\pi/3)k^3(\lambda'_1\lambda'_2\lambda'_3)^{1/2} = (2\pi/3)(\det[g'])^{1/2}$, where λ'_i ($i = 1, 2, 3$) are the eigenvalues of $[g']$.

This gives $m - 3$ equations that could be used to solve for $\dot{\theta}_k$, $k = 4, \ldots, m$. Using these $\dot{\theta}_k$'s would yield motions that have maximum or minimum $\mathcal{V}$ for all points in the $(\dot{\theta}_4, \ldots, \dot{\theta}_m)$ subspace.

We can also get alternative or additional constraint equations by altering some higher-order properties such as the components of the acceleration distribution at the point $\mathbf{p}_0$ (Ghosal 1986; Ghosal and Roth 1986).

3.2. Indeterminate α_{ji}

The above procedures for computing $\dot{\theta}_k$, $k = 4, \ldots, m$, can be used as long as the α_{ij}'s are not in the indeterminate form 0/0. In the positions where the numerator and denominator of Eq. (13) are zero, the first-order properties cannot be altered. In these positions the number of independent parameters is less than m. These positions are independent of the property being altered and are fixed for a given Ψ; i.e., they are determined by the geometry and the type of physical device. Such positions occur at the boundaries and at the singularities in the workspace of a redundant device, since at the boundaries and at the singularities the number of independent motion parameters is less than m. Hence, the conditions for indeterminate α_{ji} can yield the boundaries and singularities in the workspace of a redundant manipulator. In Section 4.1 we will show how the boundaries and singularities can be obtained for a planar 3R manipulator.

3.3. Choice of Independent $\Psi_i\dot{\theta}_i$ and Alterable Region

In our analysis, we have assumed that $\Psi_i\dot{\theta}_i$, $i = 1, 2, 3$, are the independent terms. For an actual redundant manipulator, these would be the contributions of, say, the first three joints. The alterable regions or the regions where the eigenvalues can be equal would, in general, be different if we choose terms corresponding to three other joints. If the point $\mathbf{p}(x, y, z)$ is in a region where there is only one way to choose the independent terms, we have no alternative. If there is more than one possible set of independent terms, then it is possible to choose the set that satisfies an additional criterion such as this: the set that results in the lowest maximum values for the computed $\dot{\theta}_k$'s, or the set that yields maximum area or volume of the alterable region. Alternatively, the choice of the independent set can be based on dynamics and controls considerations.

3.4. Inverse Kinematics

We have thus presented a method for using the redundancy at the velocity level. Once the $\dot{\theta}_k$'s are known, most researchers use integration to obtain the θ_k's. In this section we outline a procedure for obtaining the θ_k's that does not involve integration.

In Section 3.1 we presented the conditions for equal eigenvalues and maximum and minimum transmission ratios. It is sometimes possible to easily obtain the analytic functions that give regions in the workspace of a redundant manipulator where the eigenvalues are equal or the transmission ratio is maximum or minimum. These are equations of the form $f_i(\theta_1, \ldots, \theta_m) = 0$, $i = 1, \ldots, m - 3$. Such equations and the three equations giving (x, y, z) as functions of the independent parameters (Eq. (9)) can now be solved to give all the θ_k's. This method for inverse kinematics works only if the point $\mathbf{p}(x, y, z)$ is in the alterable region. In the next section we illustrate the method in an example of a planar 3R manipulator.

4. An Example — Three-Parameter Motion in a Plane

In this section we present the analysis for three-parameter motion in a plane. We first analyze the general case and then present an example of a 3R planar manipulator.

Mathematically, a general point trajectory in a plane due to a three-parameter motion can be represented as

$\Psi:(\theta_1, \theta_2, \theta_3) \to (x, y)$, where θ_i, $i = 1, 2, 3$, are the motion parameters, (x, y) are the coordinates of a moving point $\mathbf{p}(x, y)$ measured in a fixed reference plane $\mathbf{R}^2$, and the function Ψ depends on the actual mechanism. (The inverse function to Ψ, which gives θ_i, $i = 1, 2, 3$, for known x and y, has infinitely many solutions.) The velocity of the point is

$$\mathbf{v} = \sum_{i=1}^{3} \Psi_i \dot{\theta}_i. \tag{20}$$

For general positions (except at the boundary), two out of the three $\Psi_i\dot{\theta}_i$ are independent. We first consider the case when $\Psi_3\dot{\theta}_3$ is a linear combination of $\Psi_i\dot{\theta}_i$, $i = 1, 2$. (We will also consider the cases of $\Psi_1\dot{\theta}_1$ or $\Psi_2\dot{\theta}_2$ being dependent.) We can write

$$\Psi_3\dot{\theta}_3 = \sum_{i=1}^{2} \alpha_i \Psi_i \dot{\theta}_i. \tag{21}$$

If α_1 (or α_2) is zero, then $\Psi_3\dot{\theta}_3$ is parallel to $\Psi_2\dot{\theta}_2$ (or $\Psi_1\dot{\theta}_1$). α_1 and α_2 can be determined in terms of the $\dot{\theta}_i$'s and the dot products $\Psi_i \cdot \Psi_j$: forming the dot product of (21) with Ψ_1 and Ψ_2 yields

$$\begin{aligned}(\Psi_3 \cdot \Psi_1)\dot{\theta}_3 &= \alpha_1 g_{11}\dot{\theta}_1 + \alpha_2 g_{12}\dot{\theta}_2,\\ (\Psi_3 \cdot \Psi_2)\dot{\theta}_3 &= \alpha_1 g_{12}\dot{\theta}_1 + \alpha_2 g_{22}\dot{\theta}_2.\end{aligned} \tag{22}$$

From Eqs. (22) we have

$$\begin{aligned}\alpha_1 &= \frac{[(\Psi_3 \cdot \Psi_1)g_{22} - (\Psi_3 \cdot \Psi_2)g_{12}]\dot{\theta}_3}{(g_{11}g_{22} - g_{12}^2)\dot{\theta}_1} = a_1\left(\frac{\dot{\theta}_3}{\dot{\theta}_1}\right),\\ \alpha_2 &= -\frac{[(\Psi_3 \cdot \Psi_1)g_{12} - (\Psi_3 \cdot \Psi_2)g_{11}]\dot{\theta}_3}{(g_{11}g_{22} - g_{12}^2)\dot{\theta}_2}\\ &= a_2\left(\frac{\dot{\theta}_3}{\dot{\theta}_2}\right),\end{aligned} \tag{23}$$

where $g_{ij} = \Psi_i \cdot \Psi_j$, $i, j = 1, 2$. Substituting (21) in (20), we get

$$\mathbf{v} = \sum_{i=1}^{2} (1 + \alpha_i)\Psi_i\dot{\theta}_i. \tag{24}$$

The maximum and minimum velocities subject to the constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$ are obtained from the eigenvalues of the symmetric matrix $[g']$ associated with the standard quadratic form $\mathbf{v}^2 = \dot{\Theta}^T[g']\dot{\Theta}$. The elements of the matrix $[g']$ are given by

$$\begin{aligned}g'_{11} &= (1 + \alpha_1)^2(\Psi_1 \cdot \Psi_1) = (1 + \alpha_1)^2 g_{11},\\ g'_{12} &= (1 + \alpha_1)(1 + \alpha_2)(\Psi_1 \cdot \Psi_2)\\ &= (1 + \alpha_1)(1 + \alpha_2)g_{12},\\ g'_{22} &= (1 + \alpha_2)^2(\Psi_2 \cdot \Psi_2) = (1 + \alpha_2)^2 g_{22}.\end{aligned} \tag{25}$$

The eigenvalues $[g']$ are functions of g_{11}, g_{12}, g_{22}, α_1, and α_2. The velocity distribution is circular when the eigenvalues of $[g']$ are equal.[8] The condition for equal eigenvalues is given by

$$[(1 + \alpha_2)^2 g_{22} - (1 + \alpha_1)^2 g_{11}]^2 + 4(1 + \alpha_1)^2(1 + \alpha_2)^2 g_{12}^2 = 0. \tag{26}$$

Since the left side of the above equation is the sum of two squares, it follows that both terms must be zero for the eigenvalues to be equal. If $g_{12} \neq 0$, then

$$\alpha_1 = \alpha_2 = -1. \tag{27}$$

Otherwise we require $g_{12} = 0$. Then

$$(1 + \alpha_1)/(1 + \alpha_2) = \pm(g_{22}/g_{11})^{1/2}. \tag{28}$$

The first case results in $\mathbf{v}$ always equal to zero (from (24)) and, hence, is not of much interest. In the second case, Eq. (28), α_1 and α_2 are

$$\begin{aligned}\alpha_2 &= \frac{\pm(g_{11}/g_{22})^{1/2} - 1}{1 \mp (g_{11}/g_{22})^{1/2}[a_1\dot{\theta}_2/(a_2\dot{\theta}_1)]},\\ \alpha_1 &= (a_1\dot{\theta}_2/a_2\dot{\theta}_1)\alpha_2,\end{aligned} \tag{29}$$

where a_1 and a_2 are defined in (23), and by using (23) and $\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$, we get

$$\dot{\theta}_3^2 = [(a_1/\alpha_1)^2 + (a_2/\alpha_2)^2]^{-1}. \tag{30}$$

We observe from (29) and (30) that for any $\dot{\theta}_2/\dot{\theta}_1$ there are two values of α_1 and α_2, and for each α_1 and α_2

8. Note that when $\alpha_1 = \alpha_2 = 0$, $[g] = [g']$ and we have the nonredundant motion described in Section 2.

there are two $\dot{\theta}_3$'s equal in magnitude but of opposite sign. Hence, if we plot $\dot{\theta}_3$ (see example in Section 4.1 and Fig. 3) as a function of the $\dot{\theta}_2/\dot{\theta}_1$, we will get four curves, but two of them will be mirror images of the other two.

The above procedure (to compute $\dot{\theta}_3$) can be used as long as α_1 and α_2 are not indeterminate. Setting the numerator and denominator (of, say, α_1) in (23) to zero yields equations $g_{11}g_{22} - g_{12}^2 = 0$ and $(\Psi_1 \cdot \Psi_3)g_{22} - (\Psi_2 \cdot \Psi_3)g_{12} = 0$. These equations represent curves, and when $\mathbf{p}$ is on these curves the velocity distribution cannot be altered.

In the above procedure, we used $\Psi_3\dot{\theta}_3$ as the dependent term. Instead, if $\Psi_2\dot{\theta}_2$ is used, the alterable region will be different. We will get the condition $g_{13} = 0$ for equal eigenvalues, and similar to (28) we get

$$\alpha_3 = \frac{\pm(g_{11}/g_{33})^{1/2} - 1}{1 \mp (g_{11}/g_{33})^{1/2}[a_1\dot{\theta}_3/(a_3\dot{\theta}_1)]}, \qquad \alpha_1 = (a_1\dot{\theta}_3/a_3\dot{\theta}_1)\alpha_3, \tag{31}$$

and again with $\dot{\theta}_1^2 + \dot{\theta}_3^2 = 1$, we get

$$\dot{\theta}_2^2 = [(a_1/\alpha_1)^2 + (a_3/\alpha_3)^2]^{-1}. \tag{32}$$

In the above equations a_1 and a_3 are obtained when α_1 and α_3 are solved for in terms of the dot products $\Psi_i \cdot \Psi_j$, $\dot{\theta}_1$, and $\dot{\theta}_3$ (the result is analogous to Eqs. (23)).

In a similar manner, the region where the eigenvalues are equal, when $\Psi_1\dot{\theta}_1$ is the dependent term, is given by $\Psi_2 \cdot \Psi_3 = 0$. The resulting expression for $\dot{\theta}_1$ is similar to Eq. (30).

The inverse kinematics for the three-parameter motion of a point in a plane can also be done very easily. We have two equations, $(x, y) = \Psi(\theta_1, \theta_2, \theta_3)$, and we have the condition for equal eigenvalues, $g_{ij} = 0$, $i \neq j$. If the point $\mathbf{p}(x, y)$ is in the alterable region, we can solve for $(\theta_1, \theta_2, \theta_3)$ from the equations $g_{ij} = 0$, $i \neq j$, and $(x, y) = \Psi(\theta_1, \theta_2, \theta_3)$ As mentioned before, the procedure does not involve integration.

From the above analysis we can make the following general statement for three-parameter redundant motions of a point in $\mathbf{R}^2$: in general, except for regions where α_i is indeterminate, we can at each point determine the values of $\dot{\theta}_i$, as a function of the g_{ij}'s and the $\dot{\theta}_j$'s ($i \neq j$), that will give any required velocity distribution. Furthermore, for equal eigenvalues g_{ij}, $i \neq j$, must be zero. In addition, when the point $\mathbf{p}(x, y)$ is in the alterable region the inverse kinematics follows very easily.

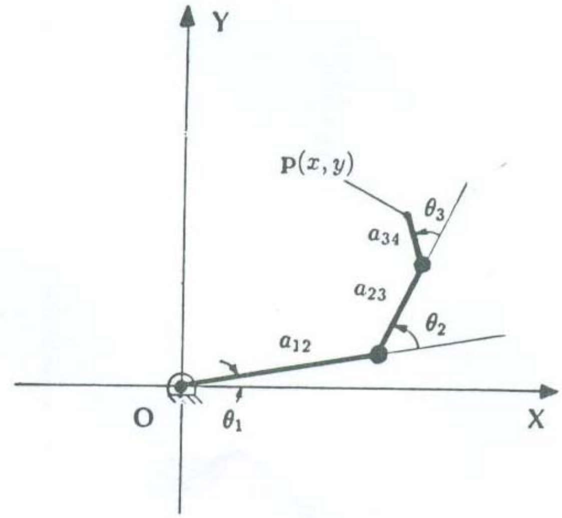


Fig. 1. Planar 3R manipulator.

4.1. A 3R Manipulator in a Plane

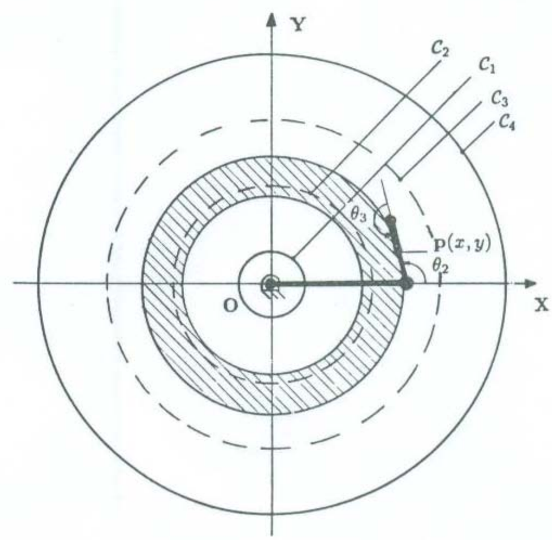
Figure 1 shows a three-degrees-of-freedom manipulator in the plane XY. There are three revolute joints, with rotations θ_1, θ_2, and θ_3. The link lengths are a_{12}, a_{23}, and a_{34}. We are interested in the motion of the point $\mathbf{p}(x, y)$. The kinematic equations Ψ: $(\theta_1, \theta_2, \theta_3) \rightarrow (x, y)$ are[9]

$$x = a_{12}c_1 + a_{23}c_{1+2} + a_{34}c_{1+2+3}, \qquad y = a_{12}s_1 + a_{23}s_{1+2} + a_{34}s_{1+2+3}. \tag{33}$$

The g_{ij}'s, given by $\Psi_i \cdot \Psi_j$, can be computed from (33). They are

9. In this paper, c_1, s_1, c_{1+2}, etc. represent $\cos(\theta_1)$, $\sin(\theta_1)$, $\cos(\theta_1 + \theta_2)$, etc., respectively.

Fig. 2. Alterable region I.



$$
\begin{aligned}
g_{11} &= a_{12}^2 + a_{23}^2 + a_{34}^2 + 2a_{12}a_{23}c_2 \\
&\quad + 2a_{12}a_{34}c_{2+3} + 2a_{23}a_{34}c_3, \\
g_{12} &= a_{23}^2 + a_{34}^2 + a_{12}a_{23}c_2 + a_{12}a_{34}c_{2+3} \\
&\quad + 2a_{23}a_{34}c_3, \\
g_{22} &= a_{23}^2 + a_{34}^2 + 2a_{23}a_{34}c_3, \\
g_{13} &= a_{34}^2 + a_{12}a_{34}c_{2+3} + a_{23}a_{34}c_3, \\
g_{23} &= a_{34}^2 + a_{23}a_{34}c_3, \\
g_{33} &= a_{34}^2.
\end{aligned} \tag{34}
$$

As expected, the g_{ij}'s are independent of θ_1. (They are invariant under rotation and translation of the coordinate system, and changing θ_1 is equivalent to rotating the coordinate system.)

The velocity of the point is given by

$$\mathbf{v} = \Psi_1\dot{\theta}_1 + \Psi_2\dot{\theta}_2 + \Psi_3\dot{\theta}_3. \tag{35}$$

We first consider $\Psi_3\dot{\theta}_3$ as a linear combination of $\Psi_1\dot{\theta}_2$ and $\Psi_2\dot{\theta}_2$. For equal eigenvalues θ_2 and θ_3 have to be such that $g_{12} = 0$. For $a_{12} = 4$, $a_{23} = 2$, $a_{34} = 1$, the expression $g_{12} = 0$ given in (34) reduces to

$$5 + 8c_2 + 4c_{2+3} + 4c_3 = 0. \tag{36}$$

From the above equation it follows that for each value of θ_3 we have two values of θ_2, given by

$$\tan(\theta_2/2) = \tfrac{1}{3}[-4s_3 \pm (55 + 24c_3 - 16c_3^2)^{1/2}]. \tag{37}$$

The region where g_{12} is zero can be obtained by eliminating θ_2 and θ_1 from Eqs. (33) and (36) and is given by the equation

$$x^2 + y^2 = 11 - 4c_3. \tag{38}$$

We can see from (38) that θ_3 could take any value between 0° and 180°, and the extreme values of θ_2 are ±138.59° and ±104.47°, respectively. Since θ_1 can take any value, the velocity distribution can be altered in the annular shaded region shown in Fig. 2. This region is bounded by circles of radii $\sqrt{7}$ and $\sqrt{15}$, respectively, and will be called the *alterable region I.* In this region, on one circle, $x^2 + y^2 = 9$, α_1 and α_2 are indeterminate, and the velocity distribution cannot be altered. The conditions for indeterminate α_1 and α_2 yield curves $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, and $\mathscr{C}_4$—four concentric circles of radii 1, 3, 5, and 7, respectively. The circles with radii 1 and 7 are the inner and the outer boundaries, respectively. The circles with radii 3 and 5 are the singularities where the number of independent motion parameters reduces to two. The four circles are shown in Fig. 2.

One set of points where the velocity distribution can be altered is $\theta_2 = 104.47°$ and $\theta_3 = 180°$. We plot $\dot{\theta}_3$ (for these points) as a function of the angle δ, defined by $\tan\delta = \dot{\theta}_2/\dot{\theta}_1$, in Fig. 3. (Since $0° \le \delta \le 360°$, it is easier to plot $\dot{\theta}_3$ as a function of δ than as a function of $\dot{\theta}_2/\dot{\theta}_1$.) As expected, there are four curves, with two of them being mirror images of the other two. The four curves in Fig. 3 give the required values of $\dot{\theta}_3$ for circular velocity distribution at points corresponding to $\theta_2 = 104.47°$ and $\theta_3 = 180°$.

Next we consider $\Psi_2\dot{\theta}_2$ as the dependent term. The alterable region is now given by $g_{13} = 0$. The alterable region is given by the equation

$$x^2 + y^2 = 17 + 16c_2. \tag{39}$$

The boundaries of the alterable region are circles of radii 1 and $\sqrt{33}$, respectively. The shaded *alterable region II* is shown in Fig. 4.

Finally, we consider the case of $\Psi_1\dot{\theta}_1$ expressed as a linear combination of $\Psi_2\dot{\theta}_2$ and $\Psi_3\dot{\theta}_3$. In this case

Fig. 3. Plot of $\dot{\theta}_3$ with respect to δ.

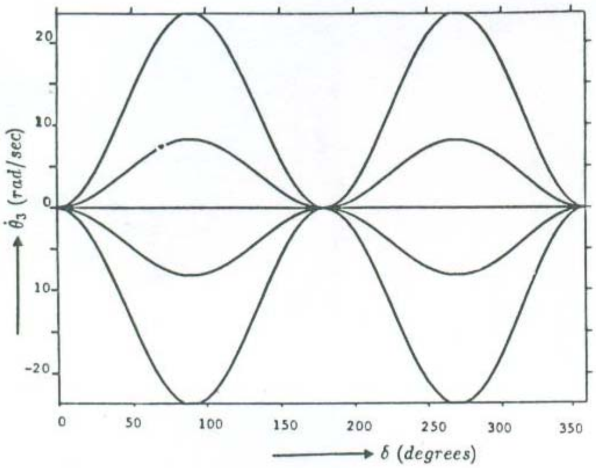
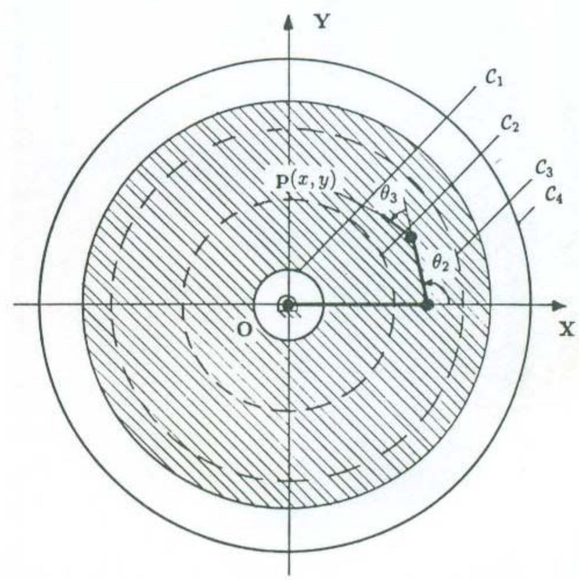


Fig. 4. Alterable region II.



$g_{23} = 0$, which yields $\cos(\theta_3) = -0.5$ and arbitrary θ_2. This yields *alterable region III,* as shown in Fig. 5. In this case the alterable region is bounded by circles of radii 2.55 and 5.44, respectively.

We next give the procedure for the inverse kinematics, assuming the point is in alterable region I. (The inverse kinematics can also be easily done for the other two alterable regions.) Squaring and adding the left side of Eq. (33), we get

$$x^2 + y^2 = a_{12}^2 + a_{23}^2 + a_{34}^2 + 2a_{12}a_{23}c_2 + 2a_{12}a_{34}c_{2+3} + 2a_{23}a_{34}c_3. \quad (40)$$

Using $g_{12} = 0$ in (40), we have

$$c_3 = (1/2a_{23}a_{34})(a_{12}^2 - a_{23}^2 - a_{34}^2 - x^2 - y^2). \quad (41)$$

The above equation gives two values of θ_3 for a given (x, y).

We can also write Eq. (33) as

$$(x - a_{12}c_1)^2 + (y - a_{12}s_1)^2 = a_{23}^2 + a_{34}^2 + 2a_{23}a_{34}c_3, \quad (42)$$

and substituting c_3 we get

$$xc_1 + ys_1 = (x^2 + y^2)/a_{12}^2. \quad (43)$$

If we substitute for c_1 and s_1 in terms of the tangent of the half-angle, we get a quadratic in $\tan(\theta_1/2)$ that yields two values of θ_1.

Finally from the condition $g_{12} = 0$, we get

$$x^2 + y^2 - a_{12}^2 = a_{12}a_{23}c_2 + a_{12}a_{34}c_{2+3}. \quad (44)$$

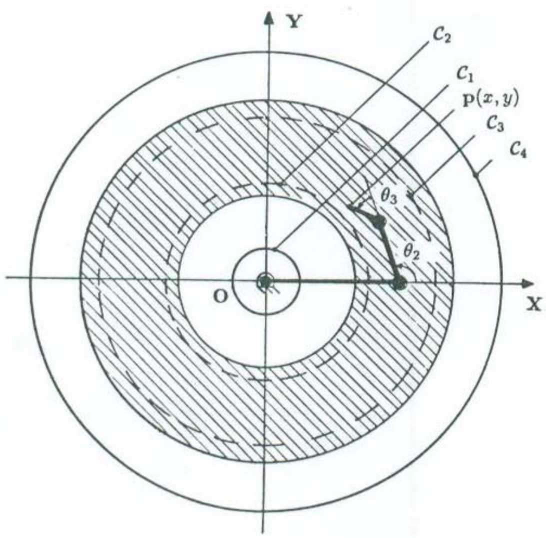
Once θ_3 is known, we can find θ_2 from the above equation.

Equations (41), (43), and (44) give θ_1, θ_2, and θ_3 for a given (x, y) lying in alterable region I. We can similarly find closed-form solutions for the inverse kinematics in the other regions.

5. Discussion

In Section 2.1 we mentioned that the g_{ij}'s are differential invariants. This implies that when all joint axes intersect, the g_{ij}'s are independent of the rotations at the joints. Hence, for a spherical mechanism, such as a wrist, the velocity distribution will be the same at every point (excluding the singularities and boundaries) in the workspace.

Fig. 5. Alterable region III.



In Section 4 we demonstrated that with one extra motion parameter for planar motion, we can make any point-positioning linkage *kinematically linear* in a large portion of its workspace. For spatial motion we need two extra motion parameters for kinematic linearity; i.e., with a five-degrees-of-freedom arm we can have a spherical velocity distribution in large portions of a reference point's workspace. (It is interesting to note that the human arm with the wrist excluded has five degrees of freedom.)

In this paper we have presented a new method for kinematic use of redundancy. The approach does not require the pseudo-inverse and is analytical. Our method relies on the concept of altering first-order properties of point trajectories generated by the redundant mechanism. We can also alter the second- and higher-order properties. A preliminary discussion on altering second-order properties is given in Ghosal (1986); however, more work needs to be done.

We have restricted $\dot{\theta}_i$ ($i = 1, 2, 3$) (($i = 1, 2$) for planar motion) to lie on a sphere of radius k (circle for planar motion) by use of the normalization condition $\dot{\Theta}^T\dot{\Theta} = k^2$. The only restriction on the rest of the $(m - 3)$ ($m - 2$ for planar motion) $\dot{\theta}_i$'s is that they be nonzero and finite. If we restrict any of these $(m - 3)$ ($m - 2$ for planar motion) $\dot{\theta}_i$'s, the alterable region will become smaller. A common restriction might be a bounded angular velocity magnitude, say $|\dot{\theta}_i| \leq k'$.

We have dealt only with the positional aspects of the motion of a rigid body. This approach could be extended to deal with the rotational motion of a rigid body, where we could consider altering the angular velocity vector. Such work is yet to be done.

Acknowledgments

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Appendix

In this appendix we give brief proofs of the statements in Section 2.1. From Eq. (6) the square of the velocity at $\mathbf{p}_0$ can be written as

$$\mathbf{v}^2 = g_{11}\dot{\theta}_1^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2^2, \qquad \text{(I.1)}$$

where $g_{ij} = \Psi_i \cdot \Psi_j$, $i, j = 1, 2$. It is known from differential geometry (Millman and Parker 1977) that the symmetric matrix $[g]$ with elements g_{ij} is positive definite.

To find the maximum and minimum $\mathbf{v}^2$ subject to the constraint $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$, we use the method of Lagrange multipliers. We solve $\partial f/\partial\dot{\theta}_1 = \partial f/\partial\dot{\theta}_2 = 0$, where

$$f = g_{11}\dot{\theta}_1^2 + 2g_{12}\dot{\theta}_1\dot{\theta}_2 + g_{22}\dot{\theta}_2^2 - \lambda(\dot{\theta}_1^2 + \dot{\theta}_2^2 - k^2). \qquad \text{(I.2)}$$

The conditions $\partial f/\partial\dot{\theta}_1 = \partial f/\partial\dot{\theta}_2 = 0$ reduce to the eigenvalue problem

$$[g]\dot{\Theta} - \lambda\dot{\Theta} = 0. \qquad \text{(I.3)}$$

The eigenvalues of $[g]$ are

$$\lambda_{1,2} = \tfrac{1}{2}[(g_{11} + g_{22}) \pm ((g_{11} + g_{22})^2 - 4(g_{11}g_{22} - g_{12}^2))^{1/2}]. \qquad \text{(I.4)}$$

Assuming $\lambda_1 > \lambda_2$, we have from (I.3), (I.1), and $\dot{\theta}_1^2 + \dot{\theta}_2^2 = k^2$ that

$$|\mathbf{v}|_{\max} = \sqrt{\mathbf{v}^2|_{\lambda=\lambda_1}} = k\sqrt{\lambda_1} \tag{I.5}$$

and

$$|\mathbf{v}|_{\min} = \sqrt{\mathbf{v}^2|_{\lambda=\lambda_2}} = k\sqrt{\lambda_2}. \tag{I.6}$$

Equations (I.5) and (I.6) prove statement (1) in Section 2.1.

To prove that the tip of the velocity vector describes an ellipse in the tangent plane, we start with Eq. (7). Dropping the modifier (Ψ) and the subscripts, we can write Eq. (7) as

$$\mathbf{v} = J\dot{\Theta} \tag{I.7}$$

or

$$J^{\mathrm{T}}\mathbf{v} = [g]\dot{\Theta}. \tag{I.8}$$

Since $[g]$ is invertible, we can write

$$\dot{\Theta} = [g]^{-1}J^{\mathrm{T}}\mathbf{v} \tag{I.9}$$

and

$$\dot{\Theta}^{\mathrm{T}}\dot{\Theta} = \mathbf{v}^{\mathrm{T}}[A]\mathbf{v}, \tag{I.10}$$

where $[A]$ denotes the matrix $(J^{\mathrm{T}}[g]^{-1})^{\mathrm{T}}(J^{\mathrm{T}}[g]^{-1})$. We can make the following observations about $[A]$:

1. $[A]$ is a 3×3 symmetric matrix.
2. $[A]$ is singular and of rank 2, since J and $[g]$ are of rank 2.
3. $[A]$ is at least positive semidefinite, since the quadratic form on the right side (being equal to $\dot{\Theta}^{\mathrm{T}}\dot{\Theta}$) is always greater than or equal to zero.

Equation (I.10), with the left side set of k^2 and the matrix $[A]$ having the above-mentioned properties, describes an infinite cylinder with an elliptic cross section (Strang 1976). However, we know that the velocity vector always lies in the tangent plane. Hence in this case, Eq. (I.10), with its left side set to k^2, describes an ellipse in the tangent plane. This proves statement (2) in Section 2.1.

The shape of an ellipse is determined by the ratio of the length of the semimajor and semiminor axes. In our case, the shape is determined by $|\mathbf{v}|_{\max}/|\mathbf{v}|_{\min} = \sqrt{\lambda_1/\lambda_2}$, which is clearly independent of k.

The area of the ellipse is π times the product of the lengths of the semimajor and semiminor axes. In our case the area is $\pi|\mathbf{v}|_{\max}|\mathbf{v}|_{\min}$. From (I.5), (I.6), and the fact that the product of the eigenvalues equals det $[g]$, the area of the ellipse becomes $\pi k^2(\det [g])^{1/2}$. The above two paragraphs prove statement (3) of Section 2.1.

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